A COURSE OF HIGHER MATHEMATICS

V. I. Smirnov

Volume III Part 2

Complex Variables - Special Functions

ADIWES INTERNATIONAL SERIES IN MATHEMATICS

A. J. LOHWATER
Consulting Editor

A COURSE OF
Higher Mathematics

VOLUME III
PART TWO

V. I. SMIRNOV

Translated by

D. E. BROWN

Translation edited by

I. N. SNEDDON

Simson Professor in Mathematics
University of Glasgow

PERGAMON PRESS

OXFORD · LONDON · EDINBURGH · NEW YORK
PARIS · FRANKFURT

ADDISON-WESLEY PUBLISHING COMPANY, INC.
READING, MASSACHUSETTS · PALO ALTO · LONDON

1964

Copyright © 1964
PERGAMON PRESS LTD.

U. S. A. Edition distributed by
ADDISON-WESLEY PUBLISHING COMPANY, INC.
Reading, Massachusetts · Palo Alto · London

PERGAMON PRESS
International Series of Monographs in
PURE AND APPLIED MATHEMATICS

Volume 60

Library of Congress Catalog Card No. 63-10134

This translation has been made from
the Russian Edition of
V. I. Smirnov's book
Курс высшей математики (Kurs vysshei matematiki),
published in 1957 by Fizmatgiz, Moscow
MADE IN GREAT BRITAIN

CONTENTS

INTRODUCTION ix

FOREWORD TO THE FOURTH EDITION x

CHAPTER I

THE BASIS OF THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE 1

1. Functions of a complex variable **2.** The derivative. **3.** Conformal transformation. **4.** The integral. **5.** Cauchy's theorem. **6.** The fundamental formula of the integral calculus. **7.** Cauchy's formula. **8.** Integrals of Cauchy's type. **9.** Corollaries of Cauchy's formula. **10.** Isolated singularities. **11.** Infinite series with complex terms. **12.** The Weierstrass theorem. **13.** Power series. **14.** Taylor's series. **15.** Laurent's series. **16.** Examples. **17.** Isolated singularities. Point at infinity. **18.** Analytic continuation. **19.** Examples of many-valued functions. **20.** Singularities of analytical functions and Riemann surfaces. **21.** The theorem of residues. **22.** Theorem on the number of zeros. **23.** The inversion of a power series. **24.** The principle of symmetry. **25.** Taylor's series on the circumference of the circle of convergence. **26.** The principal value of an integral. **27.** The principal value of an integral (continuation). **28.** Integrals of Cauchy's type.

CHAPTER II

CONFORMAL TRANSFORMATION AND THE TWO-DIMENSIONAL FIELD 120

29. Conformal transformation. **30.** Linear transformation. **31.** Bilinear transformation. **32.** The function $w=z^2$. **33.** The function $w=k(z+1/z)/2$ **34.** The biangular figure and the strip. **35.** The fundamental theorem. **36.** Christoffel's formula. **37.** Individual cases. **38.** The exterior of the polygon. **39.** The minimum property of the transformation into a circle. **40.** The method of conjugate trigonometric series. **41.** The two-dimensional established flow of liquids. **42.** Examples. **43.** The problem of flow round a contour. **44.** N. E. Zhukovskij's formula. **45.** The two-dimensional problem of electrostatics. **46.** Examples. **47.** The two-dimensional magnetic field. **48.** Schwarz's formula. **49.** The core cot $(s-t)/2$. **50.** Limiting problems. **51.** The biharmonic equation. **52.** The wave equation and analytic functions. **53.** The fundamental theorem. **54.** The diffraction of a two-dimensional wave. **55.** The reflection of elastic waves from rectilinear objects.

CHAPTER III

THE APPLICATION OF THE THEORY OF RESIDUES, INTEGRAL AND FRACTIONAL FUNCTIONS 223

56. Fresnel's integral. **57.** Integration of expressions containing trigonometric functions. **58.** Integration of a rational fraction. **59.** Certain new types of integrals containing trigonometric functions. **60.** Jordan's lemma. **61.** Contour integrals of certain functions. **62.** Examples of integrals of many-valued functions. **63.** Integration of a system of linear equations with constant coefficients. **64.** The expansion of a fractional function into partial fractions. **65.** The function cot z. **66.** The construction of meromorphic functions. **67.** Integral functions. **68.** Infinite products. **69.** The construction of an integral function from its given zeros. **70.** Integrals which depend on parameters. **71.** Euler's integral of the second class. **72.** Euler's integral of the first class. **73.** The infinite product of the function $[\Gamma(z)]^{-1}$. **74.** The representation of $\Gamma(z)$ by a contour integral. **75.** Stirling's formula. **76.** Euler's summation formula. **77.** Bernoulli numbers. **78.** Method of the steepest descent. **79.** Isolation of the principal part of an integral. **80.** Examples.

CHAPTER IV

FUNCTIONS OF SEVERAL VARIABLES AND MATRIX FUNCTIONS 313

81. Regular functions of several variables. **82.** The double integral and Cauchy's formula. **83.** Power series. **84.** Analytic continuation. **85.** Matrix functions. Preliminary propositions. **86.** Power series of one matrix. **87.** Multiplication of power series. Conversion of power series. **88.** Further investigations of convergence. **89.** Interpolation polynomials. **90.** Cayley's identity and Sylvester's formula. **91.** Analytic continuation. **92.** Examples of many-valued functions. **93.** Systems of linear equations with constant coefficients. **94.** Functions of several matrices.

CHAPTER V

LINEAR DIFFERENTIAL EQUATIONS 357

95. The expansion of a solution into a power series. **96.** The analytic continuation of the solution. **97.** The neighbourhood of a singularity. **98.** Regular singularity. **99.** Equations of Fuchs's class. **100.** The Gauss equation **101.** The hypergeometric series. **102.** The Legendre polynomials. **103.** Jacobian polynomials. **104.** Conformal transformation and the formula of Gauss. **105.** Irregular singularities. **106.** Asymptotic expansion. **107.** The Laplace transformation. **108.** The choice of solutions. **109.** The asymptotic representation of solutions. **110.** Comparison of results. **111.** The Bessel equation. **112.** The Hankel function. **113.** The Bessel functions. **114.** The Laplace transformation in more general cases. **115.** The generalized Laguerre polynomials. **116.** Positive

values of the parameter. **117.** The degeneration of the equation of Gauss. **118.** Equations with periodic coefficients. **119.** The case of analytic coefficients. **120.** Systems of linear differential equations. **121.** Regular singularities. **122.** Regular systems. **123.** The form of the solution in the neighbourhood of a singularity. **124.** Canonical solutions. **125.** The connection with regular solutions of Fuchs's type. **126.** The case of the arbitrary U_s. **127.** Expansion in the neighbourhood of an irregular singularity. **128.** Expansions into uniformly convergent series.

CHAPTER VI

SPECIAL FUNCTIONS

§ 1. Spherical functions 493

129. The determination of spherical functions. **130.** The definite expression for spherical functions. **131.** The orthogonal properties. **132.** The Legendre polynomials. **133.** The expansion in terms of spherical functions. **134.** Proof of convergence. **135.** The connection between spherical functions and limit problems. **136.** The Dirichlet and Neumann problems. **137.** The potential of voluminous masses. **138.** The potential of a spherical shell. **139.** The electron in a central field. **140.** Spherical functions and the linear representation of rotating groups. **141.** The Legendre function. **142.** The Legendre functions of the second kind.

§ 2. Bessel functions 537

143. The determination of Bessel functions. **144.** Relationships between Bessel functions. **145.** The orthogonality of Bessel functions and their zeros. **146.** Converting function and integral representation. **147.** The Fourier—Bessel formula. **148.** The Hankel and Neumann functions. **149.** The expansion of the Neumann function with an integer subscript. **150.** The case of the purely imaginary argument. **151.** Integral representatio.n **152.** The asymptotic representation of Hankel functions. **153.** Bessel functions and the Laplace equation. **154.** The wave equation in cylindrical coordinates. **155.** The wave equation in spherical coordinates.

§ 3. The Hermitian and Laguerre polynomials 584

156. The linear oscillator and the Hermitian polynomial. **157.** Orthogonality. **158.** The conversion function. **159.** Parabolic coordinates and Hermitian functions. **160.** The Laguerre polynomials. **161.** The connection between Hermitian and Laguerre polynomials. **162.** The asymptotic expression for Hermitian polynomials. **163.** The asymptotic expression for Legendre polynomials.

§ 4. Elliptic integrals and elliptic functions 604

164. The transformation of elliptic integrals into normal form. **165.** The conversionof the integrals into a trigonometric form. **166.** Examples. **167.** The conversion of elliptic integrals. **168.** General properties of elliptic functions. **169.** Fundamental lemma. **170.** The Weierstrass

function. **171.** The differential equation for $\wp(u)$ (608). **172.** The functions $\sigma_k(u)$. **173.** The expansion of a periodic integral function. **174.** The new notation. **175.** The function $\vartheta_1(v)$. **176.** The function $\vartheta_k(v)$. **177.** The properties of theta-functions. **178.** An expression for the numbers e_k in terms of ϑ_s. **179.** The elliptic Jacobian functions. **180.** The fundamental properties of Jacobian functions. **181.** The differential equation for Jacobian functions. **182.** Addition formulae. **183.** The connection between the functions $\wp(u)$ and $\mathrm{sn}(u)$. **184.** Elliptic coordinates. **185.** The introduction of elliptic functions. **186.** The Lamé equation. **187.** The simple pendulum. **188.** An example of conformal transformation.

SUPPLEMENT

THE CONVERSION OF MATRICES INTO THE CANONICAL FORM 670

189. Auxiliary hypothesis. **190.** The case of simple zeros. **191.** The first stage of the transformation in the case of repeated zeros. **192.** Conversion into the canonical form. **193.** The determination of the structure of the canonical form. **194.** Examples.

INDEX 697

INTRODUCTION

SOME observations on the aims and history of Prof. Smirnov's five-volume course on higher mathematics have been made in the Introduction to the first volume of the present English edition.

In the present volume which forms the second part of Vol. III the author comes to the discussion of one of the central subjects of modern pure mathematics, an understanding of which is essential also to engineers and physicists — the theory of functions of a complex variable. Prof. Smirnov's approach is classical and for that reason is an excellent introduction for those students who will go on to study the more modern developments of the theory of functions while at the same time presenting a complete picture of those aspects of the theory (conformal mappings, differential equations in the complex plane, calculus of residues) which are of most direct interest to applied mathematicians. An interesting feature of the book is that it was the first textbook at this level to include a chapter on the theory of functions of several complex variables, a subject which forms an essential tool to workers in quantum field theory.

Special functions continue to play an important part in the education of both pure and applied mathematicians. This is recognized by Prof. Smirnov. He follows his lucid account of the general theory of functions with a full treatment of those properties of special functions which are necessary for the proper understanding of mathematical physics and engineering.

Both sides of the coin are engraved with the author's characteristic style and the result is a work of great interest which has already been acknowledged as a classic in the countries of Eastern Europe and is now whole heartedly commended to students in the English-speaking world.

I. N. SNEDDON

FOREWORD TO THE FOURTH EDITION

IN THIS edition Volume III was divided into two parts. The second part contains material from the former Volume III, beginning with the chapter dealing with the principles of the theory of the functions of the complex variable. It was slightly rearranged and new material was added. This new material is related mainly to the treatise of the integrals of Cauchy's type and to the approximate calculation of integrals by the method of the steepest descent.

I was greatly assisted in my treatment of the latter by Professor G. J. Petrashen', to whom I am deeply grateful.

References to the first part of Volume III are indicated, for example: [III_1, 44].

V. SMIRNOV

CHAPTER I

THE BASIS OF THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

1. Functions of a complex variable. In the differential and integral calculus we assumed that the independent variable, as well as the function, could only be real. Later, in higher algebra we analysed the most elementary function, viz. the polynomial, in which the independent variable took complex as well as real values. In this chapter we shall develop the analytical basis so as to include functions of a complex variable.

Take, for example, the polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n,$$

where a_k are given complex numbers. We can assume that the independent variable z also assumes arbitrary complex values; in this event the function $f(z)$ is defined for arbitrary complex values of z.

The same can be said about the rational function:

$$\frac{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}{b_0 z^m + b_1 z^{m-1} + \ldots + b_m}$$

or even about expressions containing radicals, for example:

$$\sqrt{z - 1}.$$

In Chapter VI of Volume I we defined elementary transcendental functions for the case when the independent variable assumed complex values; thus we have for the exponential function:

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y)$$

and having thus defined the exponential function, we can also define

trigonometric functions the arguments of which assume complex values:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}; \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2};$$
$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\,\frac{e^{i2z} - 1}{e^{i2z} + 1}; \qquad \cot z = \frac{\cos z}{\sin z} = i\,\frac{e^{i2z} + 1}{e^{i2z} - 1}. \tag{1}$$

Let us recall the expression for the natural logarithm of a complex number:

$$\log z = \log |z| + i \arg z, \tag{2}$$

where $|z|$ is the modulus of z and $\arg z$ denotes the amplitude of the variable z. Similarly, by considering the functions inverse to (1), we arrive at the inverse circular functions of a complex variable:

$$\operatorname{arc\,sin} z, \quad \operatorname{arc\,cos} z, \quad \operatorname{arc\,tan} z, \quad \operatorname{arc\,cot} z.$$

It can easily be shown that these functions can be expressed by a logarithm.

Assume, for example, that

$$z = \tan w = \frac{e^{i2w} - 1}{i\,(e^{i2w} + 1)},$$

whence

$$i(e^{i2w} + 1)\,z = e^{i2w} - 1$$

or

$$e^{i2w} = \frac{1 + iz}{1 - iz}.$$

Multiplying the numerator and the denominator by i and taking logarithms, we obtain:

$$w = \operatorname{arc\,tan} z = \frac{1}{2i} \log \frac{i - z}{i + z}.$$

Similarly, if we assume that:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i},$$

we obtain a quadratic equation for e^{iw}:

$$e^{2iw} - 2iz\,e^{iw} - 1 = 0,$$

whence

$$e^{iw} = iz + \sqrt{1 - z^2},$$

and consequently:

$$w = \frac{1}{i} \log (iz + \sqrt{1 - z^2})$$

where both values of the square root must be taken.

We shall see later that all the elementary functions of a complex variable mentioned above have a derivative i.e. a definite limit of the relationship

$$\frac{f(z+\Delta z)-f(z)}{\Delta z},$$

exists when the complex expression Δz tends to zero. This chapter is devoted entirely to the study of the theory of functions of a complex variable possessing a derivative. We shall see that this theory is distinguished by great clarity and simplicity and has wide applications in many branches of natural science and engineering. In this chapter we shall give a short outline of the theory itself and in the following chapters we shall deal with its applications. In this way we hope to achieve a clearer and more compact treatment of the theoretical basis.

In future we shall frequently use the geometric interpretation of complex numbers which we mentioned earlier in [I, 170].

Let us briefly recall this method of interpretation. Consider a plane referred to perpendicular axes OX and OY; every point in this plane can be defined by two real coordinates (x, y), or by one complex coordinate $x + iy$, which we shall in fact use in future. In this sense the plane is known as the plane of the complex variable, the X axis as the real axis and the Y axis as the imaginary axis. In the following chapters, in addition to this point representation of a complex variable we shall also use the vector representation: when the complex number $x + iy$ is represented by a vector the components components of which are equal to x and y. The relationship between these interpretations becomes clear at once: if a vector is drawn from the origin to the point with complex coordinate $x + iy$ then this vector corresponds to the same complex number $x + iy$. In general, if a vector is drawn in our plane from a point A with complex coordinate $a_1 + ia_2$ to a point B with complex coordinate $b_1 + ib_2$, then this vector $\overrightarrow{AB}$ corresponds to a complex number equal to the difference between the complex coordinates of its origin and terminus:

$$(b_1 - a_1) + i(b_2 - a_2).$$

Let us recall some earlier results from [I, 171 and 172].

The addition of complex numbers follows the laws of the geometric addition of the vectors, corresponding to those numbers. The modulus of the complex number is equal to the length of the corresponding vector,

and the amplitude to the angle which this vector makes with the X axis.

If the complex variable varies, the corresponding point moves in the plane.

We say that $z = x + iy$ tends to the limit $a = a + ib$, where a and b are constants, if the modulus of the difference

$$|a - z| = \sqrt{(a - x)^2 + (b - y)^2}$$

tends to zero.

It follows from the above expression that, since the terms under the radical are positive, $|a - z| \to 0$ is equivalent to

$$x \to a \text{ and } y \to b.$$

Hence

$$x + iy \to a + ib$$

is equivalent to

$$x \to a \text{ and } y \to b.$$

Obviously the variable point M which corresponds to $z = x + iy$ will in this case tend to a limiting position represented by the point A with complex coordinate $a = a + ib$. It can easily be proved, though we shall not do so here, that the usual theorems on the limit of a sum, product and quotient apply.

Note also that it follows from the definition of the limit that $z \to 0$ is equivalent to $|z| \to 0$. Also, if $z \to a$ it is clear that $|z| \to |a|$.

Cauchy's test for the existence of a limit also applies to a complex variable.

Assume, for example, that we have a numbered sequence of values of the complex variable:

$$z_1 = x_1 + y_1 i;$$

$$z_2 = x_2 + y_2 i; \ldots;$$

$$z_n = x_n + y_n i; \ldots$$

The existence of a limit of this sequence is equivalent to the existence of limits of the real sequences x_n and y_n; the necessary and sufficient condition for such a limit to exist is that the absolute values of the differences $|x_n - x_m|$ and $|y_n - y_m|$ should be as small as possible, when n and m are sufficiently large [I, 31].

Taking into consideration that

$$|z_n - z_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}$$

and that the terms under the radical are positive, we can see that the necessary and sufficient condition for the limit of the sequence z_n to exist is that $|z_n - z_m|$ should tend to become as small as desired when n and m are sufficiently large, i.e. strictly speaking, for any given positive ε an N exists such that $|z_n - z_m| < \varepsilon$ if n and $m > N$. Everything said at the beginning [25] of Volume 1 about real variables also applies generally to the complex variable. The necessary and sufficient condition for a limit of a complex variable z to exist consists of the following [I, 31]: for any given positive ε a value of the variable z exists such that $|z' - z''| < \varepsilon$, provided z' and z'' are any two values which exceed the value mentioned. We shall say in future that the complex variable z tends to infinity when $|z| \to +\infty$.

Consider now a function of the complex variable

$$w = f(z)$$

and let us agree on our terminology. The function $f(z)$ can be defined either in the whole plane or in a definite domain of the plane of the complex variable z, for example, in a definite circle, rectangle, annulus etc. In this domain we shall distinguish interior points and points on the contour. Thus, for example, in the case of a circle, centre the origin and unit radius, interior points are determined by the condition

$$|z| < 1 \text{ or } x^2 + y^2 < 1,$$

whilst the contour is the circumference

$$|z| = 1 \text{ or } x^2 + y^2 = 1.$$

An interior point has the characteristic property that not only the point itself but also a neighbourhood of it belongs wholly to the domain, i.e. the point M will be an interior point of a domain, provided that a sufficiently small circle, centre at M, belongs wholly to that domain.

Points of the contour are not interior points, though interior points can be as close as we please to the contour. We also assume that our domain does not break down into separate parts (connectivity of the domain), or, to be more accurate, we assume that any two points in the domain can be connected by a line which also lies wholly within the domain. In future we shall understand by a domain *the set of interior points of the domain*. If the contour is added to the domain then the domain is *closed*.

We shall call a *domain bounded* if all its points are at a finite distance from the origin. In future we shall enlarge further on the characteristics of a domain.

Let us now return to the function $w = f(z)$. Assume that it is defined in a domain B, i.e. for every point z inside B $f(z)$ has a definite complex value (we are considering single-valued functions). Let z_0 be a point in B. The function $f(z)$ is said to be continuous at the point z_0 if $f(z) \to f(z_0)$ when $z \to z_0$, i.e. for every given positive ε a positive value of η exists such that $| f(z) - f(z_0) | < \varepsilon$ provided $| z - z_0 | < \eta$. A function is said to be continuous in B if it is continuous at every point of the domain B. The function $f(z)$ can be defined not only in B but also on the contour l of the domain, i.e. in the closed domain B. We say that such a function is continuous in the closed domain B provided it is continuous at every point of the closed domain B. When defining continuity at any given point z_0 on the contour l it must be remembered that the point z may tend to z_0 in any direction, though without leaving the domain B. The theorem [I, 43] on real variables is also valid: if $f(z)$ is continuous in a bounded closed domain, it is uniformly continuous in this domain, i.e. for any given positive ε a positive η exists (which is one and the same for the whole domain such that $|f(z_1) - f(z_2)| < \varepsilon$ provided $| z_1 - z_2 | < \eta$, where z_1 and z_2 belong to the given closed domain.

Let us write z and $w = f(z)$, separated into real and imaginary parts:

$$z = x + yi;$$

$$w = f(z) = u + vi.$$

To specify z implies specifying x and y, and to specify $f(z)$ implies specifying u and v, i.e. we must consider u and v as functions of x and y:

$$w = f(z) = u(x, y) + v(x, y)i. \tag{3}$$

With elementary functions the separation into real and imaginary parts can be performed by simple operations for example:

$$w = z^2 = (x + yi)^2 = (x^2 - y^2) + 2xyi.$$

Assume that $z_0 = x_0 + y_0 i$; the condition $z \to z_0$ is equivalent to $x \to x_0$ and $y \to y_0$.

It follows from the definition of continuity at the point z_0 that when $z \to z_0$, i.e. when $x \to x_0$ and $y \to y_0$ we have:

$$f(z) \to f(z_0)$$

or

$$u(x, y) + v(x, y)i \to u(x_0, y_0) + v(x_0, y_0)\, i,$$

which is equivalent to

$$u(x, y) \to u(x_0, y_0)$$

and

$$v(x, y) \to v(x_0, y_0).$$

Consequently, the continuity of $f(z)$ at the point z_0 is equivalent to the continuity of $u(x, y)$ and $v(x, y)$ at the point (x_0, y_0).

Separating the real and imaginary parts and using the property of continuity of elementary functions of the a variable, we can see that the polynomials e^z, $\sin z$, $\cos z$, are continuous functions in the whole plane of the complex variable. A rational function is continuous everywhere except at points where its denominator vanishes. Similarly, $\tan z$ is continuous everywhere except at the points where $\cos z$ vanishes. As with a real variable, the sum and product of a finite number of continuous functions is also a continuous function and the quotient of two continuous functions is also continuous, except for those values of z for which the denominator vanishes.

In our further treatment of the theory we shall first consider single-valued functions, then later we shall consider specially the problem of many-valued functions.

The function $\sqrt{z-1}$, the function (2) and the inverse circular functions are all examples of many-valued functions.

2. The derivative. Assume that $f(z)$ is defined at a point z and at all points sufficiently close to z. The derivative $f'(z)$ at the point z is determined, as already mentioned, by the usual method, as the limit of the equation

$$\frac{f(z + \Delta z) - f(z)}{\Delta z}, \tag{4}$$

and this limit should be finite and determinate when the increment Δz tends to zero in any manner.

It is easy to prove that, as with a real variable, a constant factor can be taken outside the symbol of differentiation and that the usual laws for differentiating sums, products and quotients apply [I, 47].

It is also easy to show by applying Newton's formula, that the usual law for differentiating power functions applies when the index is a positive integer:

$$(z^n)' = nz^{n-1} . \tag{5}$$

We can thus assert that a polynomial has a derivative at any point z and a rational fraction has a derivative everywhere, except for those values of z for which its denominator vanishes.

Furthermore, the usual rule for differentiating a function of a function applies:

$$F'_z(w) = F'_w(w) \cdot w'_z, \tag{6}$$

assuming, of course, that both derivatives on the right-hand side of the equation exist. As with a real variable [I, 45], the existence of a derivative at a certain point proves the continuity of $f(z)$ at that point.

Assume that the function $f(z)$ is defined in a domain B, and that it has a derivative at every point in B. If this is the case we simply say that $f(z)$ has a derivative in the domain B. This derivative $f'(z)$ will be a single-valued function in B.

We shall now introduce an important new definition. We say that $f(z)$ is *regular* or *holomorphic* in B if it is a single-valued function in B, and has a continuous derivative $f'(z)$ in B. Note that the continuity of $f(z)$ in B follows from the existence of a derivative. It is sometimes said that $f(z)$ is regular (or holomorphic) at the point z_0. This means that $f(z)$ is regular in a domain which contains z_0.

Let us turn to formula (3) in which both z and $f(z)$, are separated real and imaginary parts and let us then ask the question: what conditions must be satisfied by the functions $u(x, y)$ and $v(x, y)$ in order that $f(z)$ should be regular in the domain B. To start with we assume that $f(z)$ is regular in B and thence we draw conclusions regarding $u(x, y)$ and $v(x, y)$.

As we mentioned above when defining a derivative (the existence of which we are assuming), the increment of the independent variable $\Delta z = \Delta x + \Delta y i$ may tend to zero in any manner.

Select a point M in B, the coordinate of which is $z = x + yi$, and a variable point N with coordinate $z + \Delta z = (x + \Delta x) + (y + \Delta y)\, i$, so that N tends to M.

We shall consider two cases of N tending to M, i.e. of Δz tending to zero.

In the first case we assume that N tends to M while remaining on a straight line parallel to the X axis; we then have

$$\Delta y = 0 \text{ and } \Delta z = \Delta x, \tag{7}$$

and in the second case we assume that N tends to M while remaining on a straight line parallel to the Y-axis; we then have

$$\Delta x = 0 \text{ and } \Delta z = i\Delta y. \tag{8}$$

We can now construct the derivative $f'(z)$ in both cases. In the general case we have:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \tag{9}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y}.$$

This gives in the first case:

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right].$$

We can thus see that the real and imaginary parts on the right-hand side of the equation must have a limit, i.e. the functions $u(x, y)$ and $v(x, y)$ must have partial derivatives with respect to x, in which case the formula applies:

$$f'(z) = \frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial x} i. \tag{10}$$

Similarly, in the second case we have, from (8) and (9):

$$f'(z) = \lim_{\Delta y \to 0} \frac{1}{i} \left[\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + i \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \right]$$

or

$$f'(z) = \frac{\partial v(x,y)}{\partial y} - \frac{\partial u(x,y)}{\partial y} i. \tag{11}$$

Comparing the expressions (10) and (11) for $f'(z)$ we obtain the conditions which must be satisfied by the partial derivatives of $u(x, y)$ and $v(x, y)$:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}; \quad \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y}. \tag{12}$$

Notice that, by (10) and (11), the continuity of the partial derivatives of the first order of the functions $u(x, y)$ and $v(x, y)$

follows from the continuity of $f'(z)$. Our earlier considerations lead us to the following conclusion. In order that $f(z)$ should be regular in B the following conditions must be satisfied: *$u(x, y)$ must have continuous partial derivatives of the first order with respect to x and y in B and these derivatives must satisfy the relationship* (12).

We can now show that *these conditions are not only necessary but are also sufficient for the regularity of $f(z)$ in B.* We shall assume that the above conditions are satisfied and prove the existence of a continuous derivative $f'(z)$. Taking into account the continuity of the partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to x and y, we can write [I, 68]:

$$u(x + \Delta x, y + \Delta y) - u(x, y) = \\ = \frac{\partial u(x, y)}{\partial x}\Delta x + \frac{\partial u(x, y)}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

$$v(x + \Delta x, y + \Delta y) - v(x, y) = \\ = \frac{\partial v(x, y)}{\partial x}\Delta x + \frac{\partial v(x, y)}{\partial y}\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y,$$

where ε_k tends to zero simultaneously with Δx and Δy. Using the latter expressions for the construction of the increment of the function $f(z + \Delta z) - f(z)$ and substituting this in the equation (4), we obtain:

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \\ = \frac{\left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y}{\Delta x + i\Delta y},$$

whence, by making use of the conditions (12), we can rewrite this relationship in the form:

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \\ = \frac{\frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y)}{\Delta x + i\Delta y} + \varepsilon_5 \frac{\Delta x}{\Delta x + i\Delta y} + \varepsilon_6 \frac{\Delta y}{\Delta x + i\Delta y}.$$

where

$$\varepsilon_5 = \varepsilon_1 + i\varepsilon_3 \text{ and } \varepsilon_6 = \varepsilon_2 + i\varepsilon_4$$

tend to zero simultaneously with Δz.

It is easy to see that the last two terms on the right-hand side also tend to zero.

In fact, for example:

$$\left|\varepsilon_5 \frac{\Delta x}{\Delta x + i\Delta y}\right| = |\varepsilon_5| \frac{|\Delta x|}{\sqrt{\Delta x^2 + \Delta y^2}};$$

the first factor tends to zero and the second factor does not exceed unity.

Thus the last formula can be rewritten in the form:

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u(x, y)}{\partial x} + \frac{\partial v(x, y)}{\partial x} i + \varepsilon_7,$$

where ε_7 tends to zero simultaneously with Δz and the first two terms on the right-hand side are independent of Δz. The equation (4) thus tends to a definite limit which is defined by formula (10). Hence the above conditions for $u(x, y)$ and $v(x, y)$ are necessary and sufficient for $f(z)$ to be regular in B. The equations (12) are usually known as the *Cauchy-Riemann equations*.

It may be recalled that we have already encountered these equations: they were satisfied by the velocity potential and the stream function in the case of the steady-state plane flow of an ideal incompressible fluid [II, 74]. We can thus see that the fundamental equations of the theory of functions of a complex variable (12) are, at the same time, also fundamental equations in the study of the above hydrodynamical problem. This is the basis for numerous applications of the theory of functions of a complex variable to hydrodynamics and we shall deal with this in the next chapter.

Let us now mention a very important circumstance which follows from the equations (12). We shall see later that regular functions $u(x, y)$ and $v(x, y)$ have derivatives of all orders. Differentiating term by term the first of the equations (12) with respect to x and the second with respect to y and adding, we obtain:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad (13_1)$$

Similarly, it is easy to deduce from the equations (12) that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad (13_2)$$

This shows that *the real and imaginary parts of the regular function $f(z)$ should satisfy the Laplace equation*, i.e. they should be harmonic functions. In the following chapters we shall analyse in greater detail the connection between the theory of functions of a complex variable and the Laplace equation.

Notice another important circumstance which follows from the equations (13), viz. in the construction of a regular function we can take its real part arbitrarily, i.e. we can take for $u(x, y)$ any solution of the equation (13_1). We can show that in this case $v(x, y)$ is defined up to a constant term.

In fact, it follows from the equations (12) that

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy,$$

whence

$$v(x, y) = \int^{(x, y)} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy + C. \tag{14}$$

It remains to show that the given line integral is independent of the path and yields a certain function of its upper limit [II, 171]. Let us recall that the condition for a line integral

$$\int X\,dx + Y\,dy$$

to be independent of the path can be written as follows:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}\,.$$

Applying this to the integral (14) we obtain:

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and this condition is satisfied since we took a harmonic function for $u(x, y)$. Remember that, if $u(x, y)$ is a single-valued function then $v(x, y)$ may turn out to be a many-valued function, provided the region in which we are applying formula (14) is multiply connected [II, 72].

Let us now turn to some examples. A polynomial is obviously a regular function in the whole z-plane. A rational fraction is a regular function in any domain which does not contain the zeros of its denominator. If we take, for example, $f(z) = z^2$, then $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. It can easily be shown that these functions satisfy the equations (12).

Let us show, for example, that the exponential function

$$e^z = e^x(\cos y + i \sin y)$$

is regular in the whole plane. In this case:

$$u(x, y) = e^x \cos y; \quad v(x, y) = e^x \sin y,$$

whence it follows directly that:

$$\frac{\partial u}{\partial x} = e^x \cos y; \quad \frac{\partial u}{\partial y} = -e^x \sin y;$$

$$\frac{\partial v}{\partial x} = e^x \sin y; \quad \frac{\partial v}{\partial y} = e^x \cos y.$$

These partial derivatives are continuous and satisfy the equations (12). We evaluate the derivative in accordance with the formula (10):

$$(e^z)' = e^x \cos y + ie^x \sin y = e^x (\cos y + i \sin y), \text{ i.e. } (e^z)' = e^z.$$

We have thus obtained the same law for differentiating an exponential function as in the case of a real variable. It is now easy to prove that $\sin z$ and $\cos z$ also have continuous derivatives in the whole z-plane. These derivatives are evaluated by the same laws as those for real variables. For, on applying the rules for the differentiation of an exponential function and a function of a function, we obtain:

$$(\sin z)' = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)' = \frac{e^{iz} + e^{-iz}}{2} = \cos z,$$

$$(\cos z)' = \left(\frac{e^{iz} + e^{-iz}}{2}\right)' = i\frac{e^{iz} - e^{-iz}}{2} = -\sin z.$$

3. Conformal transformation. Let us now explain the geometric meaning of the concept of functional dependence and of a derivative. Assume that the function $f(z)$ is regular in a domain B in the XY-plane. Every value of z in the domain B corresponds to a definite value $w = f(z)$, and the set of all the values $w = u + iv$ which correspond to all the z's in B will fill a new domain B_1 which we shall draw in a new plane of the complex variable $u + iv$ (Fig. 1). We can

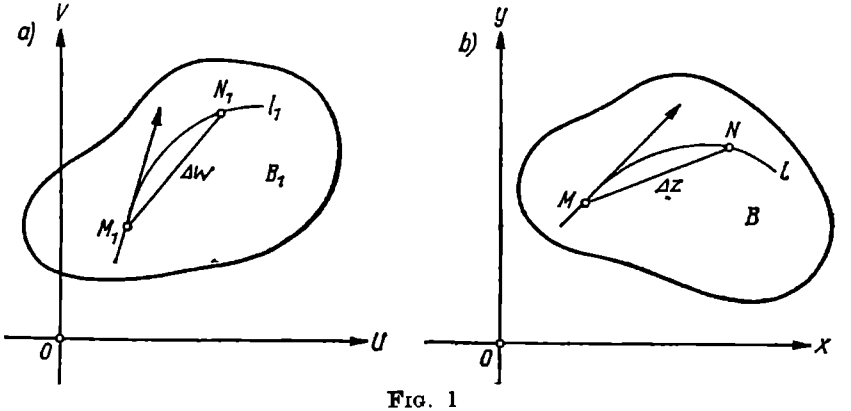

FIG. 1

therefore say that our *function* $f(z)$ *transforms the domain* B *into the domain* B_1. Strictly speaking we should have investigated more closely the dependence between the points z and w as given by our function and proved that the set of the values of w would also fill a certain domain. Later, when we have at our disposal the necessary analytic apparatus, we shall in fact carry out this closer investigation; for the moment, however, we shall restrict ourselves to making general obser vations only, which will, nevertheless, permit the reader to understand the geometric meaning of the terms introduced. We shall see later that, if the derivative $f'(z)$ does not vanish at a point z, then a sufficiently small circle, centre at z, will be transformed into a domain in the w-plane, which includes the corresponding point $w = f(z)$.

Let us now explain the geometric meaning of the *modulus* and *amplitude* of a derivative. Assume that the derivative $f'(z)$ does not vanish at the given point. Take two adjacent points z and $z + \Delta z$. Their corresponding points in the region B_1 will be w and $w + \Delta w$.

Take the lines MN and M_1N_1 which join the points z and $z + \Delta z$, and w and $w + \Delta w$ respectively. These vectors correspond to the complex numbers Δz and Δw. Thus, the ratio of the lengths of these vectors will be as follows:

$$\frac{|M_1N_1|}{|MN|} = \frac{|\Delta w|}{|\Delta z|}$$

or, remembering that the modulus of a quotient is equal to the quotient of the moduli:

$$\frac{|M_1N_1|}{|MN|} = \left|\frac{\Delta w}{\Delta z}\right|.$$

In the limit, when the point N tends to M and the point M_1 tends to N_1, we have:

$$\lim \frac{|M_1N_1|}{|MN|} = |f'(z)|,$$

i.e. *the modulus of the derivative* $f'(z)$ *gives the change in linear dimensions* at the point z during the transformation by the function $f(z)$. If, for example, $f(z) = z^2 + z + 3$, then in the course of the transformation the linear dimensions at the point $z = 1$ will be magnified three times.

Let us now explain the geometric meaning of the amplitude of a derivative. Assume that the point N tends to the point M along a line l and let l_1 be the corresponding line in the domain B_1 (Fig. 2). The amplitude of the complex number Δz gives the angle between the vector $\overline{NM}$ and the real axis and, similarly, arg w gives the angle

between the vector $\overline{M_1N_1}$ and the real axis. The difference between the amplitudes, i.e.

$$\arg \Delta w - \arg \Delta z,$$

gives the angle between the vectors $\overline{M_1N_1}$ and $\overline{MN}$ and this angle is read from the vector $\overline{MN}$ in the counter-clockwise direction. Remembering that the amplitude of a quotient is equal to the difference between the amplitudes of the dividend and the divisor, we can write:

$$\arg \Delta w - \arg \Delta z = \arg \frac{\Delta w}{\Delta z} .$$

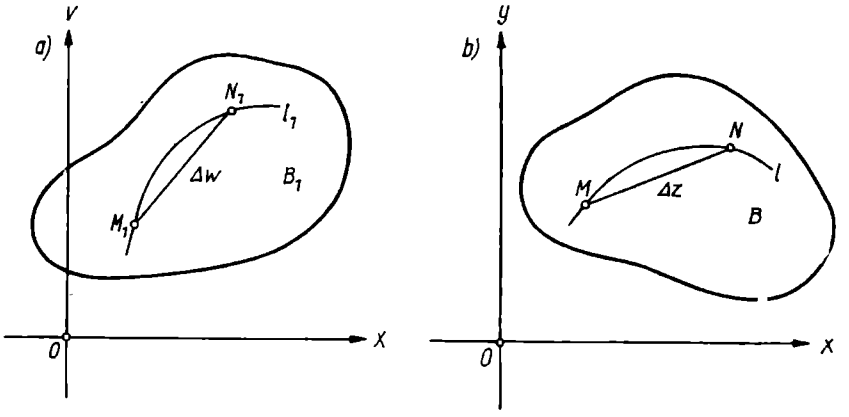

FIG. 2

In the limit the direction of the vector $\overline{MN}$ coincides with the direction of the tangent to the curve l at the point M, and the direction of the vector $\overline{M_1N_1}$ coincides with the direction of the tangent to the curve l_1 at the point M_1.

Taking the limit of the above formula we can see that *the amplitude of the derivative* $\arg f'(z)$ *gives the angle of rotation at the given point* z, made as a result of the transformation by the function $f(z)$. In other words, if an arbitrary curve l were to be drawn through z, and had a definite tangent at the point z then, as a result of the transformation, a new curve l_1 would be produced, the tangent to which would make at the corresponding point w an angle with the above tangent equal to the amplitude of the derivative. If we take two curves in the domain B, which intersect at a certain angle at the point z then, as a result of the transformation the tangents to

these curves will rotate by the same angle, equal to the amplitude of the derivative and, consequently, the angle between the transformed curves will remain the same in magnitude and direction as before, i.e. *transformation by a regular function conserves the angles at all points at which the derivative of this function does not vanish.* A transformation under which all angles are conserved is usually called *conformal.*

If we draw a net of curves in the domain B in the XY- plane then, as a result of the transformation, we shall again obtain a net of curves, which will, of course, be different; the angles between these curves will, however, remain unchanged except at points where the derivative vanishes. If we take, for example, a net of straight lines parallel to the axes in the domain B, then we shall obtain, generally speaking, a net of curves in the domain B_1; but the angles between these curves will remain right angles, as before, i.e. the net will remain orthogonal. Moreover, if we divide the domain B into small similar squares, then each one of these squares will be transformed in the domain B_1 into a small curvilinear rectangle the sides of which will be approximately equal to the product of the length of the side of the square and the modulus of the derivative at an arbitrary point in the square, i.e. this curvilinear rectangle will also be a square up to higher order terms; however the value of $|f'(z)|$ will be different at different points and therefore the curvilinear squares filling B will have sides of different lengths.

Let us consider in greater detail the problem of a function of a function:

$$F(w), \text{ where } w = f(z).$$

Let $f(z)$ be regular in the domain B and let it transform this domain into a domain B_1. We assume further that $F(w)$ is also regular in the domain B_1. In this case the function of a function $F(w)$ will be regular in the domain B and the differentiation rule, as given by formula (6), will apply for it.

4. The integral. Let l be a curve in the XY-plane. We shall always assume that a curve has a parametric equation of the type:

$$x = \varphi_1(t); \quad y = \varphi_2(t),$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are continuous functions with continuous derivatives, or that the curve consists of a finite number of sections, each one of which, from beginning to end, has the properties just mentioned.

We already know [II, 66] that the evaluation of a line integral

$$\int [X(x, y)\,dx + Y(x, y)\,dy]$$

simply involves the evaluation of the usual definite integral. It is sufficient to substitute $\varphi_1(t)$ and $\varphi_2(t)$ for x and y in the integrand, where $dx = \varphi_1'(t)\,dt$ and $dy = \varphi_2'(t)\,dt$. We now have to integrate with respect to the variable t within the limits of variation corresponding to the curve l.

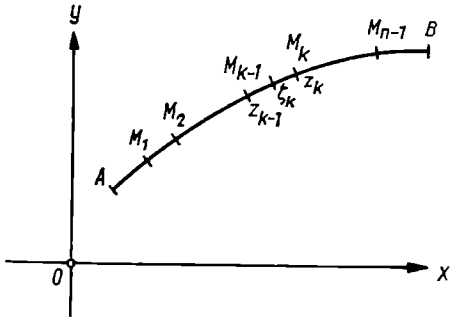

FIG. 3

Assume that a continuous function $f(z)$ is given on the curve l (Fig. 3). We shall explain the concept of the contour integral of the function $f(z)$ over the curve (contour) l. Divide the curve l into sections between the points $M_1, M_2, \ldots, M_{n-1}$, and let z_k be the complex coordinate of the point of division M_k; for the sake of symmetry we shall denote the complex coordinate of the beginning of the curve A by z_0 and of the end of curve B by z_n. Moreover, let ζ_k be a point on the arc of the curve $M_{k-1}M_k$. Let us write the sum of the products:

$$\sum_{k=1}^{n} f(\zeta_k)\,(z_k - z_{k-1}).$$

The limit of this sum, when the number n of divisions increases indefinitely and every arc $M_{k-1}M_k$ becomes indefinitely smaller, is known as *the contour integral of the function $f(z)$ over the contour l:*

$$\int_l f(z)\,dz = \lim \sum_{k=1}^{n} f(\zeta_k)\,(z_k - z_{k-1}). \tag{15}$$

Denote $z_k = x_k + y_k i$ and $\zeta_k = \xi_k + \eta_k i$. Separating the real and imaginary parts of $f(z)$ we can write:

$$\sum_{k=1}^{n} f(\zeta_k)\,(z_k - z_{k-1}) =$$

$$= \sum_{k=1}^{n} [u(\xi_k, \eta_k) + v(\xi_k, \eta_k)\,i]\,[(x_k - x_{k-1}) + (y_k - y_{k-1})i]$$

or

$$\sum_{k=1}^{n} f(\zeta_k)\,(z_k - z_{k-1}) =$$

$$= \sum_{k=1}^{n} u(\xi_k, \eta_k)\,(x_k - x_{k-1}) - v\,(\xi_k, \eta_k)\,(y_k - y_{k-1}) +$$

$$+ i \sum_{k=1}^{n} v(\xi_k, \eta_k)\,(x_k - x_{k-1}) + u(\xi_k, \eta_k)\,(y_k - y_{k-1}).$$

Bearing in mind the assumptions made with regard to the line l and the continuity of $f(z)$, both sums standing on the right-hand side of the equation tend to limits, equal to the corresponding line integrals over l; we thus obtain an expression for the integral (15) in the form of the sum of the usual real line integrals:

$$\int_l f(z)\,\mathrm{d}z = \int_l [u(x, y)\,\mathrm{d}x - v(x, y)\,\mathrm{d}y] + i \int_l [v(x, y)\,\mathrm{d}x + u(x, y)\,\mathrm{d}y]. \quad (16)$$

Above, for the sake of simplicity, we assumed that the line l has a beginning and an end; it is evident that this definition still holds when we integrate round closed contours.

The contour integral (15) possesses exactly the same properties as the usual real line integral [II, 66]. Let us recall these. A constant factor can be taken outside the sign of the integral. The integral of a sum is equal to the sum of integrals. When the direction of the contour of integration changes, the sign of the integral changes. If the contour of integration were to be divided into several separate parts, then the integral round the whole contour would be equal to the sum of the integrals over the separate parts.

We now introduce an important inequality for the magnitude of the integral (15). Assume that the modulus of the integrand does not exceed a certain positive number M on the contour l, i.e.

$$|f(z)| \leqslant M \qquad (z \text{ on } l), \quad (17)$$

and let s be the length of the contour l. In this case the following inequality holds for the integral (15):

$$\left|\int_l f(z)\,\mathrm{d}z\right| \leqslant Ms. \quad (18)$$

For let us return to the sum (15), which gives the limit of the integral.

Taking into account the fact that the modulus of a sum is less than or equal to the sum of the moduli of the terms, we obtain:

$$\left|\sum_{k=1}^{n} f(\zeta_k)\,(z_k - z_{k-1})\right| \leqslant \sum_{k=1}^{n} |f(\zeta_k)|\,|z_k - z_{k-1}|$$

or, by (17):

$$\left|\sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})\right| \leqslant M \sum_{k=1}^{n} |z_k - z_{k-1}|.$$

The factor multiplying M obviously represents the perimeter of a step line inscribed in the contour l, and by taking the limit of this latter inequality, we obtain the inequality (18).

The integral (15) satisfies a more precise inequality, viz. denoting by ds the differential of the arc of the curve l we obtain the following formula:

$$\left|\int_l f(z)\,dz\right| \leqslant \int_l |f(z)|\,ds. \tag{19}$$

This inequality can be obtained directly, if we replace $f(z)$ by $|f(z)|$ and $dz = dx + i dy$ by $|dz| = \sqrt{dx^2 + dy^2} = ds$ in the integrand expression.

5. Cauchy's theorem. We shall now put the fundamental question, viz. under what conditions does the contour integral (16) become independent of the path taken. It is obviously necessary and sufficient that both line integrals on the right-hand side, which give the real and imaginary parts of the contour integral, should also be independent of the path. Applying the criterion for independence of the path of a line integral, as given in [II, 71], we arrive at the equations

$$\frac{\partial u(x, y)}{\partial y} = -\frac{\partial v(x, y)}{\partial x}; \quad \frac{\partial v(x, y)}{\partial y} = \frac{\partial u(x, y)}{\partial x},$$

and these are precisely the Cauchy–Riemann conditions. Hence, the *conditions for the independence of a contour integral* (16) *of its path are the same as the condition for the regularity of the function* $f(z)$. This circumstance is of fundamental importance in the integral calculus of functions of a complex variable.

Note that in deducing the conditions for the independence of a line integral of its path we used the formula [II, 69]:

$$\int_l [P(x, y)\,dx + Q(x, y)\,dy] = \iint_B \left(\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y}\right) dx\,dy.$$

In deducing this formula we assumed the continuity not only of the functions $P(x, y)$ and $Q(x, y)$ themselves, but also of their partial derivatives, since they too are under the sign of the double integral. In the case under consideration these latter are in fact continuous, since, for the regular function $f(z)$, both functions $u(x, y)$ and $v(x, y)$ have continuous partial derivatives of the first order. In future we shall integrate round the contour of the domain B itself. This will be quite in order if we assume that $f(z)$ is regular in the domain B up to the contour i.e. it is regular in the closed domain B. By this we mean that $f(z)$ is regular in a somewhat larger domain, which contains the domain B together with its contour; hence $f(z)$ *is regular in the closed domain* B if it is regular in a domain containing B and its contour.

For a more detailed study of this problem it is necessary to take account of the kind of domain in which $f(z)$ is regular, i.e. in this case, as with real contour integrals [II, 72], it is most important whether the domain is simply or multiply connected. We shall recall the appropriate basic definitions and formulate the results which are analogous with those obtained for real contour integrals.

If a bounded domain in the z-plane has a closed curve as its contour (in other words, has no holes) then the domain is simply connected. If at the same time $f(z)$ is a regular function in this domain and z_0 is a point in this domain, then the integral

$$F(z) = \int_{z_0}^{z} f(z')\,dz' \tag{20}$$

(where z' denotes the variable of integration), taken over an arbitrary curve in the domain, does not depend on the path and gives a single valued function of its upper limit of z. At the same time the value of the integral round an arbitrary closed contour in the domain will obviously be equal to zero. If our function $f(z)$ is regular in a closed domain then we can integrate round the contour of the domain B itself, and we obtain zero as a result of the integration.

Let us now assume that our domain B is multiply connected and bounded by one outer closed contour and by several closed interior

contours. Assume, for the sake of simplicity, that there is only one interior contour i.e. the domain is doubly connected (Fig. 4). We make a cut λ in our connecting the outer and inner contours. The cut domain B' will now be simply connected and the expression (20) will give a single-valued function of z in B'. If we assume that $f(z)$ is regular in the closed domain, then we can integrate round the contour of the domain. We can then assert that the integral round the entire contour of the simply-connected domain B' must be zero. As indicated in the figure, we must here integrate in the counter-clockwise direction round the outside contour, in the clockwise direction round the inside contour and twice in opposite directions along the cut λ. The integrals along this cut will cancel each other and we consequently have:

$$\int_{\mathcal{C}l_1} f(z)\,dz + \int_{\mathcal{C}l_2} f(z)\,dz = 0\,, \qquad (21)$$

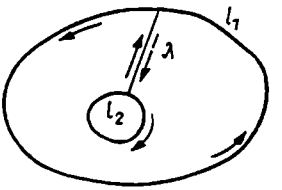

FIG. 4

where l_1 is the outside contour, l_2 is the inside contour, and the arrows indicate the direction of integration. It follows from the diagram that the direction of integration for both contours can be determined from one and the same condition, viz. when describing a circuit round the contour the domain must remain on the left-hand side. This direction will be termed *positive with respect to the domain*. Using the equation (21) we can say, that even in the case of a multiply-connected domain, the integral round the contour will be equal to zero, provided we integrate everywhere in the positive direction with respect to the domain.

If the direction of integration round the inner contour is reversed then instead of the formula (21) we can write:

$$\int_{\mathcal{C}l_1} f(z)\,dz = \int_{\mathcal{C}l_2} f(z)\,dz\,, \qquad (22)$$

i.e. the integral round the outer contour is equal to the sum of the integrals round the inner contours (in this case only one), provided we integrate round all contours in the counter-clockwise direction.

The results obtained give the fundamental theorem in the study of the theory of functions which is usually known as Cauchy's theorem. We can formulate it in several different ways.

CAUCHY'S THEOREM I. *If a function is regular in a closed simply-connected domain then its integral round the contour of this domain is equal to zero.*

CAUCHY'S THEOREM II. *If a function is regular in a closed multiply-connected domain then its integral round the entire contour of this domain in the positive direction is equal to zero.*

CAUCHY'S THEOREM III. *If a function is regular in a closed multiply-connected domain, then its integral round the outer contour is equal to the sum of the integrals round all the inner contours, provided we integrate round all contours in the counter-clockwise direction.*

Let us now explain a lemma of Cauchy's theorem which is of practical importance. Assume that different contours l' and l'' have the same beginnings and ends A and B. Assume that l' can be transformed into l'' by means of a continuous deformation without leaving the domain in which $f(z)$ is regular and without changing the beginning A and end B in any way. It follows from Cauchy's theorem that the value of the integral of $f(z)$ will not be affected i.e. *if a contour with fixed ends is continuously deformed without leaving the domain in which $f(z)$ is regular, the value of the integral of the function $f(z)$ round the contour will remain unchanged as a result of this deformation.* The same thing holds *in the case of deformation of a closed contour*, as long as it remains closed at all times.

To conclude this we shall make a statement of fundamental importance. When deducing Cauchy's theorem we assumed the continuity as well as existence of the derivative $f'(z)$. This continuity of $f'(z)$ enters into the definition of a regular function. By employing a different method of proof it is possible to prove Cauchy's theorem by using the existence alone of $f'(z)$ without assuming its continuity. However, we shall see later that it follows from Cauchy's theorem that $f(z)$ has derivatives of all orders, which means that $f'(z)$ must be continuous.

Thus the second method of proof of Cauchy's theorem, which we do not give here, is theoretically important in that it does not use the continuity of $f'(z)$; on the other hand, one of the consequences of this proof is that the existence of the derivative $f'(z)$, implies that the derivative must be continuous. In future, unless stated otherwise, we shall always integrate round a closed contour in the counter-clockwise direction.

6. The fundamental formula of the integral calculus. Assume that $f(z)$ is regular in a domain, and consider the function given by the formula (20). If our domain is multiply connected, we can still assume that $F(z)$ is a single-valued function, by making appropriate cuts. In the same way as in the integral calculus for functions of real variables [I, 96] it can be shown that $F(z)$ is a primitive for $f(z)$, i.e. $F'(z) = f(z)$.

To do so we note, first of all, that it follows from the definition of an integral as the limit of a sum that:

$$\int_l dz = \beta - \alpha,$$

where β and α are respectively the complex coordinates of the end and the beginning of the contour l. We evidently have:

$$F(z + \Delta z) - F(z) = \int_z^{z+\Delta z} f(z')\, dz',$$

where we can integrate, for example, along the straight line connecting the points z and $z + \Delta z$. We can write:

$$F(z + \Delta z) - F(z) = \int_z^{z+\Delta z} [f(z') - f(z) + f(z)]\, dz' =$$

$$= f(z) \int_z^{z+\Delta z} dz' + \int_z^{z+\Delta z} [f(z') - f(z)]\, dz',$$

where $f(z)$ is taken outside the integral, since it does not contain the variable of integration z'. The latter formula can be rewritten as follows:

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z} \int_z^{z+\Delta z} [f(z') - f(z)]\, dz'. \qquad (23)$$

It remains to be shown that the last term on the right-hand side tends to zero when $\Delta z \to 0$. Using the inequality for an integral given in [4] and taking into consideration that, in this case, the length of the path of integration is equal to $|\Delta z|$, we can write:

$$\left| \frac{1}{\Delta z} \int_z^{z+\Delta z} [f(z') - f(z)]\, dz' \right| \leqslant \frac{1}{|\Delta z|} \cdot \max |f(z') - f(z)| \cdot |\Delta z| =$$

$$= \max |f(z') - f(z)|.$$

We must take the maximum of the modulus of the difference $|f(z') - f(z)|$ when z' varies along the straight line connecting z and $(z + \Delta z)$. The continuous positive function $|f(z') - f(z)|$ of z' attains its maximum value on the given line at a point $z' = z_0'$; i.e. $\max|f(z') - f(z)| = |f(z_0') - f(z)|$. But when $\Delta z \to 0$, the point z_0', which lies on the given line, tends to z and, by virtue of the continuity of $f(z)$, the difference $f(z_0') - f(z) \to 0$; it follows that the last term on the right-hand side of the expression (23) tends to zero i.e. $F'(z) = f(z)$.

Let us now show that when the function $f(z)$ has two primitives $F_1(z)$ and $F_2(z)$ they will differ by a constant term. We have by hypothesis:

$$F_1'(z) = f(z) \text{ and } F_2'(z) = f(z),$$

i.e.

$$[F_1(z) - F_2(z)]' = 0.$$

It thus remains to be shown that *if the derivative of a function is identically zero in the domain B then this function is constant in B.* Hence let $f_1(z) = u_1(x, y) + iv_1(x, y)$ and

$$f_1'(z) \equiv 0.$$

We can write the following two expressions for the derivative:

$$f_1'(z) = \frac{\partial u_1}{\partial x} + i\frac{\partial v_1}{\partial x} = \frac{\partial v_1}{\partial y} - i\frac{\partial u_1}{\partial y} \equiv 0$$

and we have consequently:

$$\frac{\partial u_1}{\partial x} \equiv 0; \quad \frac{\partial u_1}{\partial y} \equiv 0; \quad \frac{\partial v_1}{\partial x} \equiv 0; \quad \frac{\partial v_1}{\partial y} \equiv 0,$$

from which it follows that u_1 and v_1 are independent of x and y, i.e. they are constants; hence the function $f_1(z)$ will be constant.

Assume that we have a primitive $F_1(z)$ for the function $f(z)$. It will differ from the primitive (20) only by a constant term, i.e.

$$\int_{z_0}^{z} f(z')\,dz' = F_1(z) + C.$$

To determine this constant term we assume that the end z coincides with the beginning of the path z_0, which gives

$$0 = F_1(z_0) + C \quad \text{or} \quad C = -F_1(z_0),$$

and the previous formula can be rewritten in the form:

$$\int_{z_0}^{z} f(z')\,dz' = F_1(z) - F_1(z_0), \tag{24}$$

i.e. *the contour integral is equal to the increment of the primitive over the path of integration.* We are assuming, of course, that the primitive $F(z)$ is a single-valued function and that it is regular in the domain which contains the path of integration.

Example: Consider the integral

$$\int_l (z-a)^n \, dz, \tag{25}$$

where n is an integer and l is a closed contour. If n differs from (-1) the primitive will be:

$$\frac{1}{n+1}(z-a)^{n+1}. \tag{26}$$

This will be a regular single-valued function everywhere if $n \geqslant 0$, and everywhere except at $z=a$ when $n<-1$. We assume that the contour l does not pass through $z=a$. On describing a circuit round a closed contour the single-valued function (26) will obviously receive a zero increment and, consequently, the integral (25) round an arbitrary closed contour will be equal to zero when $n \neq -1$. When $n \geqslant 0$ this follows directly from Cauchy's theorem. When $n<-1$ the result also follows from Cauchy's theorem as long as the point $z=a$ does not lie within the contour l. But the above arguments show that if n is negative and not equal to (-1) the integral (25) will be equal to zero, even if a lies within the contour l. In this case the integrand is no longer regular at the point $z=a$ since at this point it becomes infinite.

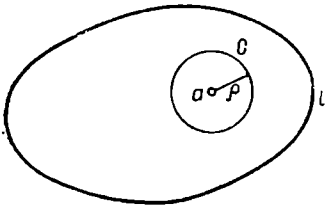

FIG. 5

Let us now consider the case when $n=-1$, i.e. consider an integral of the type

$$\int \frac{dz}{z-a}. \tag{27}$$

If a lies outside the closed contour l, then, in accordance with Cauchy's theorem, the integral will be equal to zero. Assume that the point a lies inside the contour l (Fig. 5). Let us draw a circle C, centre at a and small radius ϱ. The integrand $(z-a)^{-1}$ will be regular in the annulus between the contour l and the circle C and, consequently, in accordance with Cauchy's theorem, we can integrate round the circle C when evaluating the integral (27). On this circle

$$z-a=\varrho e^{i\varphi},$$

where ψ varies in the interval $(0,2\pi)$. Hence

$$dz = i\varrho e^{i\varphi} \, d\varphi.$$

Substituting in the integral (27) we obtain;

$$\int \frac{dz}{z-a} = \int_0^{2\pi} \frac{i\varrho e^{i\varphi}\, d\varphi}{\varrho e^{i\varphi}} = 2\pi i,$$

i.e. finally

$$\int_l \frac{dz}{z-a} = 2\pi i. \tag{28}$$

7. Cauchy's formula. Let $f(z)$ be a regular function in a closed domain B, which for the moment, for the sake of simplicity, we consider to be simply-connected. Let l be the contour and a an interior point of this domain.

Let us construct the new function:

$$\frac{f(z)}{z-a}. \tag{29}$$

This new function is also regular everywhere in B, except, perhaps, at the point $z = a$, since at this point the denominator of the fraction (29) vanishes. Isolate this point by a circle, centre at a and small radius ε, and let C_ε be its circumference. In the annulus between the contours l and C_ε our function (29) will be regular without exception and, consequently, in accordance with Cauchy's theorem, we can write:

$$\int_l \frac{f(z)}{z-a}\, dz = \int_{C_\varepsilon} \frac{f(z)}{z-a}\, dz.$$

In the integral on the right-hand side we put $f(z) = f(a) + f(z) - f(a)$, so that

$$\int_l \frac{f(z)}{z-a}\, dz = f(a) \int_{C_\varepsilon} \frac{dz}{z-a} + \int_{C_\varepsilon} \frac{f(z)-f(a)}{z-a}\, dz,$$

or, by (28):

$$\int_l \frac{f(z)}{z-a}\, dz = f(a)\, 2\pi i + \int_{C_\varepsilon} \frac{f(z)-f(a)}{z-a}\, dz. \tag{30}$$

Let us now pay attention to the following circumstance: the integral on the left-hand side of formula (30) and the first term on the right-hand side are independent of the choice of the radius ε; we can therefore assert that the second term on the right-hand side is also independent of ε. We shall now prove, however, that it tends to zero when $\varepsilon \to 0$. It then follows that it is simply equal to zero.

Applying the inequality from [4] and keeping in mind the fact that as z moves round the circumference C_ε, centre at a, $|z-a| = \varepsilon$, we get:

$$\left|\int_{C_\varepsilon} \frac{f(z)-f(a)}{z-a}\,dz\right| \leqslant \frac{\max\limits_{\text{on } C_\varepsilon} |f(z)-f(a)|}{\varepsilon} 2\pi\varepsilon = \max_{\text{on } C_\varepsilon} |f(z)-f(a)| \cdot 2\pi.$$

When ε becomes infinitesimally small, the point z on the circumference tends to a, and the maximum of the modulus of the difference $|f(z)-f(a)|$ tends to zero, i.e. the second term on the right-hand side of the formula (30) does in fact tend to zero together with ε and, in accordance with the above argument, it will be simply equal to zero. Thus formula (30) can be rewritten in the form:

$$f(a) = \frac{1}{2\pi i}\int_l \frac{f(z)}{z-a}\,dz.$$

Let us now somewhat modify our notation, viz. we shall now denote the variable of integration by z' and an arbitrary point in our domain by z. In this case the above formula takes the form:

$$f(z) = \frac{1}{2\pi i}\int \frac{f(z')}{z'-z}\,dz'. \tag{31}$$

This is Cauchy's formula and *it expresses the value of a regular function at any interior point z of the domain in terms of its value on the contour of the domain.* The integral which forms part of Cauchy's formula contains z as a parameter in the integrand in an exceptionally simple form.

The point z lies inside the domain and the variable point z' on the contour of the domain. Thus $z'-z \neq 0$ and the integral in Cauchy's formula is an integral of a continuous function; hence it can be differentiated with respect to z under the sign of the integral as many times as we please. Differentiating successively we obtain:

$$f'(z) = \frac{1}{2\pi i}\int_l \frac{f(z')}{(z'-z)^2}\,dz'; \quad f''(z) = \frac{2!}{2\pi i}\int_l \frac{f(z')}{(z'-z)^3}\,dz',$$

and, in general, when n is an arbitrary positive integer:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_l \frac{f(z')}{(z'-z)^{n+1}}\,dz'. \tag{32}$$

We can thus see that a regular function has derivatives of all orders and these derivatives are expressed by the contour values of the function in accordance with the formula (32).

Let us prove formally the possibility of differentiating under the sign of the integral in order to determine $f'(z)$. We have:

$$f(z+\Delta z)-f(z)=\frac{1}{2\pi i}\int_l \frac{f(z')}{z'-z-\Delta z}\,dz'-\frac{1}{2\pi i}\int_l \frac{f(z')}{z'-z}\,dz'=$$

$$=\frac{\Delta z}{2\pi i}\int_l \frac{f(z')}{(z'-z)(z'-z-\Delta z)}\,dz'$$

or

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{1}{2\pi i}\int_l \frac{f(z')}{(z'-z)(z'-z-\Delta z)}\,dz'.$$

If we take the limit as $\Delta z \to 0$ under the integral sign on the right-hand side we obtain the expression:

$$f'(z)=\frac{1}{2\pi i}\int_l \frac{f(z')}{(z'-z)^2}\,dz'. \qquad (32_1)$$

It remains to prove the possibility of passing to the limit under the integral sign, i.e. it must be shown that the difference

$$\delta=\frac{1}{2\pi i}\int_l \frac{f(z')}{(z'-z)^2}\,dz'-\frac{1}{2\pi i}\int_l \frac{f(z')}{(z'-z)(z'-z-\Delta z)}\,dz'$$

tends to zero when $\Delta z \to 0$.

After elementary rearrangements we obtain:

$$\delta=\frac{-\Delta z}{2\pi i}\int_l \frac{f(z')}{(z'-z)^2[z'-(z+\Delta z)]}\,dz'.$$

The function $f(z')$ is always continuous on l, and its modulus is bounded, i.e. $|f(z')|<M$. We denote by $2d$ a positive number equal to the shortest distance between the point z and the contour l, i.e. $|z'-z|\geqslant 2d$. The point $(z+\Delta z)$, when Δz is close to zero, is close to z and we have $|z'-(z+\Delta z)|>d$. Applying the usual inequality to the integral, we have:

$$|\delta|<\frac{|\Delta z|}{2\pi}\cdot\frac{M\cdot s}{4d^3},$$

where s is the length of the contour; it follows that $\delta \to 0$ when $\Delta z \to 0$. It can be shown similarly by starting with the formula (32_1), that $f'(z)$ also has a derivative

$$f''(z)=\frac{2!}{2\pi i}\int_l \frac{f(z')}{(z'-z)^3}\,dz',$$

which it was required to prove.

The formulae (31) and (32), like Cauchy's theorem, are also directly applicable to a multiply-connected domain; all we have to do is integrate round the entire contour of the domain in the positive direction, i.e. so that the domain remains on the left.

Let us now extend Cauchy's theorem to the case of an *infinite domain*. Let $f(z)$ be regular in the domain B formed by the part of the plane outside the closed contour l, and let it be subjected to an additional condition, viz. *when the point z moves to infinity the function $f(z)$ tends to zero:*

$$f(z) \to 0 \text{ when } z \to \infty. \tag{33}$$

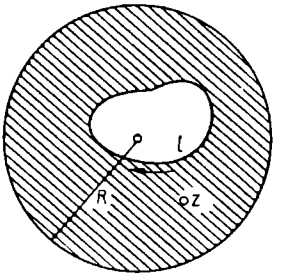

FIG. 6

We shall show that Cauchy's formula still holds:

$$f(z) = \frac{1}{2\pi i} \int_{(l)} \frac{f(z')}{z' - z} \, dz', \tag{34}$$

where we integrate so that the domain B (in this case the part of the plane outside l) remains on the left. To prove this we draw a circle, centre the origin and large radius R. Our function $f(z)$ is regular in the annulus between the contour l and the circumference C_R (Fig. 6 of the circle, and for an arbitrary point z in this annulus we have

$$f(z) = \frac{1}{2\pi i} \int_{(l)} \frac{f(z')}{z' - z} \, dz' + \frac{1}{2\pi i} \int_{(C_R)} \frac{f(z')}{z' - z} \, dz'. \tag{35}$$

As in the proof of Cauchy's formula we shall see that the second term on the right-hand side is essentially independent of R and, consequently, if we can prove that it tends to zero when R increases indefinitely, it follows that it must be identically zero and formula (35) becomes formula (34). Let us find an upper bound for the second term on the right-hand side of the formula (35). To do so we replace the modulus $|z' - z|$ in the denominator by a smaller quantity, viz. by a difference of moduli $|z'| - |z| = R - |z|$. We then obtain the upper bound in the form:

$$\left| \int_{C_R} \frac{f(z')}{z' - z} \, dz' \right| \leqslant \max_{\text{on } C_R} |f(z')| \frac{2\pi R}{R - |z|}$$

or

$$\left| \int_{C_R} \frac{f(z')}{z' - z} \, dz' \right| \leqslant \max_{\text{on } C_R} |f(z')| \frac{2\pi}{1 - \frac{|z|}{R}}.$$

If R increases indefinitely the above fraction tends to 2π and the first cfator $\max_{\text{on } C_R} |f(z')|$ tends to zero in accordance with the condition (33). We have thus proved Cauchy's formula for an infinite domain. Note that it follows from the proof that the condition (33) must be satisfied uniformly with respect to z. In other words, this condition can be fully formulated as follows: for any given ε an R_ε exists such that $|f(z)| < \varepsilon$ when $|z| > R_\varepsilon$.

Sometimes we have to deal with functions which are regular inside a domain and have definite limits on the contour of the domain so that they are continuous functions throughout the closed domain, although one cannot say that they are regular in the closed domain, i.e. that they remain regular when the domain is enlarged. Note that *Cauchy's theorem and Cauchy's formula apply to such functions i.e. those which are regular in a domain and continuous in a closed domain.* In fact, if we compress the contour slightly, the function now remains regular on the contour and therefore Cauchy's theorem evidently applies, i.e. the integral round the contour is equal to zero. If the contour is now expanded continuously so that it eventually coincides with its initial position, in the limit the value of the integral round the initial contour of the domain will also be zero. Here, of course, we can pass to the limit, because the function is uniformly continuous in the closed domain.

It can be said that almost all the further results of this chapter are a direct result of Cauchy's formula. We shall return to it on many occasions. Below we give two examples of applications of this formula.

Let us prove Cauchy's theorem in greater detail, when $f(z)$ is regular inside the circle $|z| < R$, centre the origin and radius R, and continuous in the closed circle $|z| \leqslant R$. The function $f(z)$ is regular in the closed circle $|z| \leqslant R_1$, where R_1 is any positive number less than R. Cauchy's formula is applicable, and we have:

$$\int_{|z|=R_1} f(z)\,dz = 0.$$

On the circumference of this circle, $z = R_1 e^{i\varphi}$ and $dz = R_1 i e^{i\varphi}\,d\varphi$, so that

$$iR_1 \int_0^{2\pi} f(R_1 e^{i\varphi})\, e^{i\varphi}\,d\varphi = 0.$$

Since $f(z)$ is uniformly continuous in the closed circle [1] we can prove the possibility of passing to the limit under the integral sign as $R_1 \to R$ [II, 84]; we then obtain in the limit:

$$iR \int_0^{2\pi} f(Re^{i\varphi})\, e^{i\varphi}\,d\varphi = 0,$$

or returning to the variable z we can write:

$$\int_{|z|=R} f(z)\,dz = 0,$$

which is what we wished to prove. When the contours are of a more complicated type the proof becomes more involved. Cauchy's formula for functions which are regular inside a region and continuous in a closed region follows from Cauchy's theorem, as above.

Example 1. Take the exponential function $f(z) = e^z$. This function is regular in the whole plane and we can apply formula (32); we take as l and arbitrary closed contour containing z as an interior point:

$$e^z = \frac{n!}{2\pi i}\int_l \frac{e^{z'}}{(z'-z)^{n+1}}\,dz'.$$

Let us take as l a circle, centre at z and with a fixed radius ϱ_1. We now have:

$$z'-z = \varrho e^{i\varphi};\ e^{z'} = e^z\, e^{\varrho\cos\varphi + i\varrho\sin\varphi};\ dz' = i\varrho e^{i\varphi}\,d\varphi$$

and substituting in the above formula we obtain:

$$1 = \frac{n!}{2\pi\varrho^n}\int_0^{2\pi} e^{\varrho\cos\varphi + i\varrho\sin\varphi - in\varphi}\,d\varphi,$$

whence

$$2\pi\frac{\varrho^n}{n!} = \int_0^{2\pi} e^{\varrho\cos\varphi + i(\varrho\sin\varphi - n\varphi)}\,d\varphi.$$

On separating the real part, we obtain a definite integral of a fairly complicated type:

$$\int_0^{2\pi} e^{\varrho\cos\varphi}\cos(\varrho\sin\varphi - n\varphi)\,d\varphi = 2\pi\frac{\varrho^n}{n!}. \tag{36}$$

2. Consider the rational fraction

$$\frac{\varphi(z)}{\psi(z)} = f(z), \tag{37}$$

where the degree of the polynomial $\psi(z)$ in the denominator is higher than the degree of the polynomial $\varphi(z)$. This function evidently satisfies the condition (33). Assume also that l is a closed contour containing as interior points all the zeros of the polynomial $\psi(z)$. We can then say that the function (37) is regular in the part of the plane outside l and that Cauchy's formula for an infinite region holds for it. The integration over l in this formula must be carried out in such a way that the domain outside l remains on the left, i.e. in the clockwise direction. If we integrate in the counter-clockwise direction the result will have the opposite sign and we thus obtain:

$$-\frac{\varphi(z)}{\psi(z)} = \frac{1}{2\pi i}\int_l \frac{\varphi(z')}{\psi(z')(z'-z)}\,dz'. \tag{38}$$

Consider the integrand in this latter formula. Regarded as a function of z', it ceases to be regular or, as is usually said, it has singularities inside l at points where $\psi(z')$ vanishes. The point z is not a singularity since it lies outside the contour l (in the infinite region B). The existence of these singularities, which are zeros of the polynomial $\psi(z')$, implies that the value of the integral (38) round the closed contour l is not zero.

8. Integrals of Cauchy's type. In Cauchy's formula (31) the numerator of the integrand repesented the value of the regular function $f(z')$ on the contour l. At the same time, according to Cauchy's formula, the value of the integral reproduced precisely the function $f(z)$ at a point in the domain. We shall regard the integral in Cauchy's formula as a computational device, and consider what it will yield if we substitute in the numerator of its integrand a function which is continuous and specified in a purely arbitrary way on the contour, about which nothing is known other than that it is specified and continuous on the contour. Denote this function by $\omega(z')$. Our integral will evidently be a function of z:

$$F(z) = \frac{1}{2\pi i}\int_l \frac{\omega(z')}{z'-z}\,dz'. \tag{39}$$

Bearing in mind the general assumptions made with regard to the function $\omega(z')$, the integral on the right-hand side is known as an integral of Cauchy's type. As in the previous section, we can differentiate with respect to z under the integral sign as many times as we please, and obtain formulae analogous to (32):

$$F^{(n)}(z) = \frac{n!}{2\pi i}\int_l \frac{\omega(z')}{(z'-z)^{n+1}}\,dz', \tag{40}$$

i.e. $F(z)$ is always a regular function inside the domain B, bounded by the closed contour l. We could, of course, have assumed that z lies outside the contour l. In this case we should again have obtained the formula (40) together with the formula (39), i.e. formula (39) also defines a regular function for points z which lie outside the contour l. If we assume that z lies on the contour l, the integral (39) becomes meaningless, since the integrand becomes infinite on the contour of integration. This brings us to the following result: *the integral of Cauchy's type* (39) *determines two regular functions: one inside the contour l and the other outside the contour.*

Notice that these two regular functions will generally be different. To explain this circumstance consider the simplest case, viz. when the

"density" $\omega(z')$ in the integral of Cauchy's type is the same as the value on the contour of a function $f(z)$, which is regular in the closed domain bounded by the contour l. Thus let $\omega(z') = f(z')$ be a regular function in the closed domain bounded by the contour l. If z lies inside l then Cauchy's formula (31) applies and the integral of Cauchy's type

$$\frac{1}{2\pi i}\int \frac{f(z')}{z'-z}\,dz' \tag{41}$$

gives the function $f(z)$ inside the contour. Let us now assume that z lies outside the contour l and examine the integrand in the integral (41) as a function of z'. Its numerator $f(z')$ is regular in and on l and its denominator $z - z'$ does not vanish in or on l since we have assumed that z lies outside l. We can therefore apply Cauchy's theorem and assert that the integral (41) is equal to zero provided z lies outside l, i.e. in this case the integral of Cauchy's type (41) gives $f(z)$ inside l and zero when z lies outside l.

Let us now return to Cauchy's formula (31). In this formula the "density" $f(z')$ in the integral of Cauchy's type was the same as the values of the function $f(z)$ itself on the contour l. In the general case of Cauchy's integral (39) when $\omega(z')$ is assumed to be an arbitrary continuous function on the contour l, this situation obviously no longer holds. In the case of formula (39) we have to distinguish two functions: the function $f_1(z)$ defined by the formula (39) inside l and the function $f_2(z)$ defined outside l. If z tends to the point z' on the contour l from inside, the question arises as to whether $f_1(z)$ will tend to a limit at all and if it does tend to a limit then what will be the connection between it and the values of $\omega(z')$. The same question can be asked with regard to the function $f_2(z)$ when z tends to z' from outside the contour. In this chapter we shall not concern ourselves with this problem. By making certain additional assumptions with regard to the functions $f_1(z')$ and $f_2(z')$ we find that they must have limit values, though the connection of these with $\omega(z')$ is relatively complicated. The difference between these limit values, or to be more precise, the difference between the limit values of $f_1(z)$ and $f_2(z)$ when z tends to z' along the normal to the curve l, will be exactly equal to $\omega(z')$. This is confirmed by the example of an integral of Cauchy's type given by (41). Here the interior limit is $f(z')$ and the exterior limit zero.

Integrals of Cauchy's type are frequently used in the analytical representation of functions. Note that this representation is many-valued, i.e. to be more precise, one function can be represented by

different integrals of Cauchy's type. Let us show this by the following example. Let l be a closed contour which encircles the origin $z = 0$ and let us define a regular function in l which is identically zero. This function can obviously be represented by an integral of Cauchy's type (39) with a "densisy" $\omega(z') \equiv 0$. We shall show that this function, i.e. zero, can be represented by an integral of Cauchy's type with a "density" $\omega(z') = 1/z'$. In fact consider the integral

$$F(z) = \frac{1}{2\pi i} \int_l \frac{1}{z'(z'-z)}\, dz' \tag{42}$$

and let us show that it is equal to zero whatever the position of z inside l (we recall that the origin also lies in l). On decomposing the rational fraction into partial fractions we can write:

$$\frac{1}{z'(z'-z)} = -\frac{1}{zz'} + \frac{1}{z(z'-z)}$$

and, consequently:

$$F(z) = -\frac{1}{2\pi i z} \int_l \frac{dz'}{z'} + \frac{1}{2\pi i z} \int_l \frac{dz'}{z'-z}.$$

From the example in [6] we obtain:

$$F(z) = -\frac{1}{z} + \frac{1}{z} \equiv 0.$$

Hence the integral of Cauchy's type (42) also gives zero in l. On adding this integral to another integral of Cauchy's type (39) which yields a regular function $F(z)$, we obtain another integral of Cauchy's type, yielding the same function $F(z)$. Thus we cannot conclude from the equality of two integrals of Cauchy's type:

$$\frac{1}{2\pi i} \int_l \frac{\omega_1(z')}{z'-z}\, dz' = \frac{1}{2\pi i} \int_l \frac{\omega_2(z')}{z'-z}\, dz', \qquad (z \text{ inside } l) \tag{43}$$

for any z inside l, are, that the "densities" of these integrals are the same. This will only be the case if we impose certain additional conditions on the densities. Thus, for example, the following theorem by Harnak applies: if $\omega_1(z')$ and $\omega_2(z')$ are continuous real functions and l is the circumference of a circle, the equation (43) is equivalent to the identity $\omega_1(z') \equiv \omega_2(z')$.

At the end of this chapter we shall consider the problem of the limit ing values of integrals of Cauchy's type when the contour of the domain is approached.

9. Corollaries of Cauchy's formula. Let $f(z)$ be a continuous function which is regular in the closed domain B with the contour l, or let it be regular inside the domain and continuous in the closed domain. Consider the regular function $[f(z)]^n$, where n is a positive integer, and apply Cauchy's formula to this function:

$$[f(z)]^n = \frac{1}{2\pi i} \int_l \frac{[f(z')]^n}{z' - z} \, dz' .$$

Let M be the maximum of the modulus $|f(z')|$ on the contour l and denote the minimum of the modulus $|z' - z|$ by δ, i.e. the shortest distance from the point z to the contour l.

Applying the usual inequality, we have:

$$|f(z)|^n \leqslant \frac{M^n S}{2\pi\delta} ,$$

where S is the length of the contour l. The above inequality can be rewritten as follows:

$$|f(z)| \leqslant M \left(\frac{S}{2\pi\delta}\right)^{\frac{1}{n}} .$$

When the positive integer n tends to infinity we obtain the following inequality at the limit

$$|f(z)| \leqslant M , \tag{44}$$

i.e. *if $f(z)$ is a function which is regular in a domain and continuous in a closed domain, its maximum modulus is attained on the contour*, i.e. the modulus at any interior point of the domain is not greater than its maximum modulus on the contour.

It can be shown that the sign of equality in the formula (44) can only be obtained when $f(z)$ is constant. The above property is usually known as *the principle of the maximum* [or maximum modulus theorem].

Let us now consider a second corollary of Cauchy's formula. The function e^z and a polynomial in z are examples of functions which are regular in the whole plane. We shall show that the moduli of these functions cannot be bounded except in the trivial case of $f(z)$ being constant. In other words the following theorem, generally known as Liouville's theorem, applies: *if $f(z)$ is regular in the whole plane and is bounded, i.e. a positive number N exists so that for every z*

$$|f(z)| \leqslant N , \tag{45}$$

then $f(z)$ is a constant.

We apply Cauchy's formula to $f'(z)$:

$$f'(z) = \frac{1}{2\pi i} \int_l \frac{f(z')}{(z'-z)^2} \, dz' .$$

Owing to the fact that $f(z)$ is regular in the whole plane we can take an arbitrary contour for the contour l which encircles the point z. We take a circle, centre at z and radius R for the contour l and enlarge it indefinitely. We then have:

$$|z' - z| = R$$

and therefore:

$$|f'(z)| \leqslant \frac{1}{2\pi} \frac{\max\limits_{\text{on } l} |f(z')|}{R^2} 2\pi R .$$

Taking into account the inequality (45), we obtain the following result:

$$|f'(z)| \leqslant \frac{N}{R} .$$

The left-hand side of this inequality is independent of R and the right-hand side tends to zero when R increases indefinitely. It therefore follows that $f'(z) \equiv 0$ and that $f(z)$ is a constant [6].

Take, for example, the function $\cos z$. It follows from formula (1) that its modulus increases indefinitely when z tends to infinity along the imaginary axis. For, putting $z = iy$, we have

$$\cos iy = \frac{e^{-y} + e^{y}}{2} .$$

10. Isolated singularities. Let us finally turn to a third corollary of Cauchy's formula, viz. to a study of the singularities of a regular function. Assume that $f(z)$ is a single-valued function which is regular in the neighbourhood of the point $z = a$ but not at the point $z = a$ itself. This singularity of the function is generally known as an isolated singularity. For example, for the function

$$f(z) = \frac{1}{z} ,$$

the point $z = 0$ will be an isolated singularity. We shall now consider the possible types of isolated singularities.

There can be three cases: (1) the modulus of the function $f(z)$ remains bounded when the values of z are close to a; (2) the modulus of the function $f(z)$ tends to infinity when z tends to a; (3) the modulus

$|f(z)|$ does not remain bounded when z approaches a; the function, however, does not tend to infinity but oscillates.

Let us consider the first case and show that in this event the point $z = a$ is not a singularity of the function $f(z)$. In other words *if $f(z)$ is single-valued and regular in the neighbourhood of $z = a$ and if its modulus is bounded in this neighbourhood then it is also regular at the point $z = a$.* In fact, encircle the point $z = a$ by two circles with radii R and ϱ respectively, centre at $z = a$, where $\varrho < R$. If z lies in the annulus between these circles then according to Cauchy's formula we have:

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z')}{z' - z}\, dz' + \frac{1}{2\pi i} \int_{C_\varrho} \frac{f(z')}{z' - z}\, dz'.$$

We shall show that the second term on the right-hand side tends to zero when ϱ tends to zero. As in the proof of Cauchy's formula it will follow from this that the second term is simply equal to zero. The condition the modulus of the function $f(z)$ is bounded in the neighbourhood of $z = a$ gives $|f(z)| \leqslant N$, where N is a positive number.

We have: $(z' - z) = (z' - a) - (z - a)$; we replace the modulus of this difference by the smaller quantity:

$$|(z' - a) - (z - a)| \geqslant |z - a| - |z' - a| = |z - a| - \varrho,$$

where $|z' - a| = \varrho$ on C_ϱ. We thus get the following inequality for the given term:

$$\left| \frac{1}{2\pi i} \int_{C_\varrho} \frac{f(z')}{z' - z}\, dz' \right| \leqslant \frac{2}{2\pi} \cdot \frac{N}{|z - a| - \varrho} \cdot 2\pi\varrho = \frac{N\varrho}{|z - a| - \varrho};$$

which shows that this term tends to zero when $\varrho \to 0$. Hence the above formula gives:

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z')}{z' - z}\, dz',$$

i.e. for all values of z close to a the function $f(z)$ is expressed by an integral of Cauchy's type and therefore $f(z)$ represents a function which is regular everywhere, including the point $z = a$. Strictly speaking, if $f(z)$ is single-valued and regular near $z = a$ and if also its modulus is bounded then $f(z)$ tends to a definite finite limit, when $z \to a$; if we assume that $f(a)$ is this limit then $f(z)$ will be regular everywhere, including the point $z = a$.

Let us now consider the second and third cases. The function $1(z - a)$ is an example of the second case and singularities of this type

are known as *poles*, i.e. *if $f(z)$ is single-valued and regular near the point $z = a$ and if it tends to infinity when z tends to a then the point a is a pole of the function $f(z)$.*

We shall now give an example of a singularity of the third kind. We shall show, in particular, that the point $z = 0$ will be a singularity for the function

$$f(z) = e^{\frac{1}{z}}. \tag{46}$$

In fact, if z approaches zero from the positive direction then the function (46) tends to $+\infty$, and when z approaches zero from the negative direction it tends to zero. Singularities of this type are known as *essential singularities, i.e. the point $z = a$ is an essential singularity of the function $f(z)$ if the function is single-valued and regular in the neighbourhood of the point $z = a$, yet is not bounded in this neighbourhood and does not tend to infinity when $z \to a$.*

We shall now prove a theorem on the values of a function in the neighbourhood of an essential singularity. This theorem was first proved by Sokhotskii.

THEOREM. *If $z = a$ is an essential singularity of $f(z)$ then when z varies in an arbitrarily small circle, centre $z = a$, values of $f(z)$ are obtained which can be as close as we please to any previously assigned complex number.*

The assertion of this theorem amounts to the following. Let γ be an arbitrarily chosen complex number and let ε be an arbitrarily chosen positive number. In this case there will be points z in an arbitrarily small circle, centre $z = a$, where $|f(z) - \gamma| < \varepsilon$. Let us use *reductio ad absurdum*. Assume that there is a positive number β such that at all points of a circle C, centre at a, the following inequality holds: $|f(z) - \beta| \geqslant m$, where m is a positive number. Let us construct the new function:

$$\varphi(z) = \frac{1}{f(z) - \beta}.$$

This function is regular in the circle C and its modulus is bounded:

$$|\varphi(z)| = \frac{1}{|f(z) - \beta|} \leqslant \frac{1}{m}.$$

It therefore follows from the above proof that it is regular at the point $z = a$; moreover, when $z \to a$ the function $\varphi(z)$ tends to a finite limit. Thus

$$f(z) = \beta + \frac{1}{\varphi(z)}$$

must also tend to a finite limit when $z \to a$ provided the limit of $\varphi(z)$ is not zero; or it will tend to infinity when the limit of $\varphi(z)$ vanishes; both these possibilities, however, contradict the definition of an essential singularity.

We can prove a more accurate theorem, viz:

PICARD'S THEOREM: *If $z = a$ is an essential singularity of $f(z)$ then in any small circle, centre at a, $f(z)$ assumes an infinite number of times every complex value with the possible exception of one.*

The proof of this theorem is vastly more complicated than the proof of the previous theorem and we shall not attempt to give it here. We shall only test this theorem for the function (46) which has an essential singularity at the point $z = 0$.

Take any complex number a, other than zero, and write the equation

$$e^{\frac{1}{z}} = a. \tag{46$_1$}$$

Remembering the rules for taking the logarithms of complex numbers we obtain the roots of the equation (46$_1$):

$$z = \frac{1}{\log|a| + i(\varphi + 2k\pi)},$$

where φ is the amplitude of a within the interval $(0, 2\pi)$ and k is an arbitrary integer. By taking its absolute value as large as we please, we obtain zeros of the equation (46$_1$) as close to zero as we please. Thus the function (46) assumes an infinite number of times any previously assigned number, except zero, in any circle, which can be as small as we please, centre the origin. It can be shown that the function $\sin 1/z$ assumes an infinite number of times every value without exception in a circle with the centre at the origin.

Poles and essential singularities are *isolated singularities* i.e. the function is regular in the neighbourhood of these points. In future when examining many-valued functions we shall deal with yet another type of isolated singularitiy, viz. with *branch points*.

11. Infinite series with complex terms. Having explained some fundamental points connected with the concept of the integral we shall now examine infinite series with complex terms. Consider the infinite series with complex terms:

$$(a_1 + ib_1) + (a_2 + ib_2) + \ldots + (a_n + ib_n) + \ldots \tag{47}$$

This series is said to be convergent when the sum of its first n terms

$$S_n = (a_1 + a_2 + \ldots + a_n) + i\,(b_1 + b_2 + \ldots + b_n) \tag{48}$$

tends to a finite limit as n increases indefinitely. It follows from this definition that the series (47) will only converge when the series of real terms converges

$$a_1 + a_2 + \ldots \text{ and } b_1 + b_2 + \ldots . \tag{49}$$

These series consist of the real and imaginary parts of the terms of the series (47). If we denote by A and B the sums of the series (49) then the finite sum of (48) will evidently tend to the limit $A + iB$ which represents the sum of the series (47).

Let us now explain the concept of absolute convergence of the series (47). On replacing every term in the series (47) by its modulus we obtain a series with positive terms:

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \ldots \tag{50}$$

We can show that if this series converges then the initial series (47) will also converge. In fact, from the evident inequality:

$$\sqrt{a_n^2 + b_n^2} \geqslant |a_n| \text{ and } |b_n|$$

we can see [I, 120 and 124] that the convergence of the series (50) implies the convergence (even absolute) of the series (49), and hence the convergence of the series (47).

If the series (50) *converges then the convergent series* (47) *is said to be absolutely convergent.* Such absolutely convergent series have properties analogous to those of absolutely convergent series with real terms.

If the series (47) is absolutely convergent then, as we have just seen, the series (49) will also be absolutely convergent and their sums are independent of the order of terms [I, 137]. Therefore we can say the same about the sum of the series (47).

Similarly, using arguments analogous to those in [I, 138], we can prove a theorem on the multiplication of absolutely convergent series. In fact, if we have two absolutely convergent series of complex terms

$$S = \alpha_1 + \alpha_2 + \ldots \text{ and } T = \beta_1 + \beta_2 + \ldots ,$$

then the series

$$\alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1) + (\alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1) + \ldots$$

will also be absolutely convergent and its sum will be equal to ST.

We shall not prove this theorem in detail. Cauchy's test for the existence of a limit applies for a complex variable and, as in the case of a real variable, gives the necessary and sufficient condition: for the convergence of a series of complex terms: *the necessary and sufficient condition for the series* (47) *to be convergent is that for any given positive* ε *a positive* N *exists such that*

$$\left| \sum_{k=n+1}^{n+p} (a_k + ib_k) \right| < \varepsilon ,$$

provided $n > N$, *where* p *is an arbitrary positive integer.*

Let us now examine a series with variable terms, i.e. a series the terms of which contain the variable z:

$$u_1(z) + u_2(z) + \dots \tag{51}$$

If this series converges for all values of z in a domain B (on a curve l) then it is said that the series (51) converges in the domain B (on the curve l).

Let us now introduce the concept of uniform convergence as we did for the real variable [I, 143]. *The series* (51) *is said to be uniformly convergent in the domain* B *(on the curve* l*) if for any assigned positive* ε *a positive* N *exists, which is one and the same for all values of* z *in the domain* B *(on* l*), such that*

$$\left| \sum_{k=n+1}^{n+p} u_k(z) \right| < \varepsilon , \tag{52}$$

provided $n > N$ *and* p *is an arbitrary positive integer.* Uniformly convergent series of a complex variable have the same properties as uniformly convergent series of a real variable [I, 146]. We shall give two fundamental properties which can be proved in exactly the same way as in the case of a real variable.

If the terms of the series (51) are continuous functions of z in the domain B (on the curve l) and the series is uniformly convergent in this domain (on the curve l) the sum of the series will also be a continuous function.

If the series (51), which consists of continuous functions, converges uniformly on a curve l, it can be integrated term by term along this curve.

Let us finally indicate a sufficient condition for the absolute and uniform convergence of the series (51), which is exactly analogous to that for the real variable [I, 147]. If for all values of z in the

domain B (on the curve l) the terms of the series (51) are bounded:

$$|u_k(z)| \leqslant m_k \qquad (k = 1, 2, \ldots),$$

where m_k are positive numbers which form a convergent series, then the series (51) is absolutely and uniformly convergent in the domain B (on the curve l).

Let us draw attention to one further circumstance which follows directly from the above viz. if the series (51) is uniformly convergent on the curve l and we multiply all its terms by a certain function $v(z)$, the modulus of which is bounded on the curve, e.g. is continuous, the new series will also be uniformly convergent. In fact, as a result of this multiplication we obtain instead of the series (51) the following series:

$$u_1(z)\,v(z) + u_2(z)\,v(z) + \ldots,$$

where $|v(z)| < N$. It follows from the inequality (52) that we have for the new series:

$$\left|\sum_{k=n+1}^{n+p} u_k(z)\,v(z)\right| = |v(z)|\left|\sum_{k=n+1}^{n+p} u_k(z)\right| < N\varepsilon,$$

from which follows the uniform convergence of this series, since N is a definite positive number and ε is as small as we please when n is large.

Having explained the elementary concepts concerning series with complex terms we shall now prove a fundamental theorem for series, the terms of which are regular functions of z.

12. The Weierstrass theorem. *If the terms of the series* (51) *are regular functions in a closed domain B with the contour l and if this series converges uniformly on the contour l then it converges uniformly in the whole closed domain B, its sum is a regular function in the domain B and each term of this series can be differentiated as many times as we please.*

Denote by z' a variable point on the curve l. It is given that the series

$$u_1(z') + u_2(z') + \ldots \tag{53}$$

is uniformly convergent and we therefore have the following inequality:

$$\left|\sum_{k=n}^{n+p} u_k(z')\right| < \varepsilon \qquad (\text{for } n > N \text{ and any } p > 0).$$

The above finite sum of regular functions is also a regular function in the closed domain B and therefore, in accordance with the principle of the maximum, the above inequality implies the same inequality for the whole domain [9]:

$$\left|\sum_{k=n}^{n+p} u_k(z)\right| < \varepsilon \quad (\text{for } n > N \text{ and any } p > 0),$$

from which it follows that the series (51) is uniformly convergent in the whole closed domain.

Denoting the sum of the series (53) by $\varphi(z')$ (a continuous function on l), we multiply all the terms of the series by

$$\frac{1}{2\pi i}\,\frac{1}{(z'-z)},$$

where z is an interior point of the domain B:

$$\frac{1}{2\pi i}\,\frac{\varphi(z')}{(z'-z)} = \frac{1}{2\pi i}\,\frac{u_1(z')}{(z'-z)} + \frac{1}{2\pi i}\,\frac{u_2(z')}{(z'-z)} + \dots$$

This series will also converge uniformly on the contour l and by integrating it term by term round this contour we obtain:

$$\frac{1}{2\pi i}\int_l \frac{\varphi(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\int_l \frac{u_1(z')}{z'-z}\,dz' + \frac{1}{2\pi i}\int_l \frac{u_2(z')}{z'-z}\,dz' + \dots$$

But we have Cauchy's formula for regular functions of the type $u_k(z)$ and therefore the latter formula can be rewritten in the form:

$$\frac{1}{2\pi i}\int_l \frac{\varphi(z')}{z'-z}\,dz' = u_1(z) + u_2(z) + \dots$$

This shows that the sum of the series (51) can be represented by an integral of Cauchy's type in the domain B and that it is a regular function. Denote this sum by $\varphi(z)$:

$$\sum_{k=1}^{\infty} u_k(z) = \varphi(z) = \frac{1}{2\pi i}\int_l \frac{\varphi(z')}{z'-z}\,dz'. \tag{54}$$

Note that owing to the uniform convergence of the series (51) throughout the closed domain B, which we proved above, $\varphi(z)$ is continuous in the closed domain B and formula (54) is simply Cauchy's formula for the function $\varphi(z)$.

It only remains to be shown that the series (51) can be differentiated term by term as many times as we please. To do this we multiply (51) by

the factor

$$\frac{m!}{2\pi i}\,\frac{1}{(z'-z)^{m+1}},$$

where m is a positive integer, and integrate round l:

$$\frac{m!}{2\pi i}\int_l \frac{\varphi(z')}{(z'-z)^{m+1}}\,dz' = \frac{m!}{2\pi i}\int_l \frac{u_1(z')}{(z'-z)^{m+1}}\,dz' + \frac{m!}{2\pi i}\int_l \frac{u_2(z')}{(z'-z)^{m+1}}\,dz' + \dots$$

It follows from Cauchy's formula and from formula (54) that the latter expression can be rewritten in the form:

$$\varphi^{(m)}(z) = u_1^{(m)}(z) + u_2^{(m)}(z) + \dots, \tag{55}$$

which shows that the series can be differentiated m times, term by term, inside the domain. In the next section we shall apply this theorem to a particular type of series, with which we shall deal almost exclusively in future viz. to power series.

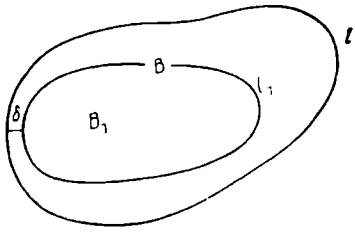

FIG. 7

Note 1. By using the usual inequality for integrals it is easy to show that *the series* (55) *which is composed from derivatives, is uniformly convergent in any domain* B_1 *which, together with its contour, lies in* B. We can construct the usual expression for the series (55)

$$\sum_{k=n}^{n+p} u_k^{(m)}(z).$$

By using the Cauchy form of the derivative we obtain

$$\sum_{k=n}^{n+p} u_k^{(m)}(z) = \frac{m!}{2\pi i}\int_l \frac{1}{(z'-z)^{m+1}} \sum_{k=n}^{n+p} u_k(z')\,dz'.$$

Let δ be the shortest distance from the contour l_1 of the domain B_1 to the contour l (Fig. 7). Applying the usual inequality to the above integral we obtain:

$$\left|\sum_{k=n}^{n+p} u_k^{(m)}(z)\right| \leqslant \frac{m!S}{2\pi\delta^{m+1}} \cdot \max_{\text{on } l}\left|\sum_{k=n}^{n+p} u_k(z')\right|,$$

where S is the length of the contour l. Owing to the uniform convergence of the series (53), the last factor on the right-hand side will be as small as we please when n is large and this gives the condition for the uniform convergence of the series (55). It is also easy to show that when B is a simply-connected domain, the series obtained by term-

by-term integration:

$$\int_a^z u_1(z')\,dz' + \int_a^z u_2(z')\,dz' + \dots,$$

where a is a point in B, will converge uniformly in B [c.f. I, 146]. The terms of this series are regular single-valued functions in B [6].

Note 2. The Weierstrass theorem could have been formulated by using sequences of functions instead of series [I, 144]: if we have a sequence of functions $s_k(z)$ $(k = 1, 2, \dots)$, which are regular in the closed domain B with the contour l, and if the sequence tends uniformly to a limit on the contour l, it will tend uniformly to a limit in the whole of the closed domain B; the limiting function $s(z)$ will be regular in B and for every positive integer m we have in B:

$$\lim_{k\to\infty} s_k^{(m)}(z) = s^{(m)}(z).$$

13. Power series. A *power series* is a series of the type

$$a_0 + a_1(z-b) + a_2(z-b)^2 + \dots, \tag{56}$$

where a_k and b are given numbers. Let us consider, first of all, the region of convergence of the series (56). We shall prove the following theorem:

ABEL'S THEOREM. *If the series* (56) *converges at a point* $z = z_0$, *then it will converge absolutely at every point* z *which lies nearer to* b *than* z_0, *i.e.*

$$|z-b| < |z_0-b|,$$

and it will converge uniformly in any circle, centre at b *and radius* ϱ *smaller than* $|z_0-b|$, *i.e. smaller than the distance from* z_0 *to* b (Fig. 8).

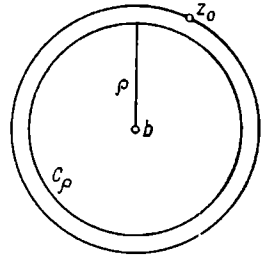

FIG. 8

It follows from the condition of this theorem that the series

$$a_0 + a_1(z_0-b) + a_2(z_0-b)^2 + \dots$$

converges and, consequently its general term tends to zero when the number of the term increases indefinitely. We can therefore assert that a positive number N exists such that for every k:

$$|a_k(z_0-b)^k| < N. \tag{57}$$

Consider now a circle C_ϱ, centre at b and radius ϱ smaller than $|z_0 - b|$, so that this radius can be denoted by $\varrho \leqslant \theta |z_0 - b|$, where $0 < \theta < 1$. We have for every z in this circle C_ϱ:

$$|z - b| \leqslant \theta |z_0 - b|. \tag{58}$$

We consider the upper bounds of the terms of the series (56) in the circle C_ϱ. From (57) and (58) we can write:

$$|a_k (z - b)^k| = |a_k (z_0 - b)^k| \left|\frac{z - b}{z_0 - b}\right|^k \leqslant N\theta^k,$$

from which it follows that in the circle C_ϱ, the terms of the series (56) have moduli less than a decreasing geometric progression composed of positive numbers, i.e. the series (56) converges absolutely and uniformly in the circle C_ϱ. It is obvious that every point z which lies nearer to b than z_0 can be considered as belonging to a circle C_ϱ and, consequently, it follows from the above, that at every such point the series (56) converges absolutely. Abel's theorem is thus fully proved. We will now discuss some corollaries of this theorem.

COROLLARY 1. If the series (56) diverges at a point $z = z_1$ then it will obviously also diverge at every other point which is further removed from b than z_1. For, if the series converges at this latter point then, as a result of Abel's theorem, it must also converge at the point z_1. We can therefore say that the following applies to the series (56): *the convergence of the series at a certain point implies its absolute convergence in the circle which passes through this point and has its centre at b; and the divergence of the series at a certain point implies its divergence outside the circle which passes through this point and has its centre at b.* It follows that for every series of the type (56) a positive number R exists such that the series (56) converges absolutely when $|z - b| < R$ and diverges when $|z - b| > R$, while in any circle with a radius smaller than R, i.e. when $|z - b| \leqslant \theta R$ $(0 < \theta < 1)$, the series (56) is convergent uniformly. This number R is known as *the radius of convergence of the series* (56) and the circle $|z - b| < R$ as *the circle of convergence of the series* (compare with analogous results obtained for a real variable) [I, 148].

Note that the above arguments do not provide conditions for the uniform convergence of the series (56) in the whole circle of convergence but only in a concentric circle drawn with a smaller radius. We express this fact by simply saying that the series (56) converges

uniformly inside its circle of convergence. In general we shall say that *a series is uniformly convergent in a domain if it converges uniformly in any closed domain which, together with its contour, lies inside the given domain.*

Note another important point in connection with the above results. The radius of convergence R can, in certain cases, be infinite. In that case the series (56) will converge absolutely at every point in the plane, and it will converge uniformly in a circle drawn with any finite radius. Note also the second extreme case, viz. when $R = 0$. In this case the series (56) diverges at every point except at the point $z = b$. At this point the series will reduce to its first term. In future we shall not deal with power series for which $R = 0$.

COROLLARY 2. The series (56) converges uniformly in its circle of convergence and, consequently, the Weierstrass theorem applies to it, i.e. the sum of the series (56) in the circle of convergence is a regular function of z and of the series can be differentiated term by term as many times as we please. By virtue of being a uniformly convergent series it can also be integrated term by term. Furthermore, as a result of their absolute convergence, power series can be multiplied term by term like polynomials.

It follows from what has been said above that term-by-term differentiation and integration of the series (56) will not affect the convergence in the circle of convergence, i.e. the series:

$$a_1 + 2a_2(z - b) + 3a_3(z - b)^2 + \ldots; \tag{59}$$

$$a_0(z - b) + \frac{a_1}{2}(z - b)^2 + \ldots \tag{59$_1$}$$

have radii of convergence which are not less than that of the series (56). It is easy to see that the radius of convergence of series (59) cannot be greater than the radius of convergence of the series (56).

For, suppose that the radius of convergence ϱ of the series (59$_1$) is greater than R, i.e. $\varrho > R$. In view of what has just been said, we do not decrease the radius of convergence on differentiating this series and we return to the series (56); hence $\varrho \leqslant R$, which contradicts $\varrho > R$. We can thus maintain that *term-by-term differentiation and integration of the series* (56) *do not alter its radius of convergence.*

Note in conclusion that nothing has been said above about the convergence of the series (56) on the circumference $|z - b| = R$ of its circle of convergence. We shall consider this problem later.

14. Taylor's series. We saw above that the sum of the series (56) is a regular function in the circle of convergence of this series. We shall now prove the converse proposition: *any function $f(z)$ which is regular in a circle $|z-b|<R$, centre at b, can be represented in this circle by a power series of the form* (56) *and this representation is unique.*

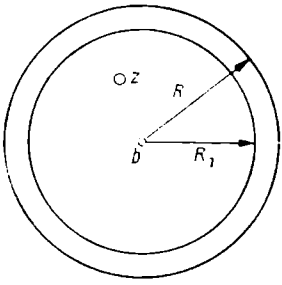

FIG. 9

Take a fixed point z in the circle $|z-b|<R$. Draw a circle C_{R_1}, centre at b and radius R_1 which is smaller than R, that z lies in C_{R_1} (Fig. 9). We can express $f(z)$ by Cauchy's formula, by integrating round C_{R_1}:

$$f(z)=\frac{1}{2\pi i}\int_{C_{R_1}}\frac{f(z')}{z'-z}\,dz'. \tag{60}$$

In C_{R_1} we have $|z'-b|=R_1$ and, on the other hand, $|z-b|<R_1$, since z lies in C_{R_1}. Using the formula which gives the sum of the terms of an infinitely decreasing geometric progression, we can write:

$$\frac{1}{z'-z}=\frac{1}{z'-b}\cdot\frac{1}{1-\dfrac{z-b}{z'-b}}=\sum_{k=0}^{\infty}\frac{(z-b)^k}{(z'-b)^{k+1}}, \tag{61}$$

where we have the following expression for the moduli of the terms of this series:

$$\left|\frac{(z-b)^k}{(z'-b)^{k+1}}\right|=\frac{1}{R_1}q^k \quad \left(q=\left|\frac{z-b}{z'-b}\right|\right),$$

and it follows from the above that $0\leqslant q<1$. Hence the infinite series (61) converges uniformly with respect to z' in C_{R_1}. Multiplying both sides by

$$\frac{1}{2\pi i}f(z')$$

and integrating term by term round to C_{R_1} we obtain from (60):

$$f(z)=\sum_{k=0}^{\infty}(z-b)^k\cdot\frac{1}{2\pi i}\int_{C_{R_1}}\frac{f(z')}{(z'-b)^{k+1}}\,dz'$$

or

$$f(z)=\sum_{k=0}^{\infty}a_k(z-b)^k, \tag{62}$$

where, by Cauchy's formula [7]:

$$a_k = \frac{1}{2\pi i} \int_{C_{R_1}} \frac{f(z')}{(z'-b)^{k+1}}\, dz' = \frac{f^{(k)}(b)}{k!}, \qquad (62_1)$$

i.e. *the value of* $f(z)$ *at an arbitrary point in the circle* $|z-b| < R$, *where* $f(z)$ *is regular, is represented by the Taylor series:*

$$f(z) = f(b) + \frac{f'(b)}{1!}(z-b) + \frac{f''(b)}{2!}(z-b)^2 + \ldots \qquad (63)$$

We can show that the representation of $f(z)$ by a power series is unique. Assume that $f(z)$ can be represented in a circle, centre at b, by a series of the type (62). We will show that the coefficients a_k are uniquely determined viz. they must be Taylor's coefficients. In fact, assuming in (62) that $z = b$, we obtain $a_0 = f(b)$. Differentiating the power series (62):

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z-b)^{k-1}.$$

On again putting $z = b$ we obtain $a_1 = f'(b)$. Proceeding in this way, we obtain in general:

$$a_k = \frac{f^{(k)}(b)}{k!},$$

and formula (62) must be the same as Taylor's series (63). Hence if we could obtain in two different ways expansions of one and the same function into power series in positive integral powers of $(z-b)$ the coefficients of the same powers of $(z-b)$ must be the same in both expansions.

The above arguments show that *Taylor's series* (63) *for the function* $f(z)$ *converges in a circle, centre at* b, *in which* $f(z)$ *is regular; in this circle the sum of the Taylor's series is equal to* $f(z)$.

Upper bounds for the coefficients in Taylor's series follow directly from the above expression. Let R be the radius of convergence of the series (62). In formula (62_1) we take a circle, centre at b and radius $(R-\varepsilon)$, for C_{R_1}, where ε is a small fixed positive number. On this circle our function $f(z)$ is regular, its modulus does not exceed a positive number M and, evidently, $|z'-b| = (R-\varepsilon)$. The usual upper bound for the integral gives:

$$|a_k| \leqslant \frac{M}{(R-\varepsilon)^k}. \qquad (64)$$

ε can be taken as close to zero as we please though the value of M evidently depends on the choice of ε.

Let us apply the Weierstrass theorem which we proved in [12] to the case of a power series. Assume that we are given regular functions in a circle C_R, centre at b:

$$u_k(z) = a_0^{(k)} + a_1^{(k)}(z-b) + a_2^{(k)}(z-b)^2 + \dots.$$

and assume that the series

$$\sum_{k=1}^{\infty} u_k(z)$$

converges uniformly in this circle. In this case, in accordance with the Weierstrass theorem, its sum is also a regular function in this circle and it can therefore be represented by a power series:

$$\sum_{k=1}^{\infty} [a_0^{(k)} + a_1^{(k)}(z-b) + a_2^{(k)}(z-b)^2 + \dots] = \\ = a_0 + a_1(z-b) + a_2(z-b)^2 + \dots$$

In accordance with the Weierstrass theorem we can differentiate this series term by term as many times as we please. Differentiating and putting $z = b$, we obtain the following expression for the sum of the coefficients of this series:

$$a_0 = \sum_{k=1}^{\infty} a_0^{(k)}; \quad a_1 = \sum_{k=1}^{\infty} a_1^{(k)}; \quad a_2 = \sum_{k=1}^{\infty} a_2^{(k)}, \dots,$$

i.e. *given the above assumptions, we can add these infinite series like ordinary polynomials.*

15. Laurent's series. It is not difficult to obtain results similar to the above for power series of a more general type:

$$\dots + a_{-2}(z-b)^{-2} + a_{-1}(z-b)^{-1} + a_0 + \\ + a_1(z-b) + a_2(z-b)^2 + \dots, \tag{65}$$

which contain negative as well as positive integral powers of $(z-b)$. The series (65) is usually known as *Laurent's series*. We shall first of all determine its domain of convergence. The series (65) consists of two series:

$$a_0 + a_1(z-b) + a_2(z-b)^2 + \dots \tag{66_1}$$

and

$$\frac{a_{-1}}{z-b} + \frac{a_{-2}}{(z-b)^2} + \dots, \tag{66_2}$$

and we have to find the domain in which these two series converge. This will be the domain of convergence of the series (65). The series (66_1) is the usual power series of the type we considered above and its domain of convergence is a circle, centre at b. Let this circle be $|z-b|<R_1$. To analyse the series (66_2) we replace z by a new variable z' in accordance with the formula $z'=(z-b)^{-1}$. This transforms the series (66_2) into the usual power series of the type:

$$a_{-1}z'+a_{-2}z'^2+\dots$$

Its domain of convergence in the z'-plane is a circle, centre the origin (zero takes the place of b). Denote the radius of this circle by $1/R_2$, so that the domain of convergence of this latter series will be $|z'|<1/R_2$ or $1/|z'|>R_2$. Returning to the former variable z, we obtain the domain of convergence in the form $|z-b|>R_2$. Thus the domain of convergence of the complete series (65) is given by two inequalities:

$$|z-b|<R_1;\quad |z-b|>R_2. \tag{67}$$

The first inequality defines the interior of the circle, centre at b and radius R_1, and this is the domain of convergence of the series (66_1). The second inequality (67) defines the part of the plane outside the circle with centre at b and radius R_2 and this is the domain of convergence of the series (66_2). When $R_1\leqslant R_2$ the inequalities (67) do not define a domain When $R_1>R_2$ the inequalities (67) define a circular annulus

$$R_2<|z-b|<R_1, \tag{68}$$

bounded by two concentric circles, centre at b and radii R_2 and R_1 respectively. *Hence the domain of convergence of a series of the type* (65) *is the circular annulus* (68).

Above we split the series (65) into two power series; it follows from the theory of power series that the series (65) converges absolutely and uniformly in its annulus of convergence, the sum of the series is a regular function and the series can be differentiated term by term. Note that in the inequality (68) which defines the dimensions of the annulus, the inner radius R_2 can vanish and in that event the series (65) will converge for all z sufficiently close to b. Similarly the outside radius R_1 can become infinite, in which case the series (65) will converge for all values of z which satisfy the condition $|z-b|>R_2$. If the annulus is defined by the inequality

$0 < |z - b| < \infty$ then the series (65) will converge in the whole z-plane except at the point $z = b$.

Note also that that part of Laurent's series (65) which contains positive powers of $(z - b)$ converges not only in the annulus (68) but also everywhere in the outer circle, i.e. where $|z - b| < R_1$; the part of the series which contains negative powers of $(z - b)$ converges everywhere outside the inner circle, i.e. where $|z - b| > R_2$. If, for example, the series contains a finite number of terms with negative powers, the condition $R_2 = 0$ applies, and if it contains a finite number of terms with positive powers of $(z - b)$, then the condition $R_1 = \infty$ applies. We must emphasize once again the fact that we are only considering Laurent's series for which $R_2 < R_1$ since otherwise the series has no domain of convergence.

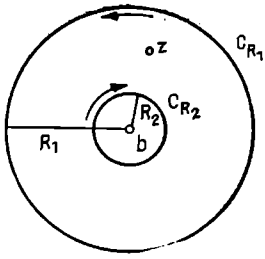

FIG. 10

The converse may be proved in the same way as for power series, viz. *if $f(z)$ is regular in the annulus* (68), *it can be represented by a Laurent's series in this annulus and this representation is unique.*

If we slightly compress the outer circumference of the annulus and slightly enlarge the inner circumference, $f(z)$ will also be regular on both contours of the annulus. Denote these contours by C_{R_2} and C_{R_1}. Applying Cauchy's formula to an arbitrary point z in this annulus (Fig. 10), we obtain:

$$f(z) = \frac{1}{2\pi i} \int_{C_{R_1}} \frac{f(z')}{z' - z}\,dz' + \frac{1}{2\pi i} \int_{C_{R_2}} \frac{f(z')}{z' - z}\,dz' . \tag{69}$$

When integrating round the circle C_{R_1} we have

$$\left| \frac{z - b}{z' - b} \right| < 1$$

and in the same way as in the proof of Taylor's theorem we can represent the fraction in the integrand by a series which converges uniformly on the circle C_{R_1}:

$$\frac{1}{z' - z} = \sum_{k=0}^{\infty} \frac{(z - b)^k}{(z' - b)^{k+1}} .$$

Multiplying by

$$\frac{1}{2\pi i} f(z') \tag{70}$$

and integrating round C_{R_1} we obtain for the first term on the right-hand side of the formula (69) a representation as a power series in positive powers of $(z-b)$:

$$\frac{1}{2\pi i}\int_{C_{R_1}}\frac{f(z')}{z'-z}\,dz' = a_0 + a_1(z-b) + a_2(z-b)^2 + \ldots,$$

where

$$a_k = \frac{1}{2\pi i}\int_{C_{R_1}}\frac{f(z')}{(z'-b)^{k+1}}\,dz'.$$

We have on the contrary, on integrating round C_{R_2}:

$$\left|\frac{z'-b}{z-b}\right| < 1,$$

and instead of the above fraction we must write another expansion, uniformly convergent on the circle C_{R_2}:

$$\frac{1}{z'-z} = -\frac{1}{z-b}\cdot\frac{1}{1-\dfrac{z'-b}{z-b}} = -\sum_{k=0}^{\infty}\frac{(z'-b)^k}{(z-b)^{k+1}},$$

whence again multiplying by the factor (70), we get an expression for the second term on the right-hand side of the formula (69) in the form of a power series in negative integral powers of $(z-b)$:

$$\frac{1}{2\pi i}\int_{C_{R_2}}\frac{f(z')}{z'-z}\,dz' = a_{-1}(z-b)^{-1} + a_{-2}(z-b)^{-2} + \ldots,$$

where

$$a_{-k} = -\frac{1}{2\pi i}\int_{C_{R_2}}(z'-b)^{k-1}f(z')\,dz'.$$

Combining both terms we obtain an expression for the function $f(z)$ in the annulus in the form of Laurent's series:

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k(z-b)^k. \tag{71}$$

It remains to be shown that this representation is unique. For this purpose we shall show, in the same way as for Taylor's series, that the formula (71) gives well-defined expressions for the coefficients of the expansion a_k. Let l be a closed contour in the annulus (68) which encircles b. On this contour the series (71) converges uniformly. We select an integer m, multiply both sides of the

equation (71) by $(z-b)^{-m-1}$ and integrate round l in the counter-clockwise direction:

$$\int_l (z-b)^{-m-1} f(z)\,dz = \sum_{k=-\infty}^{+\infty} a_k \int_l (z-b)^{k-m-1}\,dz.$$

We know from [6] that all the integrals on the right-hand side will be equal to zero except one which contains $(z-b)^{-1}$, the integrand of. This integral will be obtained in the term corresponding to $k=m$, and will be equal, as we know, to $2\pi i$. Hence the above formula gives:

$$\int_l (z-b)^{-m-1} f(z)\,dz = 2\pi i a_m,$$

whence well defined expressions for the coefficients are obtained:

$$a_m = \frac{1}{2\pi i}\int_l (z-b)^{-m-1} f(z)\,dz \qquad (m=0,\ \pm 1,\ \pm 2,\ldots). \quad (72)$$

16. Examples. Applying the expansion into Taylor's series to the elementary transcendental functions, we obtain expansions into power series which are familiar from the differential calculus; these series will also hold when the independent variable assumes complex values.

Example **1.** For the function $f(z)=e^z$ we evidently have $f^{(n)}(z)=e^z$, and consequently, $f^{(n)}(0)=1$. Formula (63), when $b=0$, gives (McLaurin's series):

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots \quad (73)$$

Our function e^z is regular in the whole plane and, consequently, *the expansion* (73) *holds in the whole plane.*

Similarly we can obtain expansions for the trigonometric functions which also hold in the whole plane:

$$\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \quad (74)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots \quad (75)$$

2. The formula for a geometric progression

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

serves as an example of a series with a circle of convergence $|z|<1$.

We replace z by $(-z)$ in this series and integrate between the limits 0 and z:

$$\varphi(z) = \int_0^z \frac{dz}{1+z} = \frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \dots \tag{76}$$

We thus obtain a new power series with the same circle of convergence $|z| < 1$. When z assumes real values its sum, as we know from [I, 132], is equal to $\log(1+z)$. We will show that the same thing applies to all complex values of z in the circle $|z| < 1$, i.e. strictly speaking, we will show that the sum of our series

$$\varphi(z) = \int_0^z \frac{dz}{1+z} \tag{77}$$

satisfies the equation

$$e^{\varphi(z)} = 1 + z. \tag{78}$$

We take the function $e^{\varphi(z)} = f(z)$ which is regular in the circle $|z| < 1$ and write its expansion into a McLaurin's series. To do so we have to find the derivatives of this function. Taking into account that $\varphi'(z) = 1/(1+z)$, we evidently have:

$$f'(z) = e^{\varphi(z)} \cdot \frac{1}{1+z} \tag{79}$$

and furthermore

$$f''(z) = e^{\varphi(z)} \cdot \frac{1}{(1+z)^2} - e^{\varphi(z)} \frac{1}{(1+z)^2} \equiv 0,$$

i.e. $f^{(n)}(z) \equiv 0$ when $n \geqslant 2$. It also follows from the formulae (77) and (79) that $f(0) = e^0 = 1$ and $f'(0) = 1$. Thus the expansion of $f(z)$ into McLaurin's series does, in fact, give us:

$$f(z) = e^{\varphi(z)} = 1 + z.$$

We can thus see that the sum of the series (76) is one of the possible values of $\log(1+z)$. This latter function is many-valued, but the power series (76) singles out a single-valued branch which is regular in the circle $|z| < 1$:

$$\log(1+z) = \frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \dots \tag{80}$$

Values of the logarithm as given by this formula are sometimes known as the principal values of the logarithm. The circumference of our circle of convergence passes through the singular point $z = -1$ which belongs to the function $\log(1+z)$. The character of this singularity will be explained later.

3. Consider the function $(1+z)^m$. When m is a positive integer its expansion in integral positive powers of z is given by the usual Newton's binomial formula. When m is a negative integer the function will have a pole at the point $z=-1$; on evaluating successively its derivatives and constructing McLaurin's series, we obtain an exp. ansion in the circle $|z|<1$ [I, 131]:

$$(1+z)^m = 1 + \frac{m}{1!}z + \frac{m(m-1)}{2!}z^2 + \frac{m(m-1)(m-2)}{3!}z^3 + \dots \tag{81}$$

When m is not an integer, our function will be many-valued. For example, when $m=1/2$ we have $\sqrt{1+z}$. In general, for an arbitrary value of the constant m we can write our function in the form [I, 176]:

$$(1+z)^m = e^{m\log(1+z)}, \tag{82}$$

and this function will be many-valued since the function $\log(1+z)$ is many-valued. Replace $\log(1+z)$ by the value given by the equation (80). In this case the function (82) will be a single-valued function which is regular in the circle $|z|<1$. Evaluating successively the derivatives of the function (82), we have from (77):

$$[(1+z)^m]' = e^{m\log(1+z)}\cdot\frac{m}{1+z} = me^{(m-1)\log(1+z)} = m(1+z)^{m-1},$$

$$[(1+z)^m]'' = m(m-1)\,e^{(m-2)\log(1+z)} = m(m-1)(1+z)^{m-2}$$

and in general:

$$[(1+z)^m]^{(k)} = m(m-1)\dots(m-k+1)\,e^{(m-k)\log(1+z)} =$$
$$= m(m-1)\dots(m-k+1)(1+z)^{m-k},$$

where $\log(1+z)$ is defined by the series (80). Note that the series (80) gives that value of $\log(1+z)$ which vanishes when $z=0$. Thus formula (82) and the subsequent formulae give:

$$(1+z)^m|_{z=0} = 1; \quad [(1+z)^m]'|_{z=0} = m$$

and

$$[(1+z)^m]^{(k)}|_{z=0} = m(m-1)\dots(m-k+1).$$

This shows that for our function (82) McLaurin's series coincides with the series (81), i.e. formula (81) gives a regular single-valued value for the function (82) in the circle $|z|<1$ for an arbitrary index m. In future we shall call formula (81) Newton's binomial formula.

4. Replacing z by $(-z^2)$ in the formula for a progression, we obtain the expansion, valid in the circle $|z| < 1$:

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - \dots$$

On integrating between the limits 0 and z we obtain a new expansion, which also holds in the same circle:

$$\int_0^z \frac{dz}{1+z^2} = \frac{z}{1} - \frac{z^3}{3} + \frac{z^5}{5} - \dots \tag{83}$$

We shall see later that the sum of the above series gives one of the possible values of arc tan z, and formula (83) thus defines one branch of a many-valued function in the circle $|z| < 1$, namely a branch which is regular and single-valued in the given circle.

The expansion for one of the branches of the many-valued function arc sin z in the same circle can be obtained similarly:

$$\int_0^z \frac{dz}{\sqrt{1-z^2}} = \frac{z}{1} + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \dots \tag{84}$$

5. Consider the function

$$f(z) = \frac{1}{z(z-1)(z-2)}.$$

The poles $z = 0$, $z = 1$, $z = 2$, are singularities of this function but otherwise it is single-valued and regular in the whole plane. Consider three circular annuli, centre the origin:

$$(K_1)\quad 0 < |z| < 1; \quad (K_2)\quad 1 < |z| < 2; \quad (K_3)\quad 2 < |z| < +\infty.$$

In each one of these our function can be expanded into a Laurent's series in integral powers of z. For example, on decomposing $f(z)$ into partial fractions in the annulus K_2, we have:

$$f(z) = \frac{1}{2}\frac{1}{z} - \frac{1}{z-1} + \frac{1}{2}\frac{1}{z-2},$$

whence, since $1 < |z| < 2$, we have in the annulus:

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}};$$

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k},$$

and finally, in the annulus K_2 we have:

$$f(z) = -\frac{1}{2}\cdot\frac{1}{z} - \sum_{k=2}^{\infty}\frac{1}{z^k} - \frac{1}{4}\sum_{k=0}^{\infty}\frac{z^k}{2^k}.$$

Similarly, in the annulus K_3, where $|z| > 2$, we have an expansion in negative powers of z only:

$$\frac{1}{z-1} = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}}; \qquad \frac{1}{z-2} = \frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} = \sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}}$$

or

$$f(z) = \sum_{k=2}^{\infty}(2^{k-1}-1)\frac{1}{z^{k+1}}.$$

Our function will also be regular, for example, in an annulus, centre at $z = 1$, inner radius $R_2 = 0$ and outer radius $R_1 = 1$. In this annulus it can easily be represented by a Laurent's series in integral powers of $(z-1)$.

6. Consider the quotient of two power series:

$$\frac{b_0 + b_1 z + b_2 z^2 + \dots}{a_0 + a_1 z + a_2 z^2 + \dots} \tag{85}$$

Let the radii of convergence of both series be not less than a positive number ϱ. Assume also that the constant term a_0 of the series which appears in the denominator does not vanish. In that case the function in the denominator will not vanish at the origin or in some circle with centre at the origin. Assume that it is regular and not equal to zero in the circle $|z| < \varrho_1$. We can assert that the fraction (85) will be regular in a circle, centre the origin, the radius of which is equal to the lesser of the two numbers ϱ and ϱ_1 (or, perhaps, even in a larger circle). In this circle we have an expansion of the function into a power series:

$$\frac{b_0 + b_1 z + b_2 z^2 + \dots}{a_0 + a_1 z + a_2 z^2 + \dots} = c_0 + c_1 z + c_2 z^2 + \dots$$

To evaluate the coefficients c_k we multiply the quotient by the divisor and obtain the product in the form of a power series·

$$a_0 c_0 + (a_1 c_0 + a_0 c_1) z + (a_2 c_0 + a_1 c_1 + a_0 c_2) z^2 + \dots = b_0 + b_1 z + b_2 z^2 + \dots$$

The product obtained must coincide with the dividend and by virtue of the uniqueness of the expansion into a power series, we can simply equate the coefficients of like powers of z. This gives a series of equations for the coefficients c_k of the quotient:

$$\left.\begin{aligned} a_0 c_0 &= b_0 \\ a_1 c_0 + a_0 c_1 &= b_1 \\ a_2 c_0 + a_1 c_1 + a_0 c_2 &= b_2 \\ \cdots\cdots\cdots\cdots & \end{aligned}\right\} \tag{86}$$

The coefficients c_k can be evaluated successively from the above formulae. The first $(n+1)$ equations (86) can be treated as a system of $(n+1)$ equations

in the unknowns $c_0, c_1, \ldots, c_n$. On solving them with the aid of Cramer's formula we can write an expression for the coefficient c_n in the form of a quotient of two determinants:

$$c_n = \frac{\begin{vmatrix} a_0, & 0, & 0, & \ldots, & 0, & b_0 \\ a_1, & a_0, & 0, & \ldots, & 0, & b_1 \\ a_2, & a_1, & a_0, & \ldots, & 0, & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1}, & a_{n-2}, & a_{n-3}, & \ldots, & a_0, & b_{n-1} \\ a_n, & a_{n-1}, & a_{n-2}, & \ldots, & a_1, & b_n \end{vmatrix}}{\begin{vmatrix} a_0, & 0, & 0, & \ldots, & 0, & 0 \\ a_1, & a_0, & 0, & \ldots, & 0, & 0 \\ a_2, & a_1, & a_0, & \ldots, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1}, & a_{n-2}, & a_{n-3}, & \ldots, & a_0, & 0 \\ a_n, & a_{n-1}, & a_{n-2}, & \ldots, & a_1, & a_0 \end{vmatrix}}. \tag{87}$$

On applying these arguments to the expansion

$$\tan z = \frac{\sin z}{\cos z} = \frac{\frac{z}{1!} - \frac{z^3}{3!} + \ldots}{1 - \frac{z^2}{2!} + \ldots},$$

we obtain a power series for $\tan z$ in the circle $|z| < \pi/2$, since, as we shall see later, the function $\cos z$ has only the real roots which are familiar from trigonometry.

17. Isolated singularities. Point at infinity. Assume that the function $f(z)$ is single-valued and regular in the neighbourhood of the point $z = b$, but not at the point $z = b$ itself. It will thus be regular in an annulus, centre at b and inner radius zero, and it can expanded in this annulus, i.e. in the neighbourhood of the point b, into a Laurent's series in integral powers of $(z - b)$. In this case there are three possibilities: (1) the series does not contain terms with negative powers of $(z - b)$, (2) the series contains a finite number of terms with negative powers of $(z - b)$ and (3) the series contains an infinite number of terms with negative powers of $(z - b)$.

In the first case the series which gives the function $f(z)$ and does not contain negative powers of $(z - b)$ is simply Taylor's series and our function will in fact be regular at the point $z = b$. Let us consider the second case, when the series has the form:

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - b)^k, \tag{88}$$

where the coefficient a_{-m} does not vanish. Formula (88) can be rewritten in the form:

$$f(z) = \frac{1}{(z-b)^m}\left[a_{-m} + a_{-m+1}(z-b) + a_{-m+2}(z-b)^2 + \dots\right].$$

When z tends to b the factor outside the square bracket tends to infinity and the square bracket tends to the limit a_{-m}, which is finite and non-zero (the sum of a power series is a continuous function); therefore the product tends to infinity. Thus in the second case, in accordance with our earlier terminology [10], the point b is a pole of the function $f(z)$. Let us introduce a new terminology which is in general use, viz. in the expansion (88) the point b is known as a *pole of order m;* the sum of terms in negative powers

$$\frac{a_{-m}}{(z-b)^m} + \frac{a_{-m+1}}{(z-b)^{m-1}} + \dots + \frac{a_{-1}}{z-b} \qquad (a_{-m} \neq 0)$$

is known as *the infinite part corresponding to this pole.* The coefficient of $(z-b)^{-1}$, viz. a_{-1} has a special name, viz. it is known as *the residue of the function $f(z)$ at the pole b.*

We will now show that an expansion of the form (88) always holds when b is a pole of the function. Thus, let $f(z)$ be single-valued and regular in the neighbourhood of b and let it tend to infinity when $z \to b$. We will show that an expansion of the form (88) holds. Consider the function

$$\varphi(z) = \frac{1}{f(z)}.$$

It is regular in the neighbourhood of the point b and tends to zero when $z \to b$. Consequently $\varphi(z)$ is also regular at the point b [10], where it vanishes. Let us write its expansion into a Taylor's series. In this expansion the constant term will certainly be absent. Assume that the first term of this expansion, which is not zero, contains $(z-b)^m$, i.e.

$$\varphi(z) = b_m(z-b)^m + b_{m+1}(z-b)^{m+1} + \dots \qquad (b_m \neq 0).$$

From above we have the following formula for $f(z)$:

$$f(z) = \frac{1}{\varphi(z)} = \frac{1}{(z-b)^m} \cdot \frac{1}{b_m + b_{m+1}(z-b) + \dots}.$$

The denominator of the second fraction written does not vanish when $z = b$ and therefore this fraction can be expanded into a Taylor's series in positive powers of $(z-b)$. Dividing this Taylor's series by

$(z - b)^m$ we obtain an expansion for $f(z)$ in the form (88). Comparing the last result with the one above, we can assert that the term "pole", introduced by us in [10] is equivalent to the concept of a singularity in the neighbourhood of which the function can be expanded into a Laurent's series with a finite number of terms in negative powers of $(z - b)$. Consequently an essential singularity will be a point in the neighbourhood of which the function $f(z)$ can be expanded into a Laurent's series with an infinite number of terms in negative powers of $(z - b)$. Here, as with a pole, the coefficient of $(z - b)^{-1}$ is known as the residue of $f(z)$ at the essential singularity b.

Note that in the expansion of $\varphi(z)$ there must be a non-vanishing coefficient b_m, since otherwise $\varphi(z)$ would be identically zero in a circle, centre at b, and this contradicts the equation $\varphi(z) = 1/f(z)$, since $f(z)$, according to the given conditions, must be regular in the neighbourhood of $z = b$.

We shall now introduce the concept of the *point at infinity*. We consider that a plane has one point at infinity. The neighbourhood of this point at infinity is defined as the part of the plane outside a circle, centre the origin. This neighbourhood is defined by an inequality of the form $|z| > R$. We could, of course, take the centre of the circle at a point other than the origin, i.e. instead of the above inequality the neighbourhood of the point at infinity could be defined by an inequality of the form $|z - a| > R$, and this would not cause any fundamental changes. However, we shall use the first condition $|z| > R$.

Let $f(z)$ be a single-valued function which is regular in the neighbourhood of the point at infinity. We can regard this neighbourhood as a circular annulus, centre the origin, inner radius R and outer radius infinity. In this annulus it must be possible to expand $f(z)$ into a Laurent's series in integral powers of z and, as before, three different cases arise.

In the first instance we consider the case when the Laurent's series contains no terms in positive powers of z, i.e. when the expansion has the form:

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{89}$$

When z tends to infinity, $f(z)$ tends to a finite limit a_0 and it is said that $f(z)$ *is regular at the point at infinity*, where $f(\infty) = a_0$.

In the second instance we consider the case when the expansion of $f(z)$ into a Laurent's series contains a finite number of terms in

positive powers of z:

$$f(z) = a_{-m}z^m + a_{-m+1}z^{m-1} + \ldots + a_{-1}z + a_0 + \frac{a_1}{z} + \\ + \frac{a_2}{z^2} + \ldots \qquad (a_{-m} \neq 0). \qquad (90)$$

Taking z^m outside the bracket, we can see, as above, that $f(z)$ tends to infinity when $z \to \infty$, while the quotient $f(z)/z^m$ tends to a finite non-zero limit a_{-m}. In this case *the point at infinity is known as a pole of* $f(z)$ *of order* m and the set of points $(a_{-m}z^m + \ldots + a_{-1}z)$ is known as *the infinite part at this pole.*

Finally, if the expansion contains an infinite number of terms in positive powers of z:

$$f(z) = \ldots + a_{-2}z^2 + a_{-1}z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots, \qquad (91)$$

then the point at infinity is known as *an essential singularity of the function* $f(z)$. If we replace z by a new independent variable t according to the formula

$$z = \frac{1}{t}; \qquad t = \frac{1}{z},$$

the neighbourhood of the point at infinity of the z-plane will be transformed into the neighbourhood of the origin in the t-plane and the expansion (91) will contain an infinite number of terms in negative powers of t. It follows that if, $z = \infty$ is an essential singularity of $f(z)$, on variation of z outside any circle, centre the origin, radius as large as we please, a value of $f(z)$ may be obtained as close as we please to any arbitrary, previously assigned complex number, and indeed, $f(z)$ takes any complex value, with the possible exception of one [10] an infinite number of times. In all three cases the coefficient a_1 of z^{-1} with the reverse sign is known as the residue at the point at infinity, i.e. $(-a_1)$. The purpose of this definition of the residue will be explained later.

Note that, when the point $z = a$ is a pole of the function $f(z)$, we write $f(a) = \infty$ and say that $w = f(z)$ transforms $z = a$ into the point at infinity. When $z = \infty$ is a pole of $f(z)$, this is usually written as $f(\infty) = \infty$ and it is said that $w = f(z)$ transforms the point at infinity into itself, i.e. the point remains in its former position.

Returning to [7] we can see that the condition for the applicability of Cauchy's formula to a domain containing the point at infinity can be formulated as follows:

$$f(z) \to 0 \text{ uniformly when } z \to \infty,$$

and means that $f(z)$ is regular at the point at infinity, whilst in the expansion (89) $a_0 = 0$, i.e. $f(\infty) = 0$.

Example **1**. We said earlier with reference to the function e^z that it is regular in the whole plane, but in saying this we excluded the point at infinity. The expansion of the function e^z holds everywhere and, in particular, in the neighbourhood of the point at infinity. It contains an infinite number of terms in positive powers of z and, consequently, the point at infinity is an essential singularity of e^z. The same can be said, for example, about $\sin z$ and $\cos z$.

2. Every polynomial is a regular function in the whole plane and evidently has a pole at infinity, the order of which is equal to the order of the polynomial.

Consider the following rational function which is a quotient of two polynomials:

$$\frac{\varphi(z)}{\psi(z)} = f(z),$$

where the fraction cannot be simplified, i.e. the zeros of the numerator and the denominator are different. Our function will have singularities at a finite distance these being the zeros of the polynomial $\psi(z)$, and these points will be the poles of the function. The behaviour of the function at the point at infinity will depend on the degrees of the polynomials in the numerator and the denominator. If the degree of $\varphi(z)$ is higher than the degree of $\psi(z)$ by m, then $f(z)$ will tend to infinity when $z \to \infty$, but the ratio $f(z)/z^m$ will tend to a finite non-zero limit, i.e. our function will have a pole of order m at infinity. However, if the degree power of $\varphi(z)$ is higher than the degree of $\psi(z)$ the function will be regular at infinity.

18. Analytic continuation. If a function $f(z)$ is regular in a domain B, the question arises as to whether is it permissible to extend the domain in which the function is defined, i.e. is it permissible to create a larger domain C which contains B within itself and to define the regular function $F(z)$ in this larger domain so that it coincides with $f(z)$ in the original domain B. *This extension of the domain in which the regular function is defined*, or, as it is sometimes said, the extrapolation of the regular function, is known as *analytic continuation of the function*. It happens that, if analytic continuation is possible, it will be fully defined and unique. In this respect regular functions of a complex variable differ fundamentally from say continuous functions of a real variable. For suppose we are given a continuous

function $\omega(x)$ of the real variable x in the interval $a \leqslant x \leqslant b$. We can evidently continue the graph of this function outside the given interval in an infinite number of ways without affecting its continuity. However, in the case of a regular function $f(z)$ of a complex variable the values in the original domain B will fully determine the values of the function outside this domain provided such extension of the domain, i.e. analytic continuation, is at all possible. It only needs to be remarked that in the course of analytic continuation one can arrive at many-valued functions. In this section all the circumstances which can arise during analytic continuation are explained and the proof of the uniqueness of this continuation is given.

To begin with let us explain some properties of regular functions.

Let the point $z = b$ be a zero of the regular function $f(z)$. In this case the constant term will be absent in the Taylor's series with the centre at b and, perhaps, some of the succeeding terms will also be missing. Assume that the first non-zero term is of the order $(z-b)^m$, i.e.

$$f(z) = a_m(z-b)^m + a_{m+1}(z-b)^{m+1} + \ldots \qquad (a_m \neq 0) \qquad (92)$$

or

$$f(z) = (z-b)^m [a_m + a_{m+1}(z-b) + \ldots]. \qquad (93)$$

In this case $z = b$ is known as a zero of multiplicity m. Let us turn to the formula (93) and assume that z is equal to a number which is close to, but different from, b. In this case the factor $(z-b)^m$ will be non-zero, and the value of the sum inside the square brackets will be close to a_m, which is also non-zero, i.e. the sum is non-zero. In other words, at all points sufficiently close to the zero of a regular function, the function must be non-zero. Hence *the zeros of a regular function are isolated points*. We naturally assumed in the above arguments that the expansion into Taylor's series (92) contains at least one term which is non: zero. Otherwise, we must evidently regard the function as identically zero, at least in the circle in which Taylor's expansion holds. Keeping this in mind we can now proceed to prove the theorem which is of fundamental importance in the problem of the uniqueness of analytic continuation.

THEOREM. *If $f(z)$ is regular in a domain B and vanishes in a domain β, which is part of B, then $f(z)$ is identically zero in the whole domain B.*

We shall use *reductio ad absurdum*. Assume that $f(z)$ does not vanish at the point c in the domain B. Let us take a point b in β and connect this point with the point c by the curve l, which belongs

to the domain B. On a section of this curve, adjacent to the point b, our function will vanish and on another section, adjacent to the point c, it will be non-zero. Therefore there must be a point d on the curve l, such that our function is zero along the whole section bd; whilst there are points on dc as close as desired to d at which the function is non-zero. A regular function is at the same time continuous and, consequently, there must be a zero at the point d itself. However this zero is not isolated since the whole arc bd of the curve l consists of zeros of our function. It follows from earlier considerations that the expansion of our function into a Taylor's series with centre at d must be identically zero, and consequently, our function must vanish in a circle with centre at d, i.e. it must also vanish on a section of the curve adjacent to the point d which is part of the section dc. This circumstance contradicts the property of the point d, that on the section dc there are points which can be as near as we please to d, where $f(z)$ does not vanish. The theorem is thus proved.

Note. In the proof of the above theorem the conditions could have been restricted to the fact that $f(z)$ vanishes on a curve in B. In this case the function would have vanished in a circle with the centre at one of the points of the curve.

It is even sufficient to assume that the zeros of $f(z)$ have a point of accumulation in B, i.e. there is a point b in B such that in a circle, centre at b and radius, as small as we please, $f(z)$ has an infinite number of zeros. By virtue of our earlier considerations, Taylor's series for $f(z)$ with centre at b would, in this case, be identically zero, i.e. $f(z)$ would vanish in a circle, centre at b, i.e. would vanish everywhere in B.

COROLLARY. Let the two functions $f_1(z)$ and $f_2(z)$ be regular in B and coincide in a part β of this domain or on a curve. Their difference must be equal to zero in β and hence, by the above theorem, it must be equal to zero in the whole domain, i.e. *if two functions, which are regular in a domain, coincide in part of that domain (or on a curve) then they coincide in the whole domain.*

Suppose that all the values of our two functions and the values of all their derivatives coincide at a point b in the domain B. In this case their expansions into Taylor's series with the centre at b will also coincide, i.e. our functions will coincide in a circle, centre at b, and therefore they will coincide in the whole domain; *hence if the values of two functions and of all their derivatives are the same at a given point*

of a domain, the values of the functions will be the same throughout the domain in which the functions are regular.

Let us now turn to the problem of analytic continuation. Let $f_1(z)$ be regular in the domain B_1 and suppose that we have succeeded in constructing a new domain B_2 which has the part $B_{1,2}$ in common with the domain B_1 (Fig. 11), and that we have defined the regular function $f_2(z)$ in the domain B_2 which coincides with $f_1(z)$ in $B_{1,2}$. We may term $f_2(z)$ the direct analytic continuation of $f_1(z)$ from B_1 into B_2 via $B_{1,2}$. The function defined as $f_1(z)$ in B_1, and as $f_2(z)$ in B_2, gives a unique regular function throughout the extended domain. We will show that there cannot be two different analytic continuations. In fact, assume that we have two different analytic continuations of $f_2(z)$ from B_1 into B_2 via $B_{1,2}$. These two functions $f_2^{(1)}(z)$ and $f_2^{(2)}(z)$ which are regular in B_2 must coincide with $f_1(z)$, and consequently, they must coincide with each other in $B_{1,2}$. But now, by what has been proved above, they must coincide in the whole domain B_2, i.e. they give the same analytic continuation.

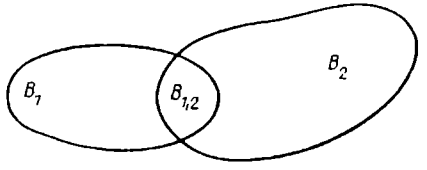

FIG. 11

Suppose now that we have a chain of domains $B_1, B_2, B_3, \ldots$ where B_1 and B_2 have the part $B_{1,2}$, in common, B_2 and B_3 have the part $B_{2,3}$ in common etc. In the domain B_2 we have a regular function $f_2(z)$ which coincides with $f_1(z)$ in $B_{1,2}$. In the domain B_3 we have a regular function $f_3(z)$ which coincides with $f_2(z)$ in $B_{2,3}$, etc. Here we have *the analytic continuation of* $f_1(z)$ *via a chain of domain* and this analytic continuation is unique. Note that, generally speaking, the overlapping of the domains B_s need not be confined to the parts $B_{k,k+1}$ which we have mentioned above. Let us consider, for example, a chain of domains consisting of three domains B_1, B_2 and B_3, and assume that B_3 overlaps with B_1 (Fig. 12). In this common part, which is shaded in the diagram, the values of $f_1(z)$ as defined in B_1 and the values of $f_3(z)$ as defined in B_3, can be different. Here we obtain a many-valued function as a result of the analytic continuation. However we can avoid this geometrically, viz. if the values of $f_1(z)$ and $f_3(z)$ are different in the shaded part then we assume that this shaded part consists of two sheets, as it were, one of which belongs to B_1 and the other to B_3.

The problem of many-valuedness can arise even during the first step of the analytic continuation. Suppose we have the analytic continuation of $f_1(z)$ from B_1 into B_2 via $B_{1,2}$ (Fig. 13), but that B_1 and B_2 have also a part β in common. In β the values of $f_2(z)$ may or may not coincide with the values of $f_1(z)$. The set of all values, obtained as a result of all the possible analytic continuations of

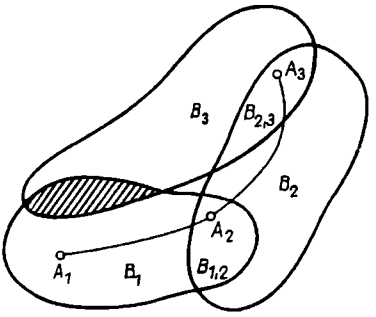

FIG. 12

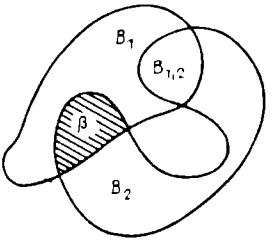

FIG. 13

the initial function $f_1(z)$, yields a unique function which we shall call *an analytic function* and which we denote by $f(z)$. As we have said already, this function $f(z)$ may prove to be many-valued.

Instead of analytic continuation via a chain of domains one frequently speaks about *analytic continuation along a curve.* Consider a line l divided into successive sections: P_1Q_1, P_2Q_2, ..., P_nQ_n, so that the sections P_kQ_k and $P_{k+1}Q_{k+1}$ have the part $P_{k+1}Q_k$ in common (Fig. 14). Assume that this curve l is covered by a chain of domains $B_1, B_2, \ldots, B_k, \ldots$, so that the section P_kQ_k lies in B_k. Denote by $B_{k,k+1}$ the domain in which B_k and B_{k+1} overlap and which contains the section $P_{k+1}Q_k$ of the line l. (There can be several or even an infinite number of domains in which B_k and B_{k+1} overlap, but we only take the one which contains $P_{k+1}Q_k$.)

Suppose there exists a regular function $f_1(z)$ in B_1 and that this function can be continued with the aid of the chain of domains $B_1, B_2, \ldots, B_k, \ldots, B_n$ via $B_{1,2}, B_{2,3}, \ldots, B_{n-1,n}$. Instead of this, we say that $f(z)$ *can be continued along the line* l. The values of the function on the section P_1Q_1 (and in the neighbourhood of this section) are given and by applying the fundamental theorem of this paragraph we can see, as before, that there can be only one analytic continuation

along l. It does not depend on the way in which we divide l into sections and cover it by domains possessing the above properties.

Let us return to the analytic continuation along l via a definite chain of domains B_k. In the neighbourhood of every point on the line l the analytic function $f(z)$ will have definite representation as a Taylor's series. We shall call this series the *function-element at the corresponding point on the line l.* If this line l is slightly deformed while its ends P_1 and

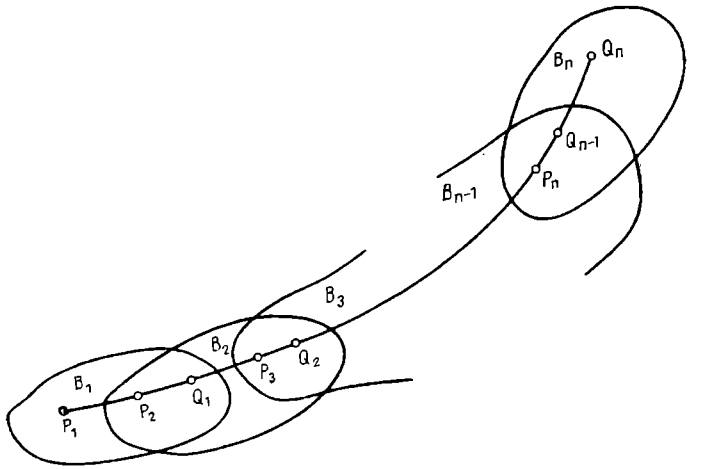

FIG. 14

Q_n remain fixed, it will remain in the domains B_k and the function-element of $f(z)$ at the point Q_n will be the same as before. This shows clearly that, *in general, if we deform this curve continuously whilst keeping its ends P_1 and Q_n fixed, and if the analytic continuation of the initial element at the point P_1 along the line is possible in any of its positions, the element obtained at the point Q_n as a result of the analytic continuation will always be the same.*

Suppose that, on analytic continuation along a curve l from the point P_1, we can only proceed as far as a point C, beyond which analytic continuation along this curve is no longer possible. In this case the point C is known as *a singularity of the function.* But note an important circumstance, viz. had we performed the analytic continuation from the point P_1 to the point C along another curve l_1 instead of the curve l, the point C might not have been a singularity, i.e. generally speaking, *a singularity is defined not only by its position in the plane but also by the path by which it is reached*

in the course of the analytic continuation (c.f. the example in [19]). In future we shall almost always be concerned with the simpler case, when the position of the singularities can be fixed in advance and they do not depend on the path of the analytic continuation.

A theorem of great importance in the theory of analytic continuation follows at once from the above. It is known as the *uniqueness theorem. If the analytic continuation of an initial function-element is possible along any path in a simply connected domain B, these analytic continuations along paths in B yield a unique function in B.*

In fact, let the initial function-element be defined in the neighbourhood of a point P_1 and let us take two different directions l_1 and l_2 for the analytic continuation from P_1 to Q_n. Owing to the fact that the domain is connected we can, by means of a continuous transformation, transform the contour l_1 into the contour l_2 without leaving the domain B and, by hypothesis such analytic continuation along the contour will always be possible. But, as we have said already, the final result of the analytic continuation at the point Q_n will be the same, i.e. in fact the different paths selected for the analytic continuation give the same final result and we obtain a unique function $f(z)$.

In the above arguments we have sometimes restricted ourselves to the simple outlines without going into the details of the proof since this would require much more space. We hope, however, that the reader formed a picture the basic ideas about analytic continuation. We must emphasize that all that has been said above is only theoretical in character and does not give any indication as to the practical methods of analytic continuation.

Let us now mention another principle in the theory of functions which is closely connected with analytic continuation and which is known as the *principle of permanency*. Assume that the initial function-element $f_1(z)$ satisfies a certain equation, e.g. the differential equation of the second order:

$$p_0(z)\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} + p_1(z)\frac{\mathrm{d}f(z)}{\mathrm{d}z} + p_2(z)f(z) = 0, \tag{94}$$

the coefficients $p_k(z)$ of which are given polynomials in z. On analytical continuation of $f_1(z)$, the derivatives $f_1'(z)$ and $f_2''(z)$, and also the whole of the left-hand side of our equation are analy-

tically continued. Consequently, if the left-hand side vanished in the initial domain, it would also vanish on analytic continuation, in other words, if the initial analytic function-element satisfies the equation (94) the equation will be satisfied by the analytic function, obtained from the initial element as a result of the analytic continuation.

Let us now turn to a definite method of analytic continuation. Here we shall only use *circular domains and Taylor's expansion* in such

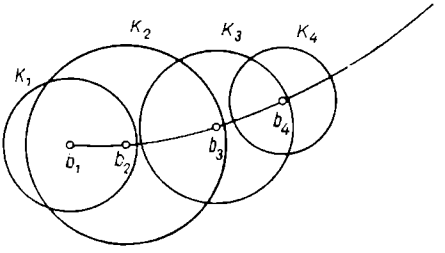

FIG. 15

domains (Fig. 15). Let the initial function-element be given in the form of a Taylor series, centre at b_1:

$$f_1(z) = \sum_{k=0}^{\infty} a_k^{(1)} (z - b_1)^k. \tag{95}$$

Let us draw a contour l from the point b_1 and perform the analytic continuation of our function along this contour. We proceed as follows: we take a point b_2 on the curve l such that the arc $b_1 b_2$ lies in the circle K_1 which is the circle of convergence of the series (95). By using this series we can evaluate the derivatives $f_1^{(n)}(b_2)$ and write the expansion of our function, centre at b_2:

$$f_2(z) = \sum_{k=0}^{\infty} a_k^{(2)} (z - b_2)^k = \sum_{k=0}^{\infty} \frac{f_2^{(k)}(b_2)}{k!} (z - b_2)^k. \tag{96}$$

This new function will be defined in a circle K_2, centre at b_2. If this circle lies outside the circle K_1, the function (96) will provide an analytic continuation of $f_1(z)$. At the point b_2 the values of the functions $f_1(z)$ and $f_2(z)$ and the values of all their derivatives coincide, and the functions will be the same in the overlapping portion of the

two circles. Notice that the series (96) can be obtained from the series (95) as follows: we rewrite the series (95) in the form:

$$\sum_{k=0}^{\infty} a_k^{(1)} [(z - b_2) + (b_2 - b_1)]^k. \tag{97}$$

On expanding $| (z - b_2) + (b_2 - b_1) |^k$ by the binomial formula and collecting terms in like powers in the sum (97) of $(z - b_2)$, we obtain the series (96).

Having performed the first analytic continuation we proceed to the next. We choose on the curve l a new point b_3 such that the arc $b_2 b_3$ belongs to the circle K_2. The series (96) can be rearranged, as shown above, in powers of $(z - b_3)$, when a new function-element is obtained:

$$f_3(z) = \sum_{k=0}^{\infty} a_k^{(3)} (z - b_3)^k,$$

which is defined in a circle K_3, centre at b_3, etc. As a simple example, consider the series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots \tag{98}$$

This series is convergent and defines a function which is regular only in the circle $|z| < 1$. But its sum $1/(1-z)$ is a regular function in the whole plane except at the point $z = 1$ and, consequently, we can continue the series (98) in the whole plane. If we take a point b_2 in the circle $|z| < 1$ and reconstruct the series (98) in powers of $(z - b_2)$, we obtain a new series of the form:

$$\sum_{k=0}^{\infty} \frac{1}{(1 - b_2)^{k+1}} (z - b_2)^k.$$

This series converges in a circle, centre at b_2 and radius equal to the distance from this point to the point $z = 1$. If the point b_2 does not lie on the segment (0, 1) of the real axis, this new circle will lie outside the old circle and we obtain an analytic continuation which can be continued further. In practice, in a case of this kind, it is evidently unnecessary to use analytic continuation of the series (98), — it is natural to use the finite form $1/(1-z)$. However, if the function is only given in the form of a power series and no other expression is known for it only the process of analytic continuation

remains open. Many attempts have been made in this field to find an easier practical way of performing analytic continuation. Later we shall give one practical method for particular case. For the present we take the analytic continuation of elementary many-valued functions as an example.

19. Examples of many-valued functions. Consider the function

$$z = w^2 \tag{99}$$

and suppose that the variable w varies in the upper half-plane i.e. in that part of the plane in which the coefficient of the imaginary part is positive (above the real axis), so that $\arg w$ varies from 0 to π. On squaring, the modulus $|w|$ is squared, and the amplitude is multiplied by two i.e. the values of z will fill the whole plane, and both the positive and negative parts of the real axis in the w-plane will be transformed into the positive part of the real axis in the z-plane. We can thus see that, as a result of the transformation (99), *the upper half of the w-plane is transformed into the whole z-plane with a cut along the positive part of the real axis from* 0 *to* $+\infty$. Denote the plane with this cut by T_1. Conversely, we can regard w as a single-valued function of z in the domain T_1:

$$w = \sqrt{z}, \tag{100}$$

where those values of the radical must be taken which give the positive coefficient of the imaginary part of $\sqrt{z}$. Real positive values of z lie both on the upper and the lower edges of our cut. On the upper edge positive values of the function $\sqrt{z}$ must be taken, and on the lower edge negative values must be taken. The limit of the ratio $\Delta w/\Delta z$ will evidently be equal to the reciprocal of the ratio $\Delta z/\Delta w$, i.e. the usual law for differentiating inverse functions will apply and the function (100) will be regular in our domain:

$$\frac{dz}{dw} = 2w; \quad \frac{dw}{dz} = \frac{1}{2\sqrt{z}}. \tag{101}$$

Returning to the beginning of our discussion, we now assume that w varies in the lower half-plane. On squaring, we evidently obtain for z a second copy of the same former region T_1. Let us denote this by T_2. In this new region T_2 our function (100) will again be regular and single-valued, provided that value of the radical is taken which gives the negative coefficient of the imaginary part of $\sqrt{z}$.

It also follows from the above that the values of the function (100) along the upper edge of the cut in the domain T_1 coincide with the values of that function along the lower edge of the cut in the domain T_2 and vice versa.

We can thus see that, by cutting from 0 to $+\infty$, we obtain a domain in which our function (100) is single-valued, but to obtain all the values of the function we must regard it as two different functions, which are defined in the domains T_1 and T_2 respectively as above. Such a division of the function (100) into two separate single-valued functions appears artificial and we shall now combine these two functions into a single analytic function which is single-valued and regular in a two-sheeted plane. In order to produce this two-sheeted plane T, imagine the copy T_1 on top of the copy T_2 with their edges joined together cross-wise along the cut, viz. the upper edge of the cut in T_1 is joined to the lower edge of the cut in T_2 and vice versa. We assume that the point $z = 0$ coincides on both copies. The constructed *two-sheeted domain* T is evidently obtained from the w-plane as a result of the transformation (99) and the function (100) will be regular and single-valued in the entire domain T, except at the point $z = 0$. Note the special importance of this point. If, starting from a point z_0 we draw a closed contour about $z = 0$, on returning to the point z_0 we find ourselves on another sheet as compared with that from which we started to draw our contour. Here, the values of the function $\sqrt{z}$, as defined above on our contour, will evidently give the analytic continuation of the function along that contour, and the final value of the function at the point z_0 will be of opposite sign as compared with the initial element at that point. The point $z = 0$ has the property that the function $\sqrt{z}$ is continuous and has a derivative in the neighbourhood of the point, but on the analytic continuation round a closed contour about this point it changes its values. Such a point is known as *a branch-point of the function.* In the case under consideration we return to the original values of the function by describing another circuit about the point $z = 0$ and such a branch-point is known as *a branch-point of the first order.* The domain T evidently represents the total domain of existence of the function (100). In this case we have been able to obtain this domain relatively simply, since the function (100) is the inverse of the very simple function (99). Figure 16 shows the appearance of a two-sheeted plane near a branch-point of the first order.

Generally speaking, if the function

$$z = \varphi(w) \tag{102}$$

is single-valued and regular in the whole w-plane, as a result of the transformation (102) this plane may change into the many-sheeted z-plane and the inverse function, to (102):

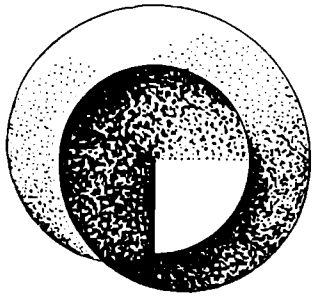

FIG. 16

$$w = f(z) \tag{103}$$

will be regular in this many-sheeted plane and will have the derivative

$$f'(z) = \frac{1}{\varphi'(w)}.$$

This regularity will only be lost at points which correspond to values of w where $\varphi'(w) = 0$. These points correspond to the branch-points of the inverse function (103). We will explain this in greater detail in one of the following paragraphs. The above many-sheeted planes are usually known as *Riemann surfaces* (Riemann was a mid-nineteenth century German mathematician). Consider the following example:

$$f(z) = \frac{1}{\sqrt{z} + 2}. \tag{104}$$

The many-valuedness of this function is solely due to the presence of $\sqrt{z}$ and, consequently, in the two-sheeted plane T, which we constructed above for the function (100), the function (104) will also be single-valued. This function will have a singularity at the point $z = 0$ (branch-point) and also at one of the points $z = 4$. There will be two of these points (on both sheets T_1 and T_2). On one of the sheets $\sqrt{4} = +2$ and on the other $\sqrt{4} = -2$. On the latter sheet the point $z = 4$ will be a pole of the function (104). Had we not the two-sheeted plane T, as a result of the analytic continuation of (104) we would have obtained different values for this function and the point $z = 4$ would have been a singularity for those paths of the analytic continuation for which $\sqrt{z}$ is equal to -2 when $z = 4$.

The function (104) can be regarded as the inverse of the function

$$z = \frac{(2w - 1)^2}{w^2}, \tag{104_1}$$

which is regular in the whole plane except at $w = 0$, where it has a pole of the second order; it can be shown that this function transforms the w-plane into a two-sheeted T-plane of the type described above. In the case under consideration the point $w = 1/2$ will be transformed into the branch-point $z = 0$; the point $w = 0$ will be transformed into the point $z = \infty$. The point $w = \infty$ gives the point $z = 4$ on one of the sheets. A point with the same coordinate $z = 4$ on the other sheet will be obtained when $w = 1/4$. Note that in the given two-sheeted plane we must regard not only the point $z = 0$ but also $z = \infty$ as coinciding on both sheets, i.e. the points $z = \infty$ and $z = 0$ are both branch-points of the first order. The first of these points can be obtained from the formula (99) only when $w = 0$, and from the formula (104_1) only when $w = 1/2$. The point $z = \infty$, can be obtained from the formula (99) only when $w = \infty$, and from the formula (104_1) only when $w = 0$.

Consider a function of the following kind:

$$w = f(z) = \sqrt{(z-a)(z-b)}. \tag{105}$$

For this function the points a and b are branch-points. By describing a circuit round a closed contour encircling one of these points we change the sign of the expression (105) but by describing simultaneously circuits round both points we leave the function unaltered. In fact, assume:

$$z - a = \varrho_1 e^{i\varphi_1}; \qquad z - b = \varrho_2 e^{i\varphi_2},$$

whence

$$f(z) = \sqrt{\varrho_1 \varrho_2}\, e^{i\frac{\varphi_1+\varphi_2}{2}}.$$

If we describe a circuit round a closed contour l encircling both points in the counter-clockwise direction, 2π will be added to the amplitudes φ_1 and φ_2, the sum $(\varphi_1 + \varphi_2)$ will receive an increment of 4π and the amplitude of the expression (105) an increment of 2π, i.e. the value of the function will be unchanged. In order to make the function (105) single-valued it is sufficient to make a cut from the point a to the point b. This cut prevents us, as it were, from making separate circuits round the points a and b. The function (105) is two-valued at all points except $z = a$ and b; in order to obtain all the values of this function we must take two copies of the plane cut in the way described above. On each of these (105) will be a single-valued function and the values of this function on different copies will differ

from each other in sign only. If we superimpose one copy on top of the other and imagine the edges of the cuts to be joined cross-wise, we obtain a two-sheeted Riemann surface, with branch-points of the first order at a and b; on this surface the function (105) will be single-valued and regular (except at the branch points). The point at infinity will not be a branch-point and each sheet will have its own point at infinity. In the neighbourhood of this point at infinity we can rewrite the function (105) in the form:

$$f(z) = \pm z\left(1 - \frac{a}{z}\right)^{\frac{1}{2}}\left(1 - \frac{b}{z}\right)^{\frac{1}{2}}.$$

Expanding the differences by the binomial formula, which is possible since in the neighbourhood of the point at infinity $|\, a/z \,|$ and $|\, b/z \,|$ are smaller than unity, we obtain our function in the following form in the neighbourhood of the point at infinity:

$$f(z) = \pm z\left(1 - \frac{1}{2}\frac{a}{z} - \frac{1}{2\cdot 4}\frac{b^2}{z^2} - \frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{a^3}{z^3} - \cdots\right)\times$$
$$\times\left(1 - \frac{1}{2}\frac{b}{z} - \frac{1}{2\cdot 4}\frac{a^2}{z^2} - \frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{b^3}{z^3} - \cdots\right),$$

i.e. on multiplying the series we can see that the point at infinity is a pole of the first order on both sheets.

Notice that, on solving the equation (105) with respect to z, we obtain a many-valued function w, i.e. the function (105) is not the inverse of a function which is single-valued in the whole plane. The Riemann surface, on which it is single-valued, will have two branch-points $z = a$ and $z = b$ of the first order. This Riemann surface can be obtained by the transformation

$$z = \frac{bw^2 - a}{w^2 - 1}$$

of the w-plane and the inverse function of the above:

$$w = \sqrt{\frac{z-a}{z-b}},$$

will have the same Riemann surface as the function (105).

Consider the function

$$f(z) = \sqrt[n]{z-a} \tag{106}$$

where n is a positive integer. Every circuit round the point $z = a$ changes the value of the function, and on performing n such circuits

in the same direction, we return to the original value of the function, i.e. for the function (106) the point a will be a branch-point of the order $n-1$. For, denoting the modulus and amplitude of $z-a$ by ϱ and φ respectively, we obtain:

$$\sqrt[n]{z-a} = \sqrt[n]{\varrho}\, e^{i\frac{\varphi}{n}}.$$

On describing a circuit round $z=a$, n times in the positive direction, we add $2n\pi$ to φ and, consequently, the amplitude of $\sqrt[n]{z-a}$ receives an increment of 2π, which does not alter the value of the function.

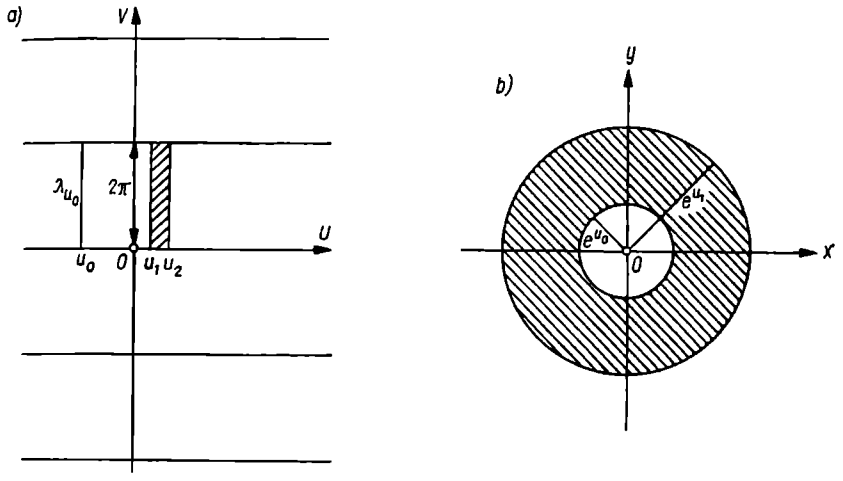

FIG. 17

We shall now consider another many-valued function which is of special importance in the theory of functions, viz. the logarithm. This function is obtained as a result of the inversion of an exponential function

$$z = e^w. \tag{107}$$

We shall first explain some properties of the exponential function. It is easy to see that it has a *pure imaginary period of* $2\pi i$. In fact:

$$e^{w+2\pi i} = e^w e^{2\pi i} = e^w (\cos 2\pi + i \sin 2\pi) = e^w.$$

We divide the plane $w = u + iv$ into strips, 2π in width, by straight lines parallel to the real axis. As the fundamental, strip take say the strip U bounded by the lines $v=0$ and $v=2\pi$. We can transform this funda-

mental strip into another strip by adding $2n\pi i$ to w, where n is an integer. The values of the function (107) thereby remain unaltered because of the periodicity, i.e. the values of the function in every strip will be the same as in the initial strip. We shall now consider the way in which the fundamental strip is transformed by the function (107). Draw in this strip a line λ_{u_0}, parallel to the imaginary axis with the abscissa $u = u_0$. Along this line we have:

$$u = u_0 \qquad (0 \leqslant v \leqslant 2\pi)$$

and, consequently:

$$e^w = e^{u_0} e^{iv} \qquad (0 \leqslant v \leqslant 2\pi),$$

i.e. our line will be transformed into a full circle, centre the origin and radius e^{u_0}, whilst the same point on the circle corresponds to the ends of the line λ_{u_0}. If we take the part of the strip U bounded by two lines, parallel to the axis $u = 0$ with the abscissae $u = u_1$ and u_2, as a result of the transformation (107) we obtain a circular annulus in the z-plane, centre the origin and radii e^{u_1} and e^{u_2} (Fig. 17). Finally, the entire strip U will be transformed into the z-plane with the exception of the origin. The upper and lower edges of the strip will be transformed into the positive part of the real axis. We make a cut along this part of the real axis. We can say then that the upper edge of this cut corresponds to the lower edge of the strip and the lower to the upper edge of the strip. Denote by T_1 the plane with the cut without the origin. In this domain T_1 the function inverse to (107):

$$w = \log z \tag{108}$$

will be single-valued and regular and its derivative can be found by the usual rule for differentiating inverse functions:

$$\frac{dw}{dz} = \frac{1}{(e^w)'} = \frac{1}{e^w} = \frac{1}{z}. \tag{109}$$

We have, as we know:

$$\log z = \log|z| + i \arg z.$$

On analytical continuation of this function along a contour we must preserve continuous variation of the amplitude $\arg z$.

In the domain T_1 variations of the amplitude is restricted to $0 \leqslant \arg z \leqslant 2\pi$, and we obtain a single-valued definition for the function (108). The point $z = 0$ is evidently a branch-point for our

function (108), viz. on analytical continuation of this function round a closed contour about the origin and on encircling the origin n times in the counter-clockwise direction, we add $2n\pi i$ to the function (108); every subsequent circuit will give new values of the function, i.e. in this case the point $z = 0$ is a *branch-point of infinite order*.

Let us return to the transformation of the w-plane by the function (107). Every strip into which we have divided the w-plane gives a new domain T_1 in the z-plane and there will consequently be an infinite number of such domains. We superimpose them on top of each other and number them so that the domain corresponding to the initial strip is number one, the next one placed on top of it is number two, etc. while domains corresponding to the lower strips are numbered 0, (-1), (-2), ... Let us now join in imagination the edges of the cuts as follows: join the upper cut of T_1 with the lower cut of T_0, and the lower cut of T_1 with the upper cut of T_2; join the upper cut of T_0 with the lower cut of T_{-1} and the lower cut of T_2 with the upper cut of T_3 etc. In this way we obtain a Riemann surface with an infinite number of sheets and with branch-points of an infinite order at $z = 0$ and $z = \infty$. On this Riemann surface T our function (108) is regular and single-valued. The surface T is obtained from the w-plane as a result of the transformation (107).

The function $w = \log(z - a)$ evidently has branch-points of an infinite order at $z = a$ and $z = \infty$. Let us also consider the function

$$w = \log \frac{z-a}{z-b} = \log(z-a) - \log(z-b). \tag{110}$$

This function has branch-points at $z = a$ and $z = b$. If we describe a circuit in the positive direction round a closed contour which encircles both points, both terms in the above expression receive an increment of $2\pi i$ and the difference remains unchanged, i.e. infinity is not a branch-point of the function (110).

We can rewrite our function in the following form:

$$w = \log\left(1 - \frac{a}{z}\right) - \log\left(1 - \frac{b}{z}\right),$$

and we can expand both terms according to formula (80) for all values of z, the moduli of which are greater than $|a|$ and $|b|$. As a result we obtain the following form for our function in the neighbourhood of the point at infinity:

$$w = \sum_{k=1}^{\infty} \frac{a_k}{z_k}, \tag{111}$$

where

$$a_k = \frac{b^k - a^k}{k}.$$

Formula (111) gives one of the branches of our many-valued function in the neighbourhood of infinity. To obtain the remaining branches it is sufficient to add $2n\pi i$ to the above expression. For each fixed integer n we obtain another branch of our function.

Consider the function

$$w = \arctan z = \frac{1}{2i} \log \frac{i - z}{i + z},$$

which has branch-points of infinite order at $z = i$ and $z = -i$. The derivative of this function, as in the case of the real variable, is

$$\frac{dw}{dz} = \frac{1}{1 + z^2}$$

or

$$\frac{dw}{dz} = -\frac{1}{(i + z)(i - z)}.$$

20. Singularities of analytic functions and Riemann surfaces. In the preceding paragraphs we dealt with examples of many-valued functions and constructed Riemann surfaces for these functions, on which they were single-valued. We shall now consider the general problem. Owing to lack of space we shall not go into details. To begin with we shall explain the concept of an isolated singular point in analytic continuation.

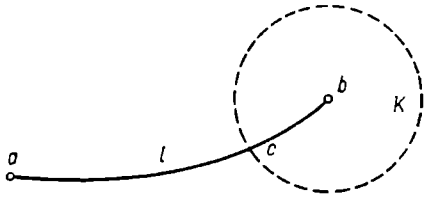

FIG. 18

Assume that we are given the initial analytic function-element $f(z)$ at a point $z = a$ and that we propose to continue it along a line l. Suppose that analytic continuation is possible up to point $z = b$ but no further, so that the point $z = b$ is a singularity in the analytic continuation along l [18]. Assume the existence of a circle K, centre at $z = b$, such that the function-elements $f(z)$, corresponding to points on the section cb of the line l in K, (Fig. 18) can be analytically continued along any line in K, which does not pass through the point $z = b$. In this case the point $z = b$ is known as *an isolated singularity* of $f(z)$

(corresponding to the path l). The analytic continuation along any line in K can lead to single-valued or many-valued functions in K. In the first case the single-valued function obtained in K is regular everywhere in K except at $z=b$ and it can be expanded into a Laurent's series in integral powers of $(z-b)$; the point $z=b$ is either a pole or an essential singularity of our analytic function $f(z)$ (on analytic continuation along l). In the second case, when the function in K is many-valued, the point $z=b$ is known as *a branch-point*. Suppose that, given all the possible analytic continuations in K, we obtain a finite number of different elements at a point $z=a$ inside K. Denote this number by m. It is easy to see that at any other point $z=\beta$ in K, we again obtain m different elements. This is due to the fact that, on analytic continuation of different initial elements along the same path from a to β or from β to a, different elements are obtained at the end-point. In the case under consideration the point $z=b$ is known as *a branch-point of the* $(m-1)$*th order*. If the number of different elements obtained on analytical continuation in K is not finite at every point in K then $z=b$ is a *branch-point of infinite order*.

Let us consider in greater detail the case of a branch-point of the finite $(m-1)$th order. By hypothesis, analytic continuation can be performed in K, except at the point $z=b$. A circle K with an isolated point $z=b$ is a doubly-connected domain. Take m copies of the circle K and cut each copy along the same radius. Such a circle K_1, cut along the radius, is a simply connected domain. Let us take the same point $z=a$ in each copy K_1. At that point we have m elements of our analytic function. In each copy we take a definite element at the point $z=a$ and continue it analytically in K_1. In accordance with the uniqueness theorem [18] we obtain a definite single-valued function in every copy. We shall term one edge of the cut in every copy the left-hand edge and the other the right-hand edge. For example, we shall describe as the right-hand edge that edge from the points of which we can reach the left-hand edge by moving inside K_1 and encircling the point $z=b$ in the counter-clockwise direction. Take a copy of the circle K_1 with a single-valued function defined in it. Call this copy the first copy and denote the definite single-valued function by $f_1(z)$. The values of the function $f_1(z)$ on the left-hand edge coincide with the values of our function on the right-hand edge of the cut in some other copy of K_1. Call this latter copy of K_1 the second copy and denote the function defined in it by $f_2(z)$. Join in imagination the left-hand edge of the

first copy with the right-hand edge of the second copy. The values of the function $f_2(z)$ on the left-hand edge of the second copy coincide with values of our function on the right-hand edge of another copy of K_1. Let us call this copy the third copy and denote the function defined in this copy by $f_3(z)$. Join in imagination the left-hand edge of the second copy with the right-hand edge of the third copy. Continuing in this way we eventually reach the last copy m. It can easily be seen that the values of the function $f_m(z)$ on the left-hand edge of the mth copy coincide with the values of $f(z)$ on the right-hand edge of the first copy. Join in imagination these two edges. We thus obtain an m-sheeted circle L with the branchpoint $z=b$ of the $(m-1)$th order. This point must coincide on all copies. In the m-sheeted circle L our function will be single-valued everywhere except at the point $z = b$. Replace z by a new independent variable

$$z' = \sqrt[m]{z-b} = \sqrt[m]{\varrho}\, e^{i\frac{\varphi}{m}}, \tag{112}$$

where $\varrho = |z-b|$ and $\varphi = \arg(z-b)$, where φ is fixed in a definite way at a point of L. The point $z=b$ will be transformed into the point $z' = 0$. In general, on describing a circuit round the point $z = b$, the amplitude changes by $2\pi m$ and on describing a circuit round the point $z'=0$ it changes by 2π. In the z'-plane the m-sheeted circle L will be transformed into a one-sheeted circle C, centre at $z' = 0$ and radius $\sqrt[m]{R}$, where R is the radius of L. In this one-sheeted circle C our function will be single-valued and regular with the possible exception of the point $z' = 0$. Consequently it is possible to expand it in C into a Laurent's series:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z'^n$$

or, returning to the former variable:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (\sqrt[m]{z-b})^n = \sum_{n=-\infty}^{+\infty} a_n (z-b)^{\frac{n}{m}}, \tag{113}$$

i.e. in the neighbourhood of the branch-point of the $(m-1)$th order our function can be expanded in integral powers of the argument (112). The value of the argument (112) at a point z, which lies in the neighbourhood of the point $z = b$, can be fixed in an arbitrary yet definite manner. The expansion (113) can have different forms. It can happen that the expansion contains no terms with negative values of n:

$$f(z) = a_0 + a_1 \sqrt[m]{z-b} + a_2 (\sqrt[m]{z-b})^2 + \dots$$

It is evident here that $f(z) \to a_0$ when $z \to b$, while z may tend to b in any manner as long as it remains in L. In the case under consideration we assume that $f(b) = a_0$ and call the point $z = b$ *a branch-point of the regular type*. If the expansion (113) contains only a finite number of terms with negative values of n, then $f(z) \to \infty$ when $z \to b$. In this case we put $f(b) = \infty$ and call the point $z = b$ *a branch-point of the polar type*. If the expansion (113) contains an infinite number of terms with negative values of n, the point $z = b$ is *a branch-point of the essential singularity type*.

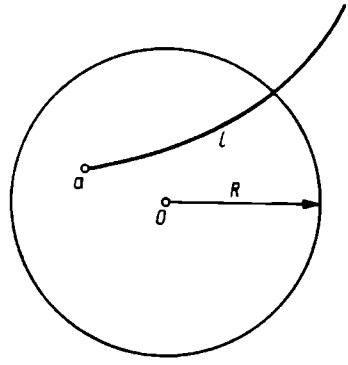

FIG. 19

All these definitions can be extended to the point at infinity. Suppose that $f(z)$ is continued analytically along the contour l and that a neighbourhood $K(|z| > R)$ of the point at infinity (Fig. 19) exists, such that the function-elements $f(z)$, corresponding to points on the arc l in K, can be analytically continued along any path inside K. If this analytic continuation gives a single-valued function then the point $z = \infty$ will be either a regular point of $f(z)$, or a pole or an essential singularity [10]. If the analytic continuation produces a many-valued function then $z = \infty$ is a branch-point. If this branch-point is of the finite $(m - 1)$th order then in its neighbourhood the expansion given below applies:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n \left(\frac{1}{\sqrt[m]{z}}\right)^n = \sum_{n=-\infty}^{+\infty} a_n z^{-\frac{n}{m}},$$

where everything that was said above regarding such an expansion again applies. The character of the point $z = \infty$ may, of course, depend on the path l of the analytic continuation, by which we reached the neighbourhood of the point at infinity.

We shall now explain in general terms the concept of a Riemann surface for a given many-valued analytic function $f(z)$. Suppose that, on analytic continuation of the initial element, we have reached a point $z = a$. At this point we are given a certain element, i.e. a series expanded in positive integral powers of $(z - a)$. This series can be rearranged in positive integral powers of $(z - \beta)$, where β is any point in the neighbourhood of $z = a$, i.e. the element at the

point $z = a$ also gives the elements at all points which are sufficiently close to a. To every such element we ascribe a point z which acts as the centre of the corresponding circle of convergence of the power series (element). To the given element with the centre at $z = a$ we ascribe the point a. To elements obtained from it at the neighbouring points $z = \beta$, we ascribe points $z = \beta$, which belong to the neighbourhood of $z = a$, i.e. which lie on the same sheet as the point $z = a$. In the course of analytic continuation we keep on obtaining further elements and, consequently, further points z on the Riemann surface. If, on returning to the point $z = a$, we obtain the function-element which we had earlier, we identify it with the former point $z = a$. If, however, this element proves to be different from the former, then we regard the new point $z = a$ as being different from the former point $z = a$ (we consider it to be situated on a different sheet) i.e. on analytic continuation we regard two points z, which have the same complex coordinates, as different if we have different elements of our analytic function at those points. In this way we construct a Riemann surface during the analytic continuation, which corresponds to the given analytic function $f(z)$. On this surface $f(z)$ is single-valued and regular. A Riemann surface usually includes the poles of $f(z)$ as well as the finite order, branch-points of the regular and polar types. Note that, in general, the Riemann surface cannot be obtained from the w-plane as a result of the transformation $z = \varphi(w)$, where $\varphi(w)$ is single-valued and regular in the w-plane (with possible poles), as could be done in the simpler cases in [19].

Above we have only considered isolated singularities. It can happen that these points completely fill certain lines in the course of analytic continuation. For example, it can happen that the initial element, given by a power series, cannot be continued in any direction, i.e. that any point of the circumference of the circle of convergence of this power series is a singularity. The series given below can serve as an example of a series which cannot be continued:

$$\sum_{n=1}^{\infty} z^{n!} = z + z^{1\cdot 2} + z^{1\cdot 2\cdot 3} + z^{1\cdot 2\cdot 3\cdot 4} + \ldots = z + z^2 + z^6 + z^{24} + \ldots$$

21. The theorem of residues. Let us now return to the expansion of a function into a Laurent's series in the neighbourhood of a singularity (a pole or an essential singularity). In this expansion we separated the coefficient of $(z - b)^{-1}$ and called it the residue of the function at the given singularity. We shall now explain the significance of this

coefficient. Thus, let us assume that in the neighbourhood of the point b the following expansion holds:

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-b)^k.$$

Integrate this formula round a small closed contour l_0, which surrounds the point b, and on which the given expansion converges uniformly:

$$\int_{l_0} f(z)\,dz = \sum_{k=-\infty}^{+\infty} a_k \int_{l_0} (z-b)^k\,dz.$$

As we saw earlier [6], all the integrals on the right-hand side are equal to zero, except one, which corresponds to $k=-1$; this integral is equal to $2\pi i$, i.e. we have:

$$\int_{l_0} f(z)\,dz = a_{-1} \cdot 2\pi i.$$

Let us now consider the more general case. Assume that $f(z)$ is regular in a closed domain B with the contour l, save for a finite number of points $b_1, \ldots, b_m$ in that domain, which are poles or essential singularities of the function. Denote by $a_{-1}^{(s)}$ $(s = 1, \ldots, m)$ the residues at these singularities. Isolate each singularity by a small closed contour l_s. In accordance with Cauchy's theorem we can write:

$$\int_l f(z)\,dz = \sum_{s=1}^{m} \int_{l_s} f(z)\,dz.$$

But, as we saw above, the value of every integral round each contour l_s is equal to $a_{-1}^{(s)} \cdot 2\pi i$ and, consequently, the above equation expresses the value of the integral round the contour of the domain in terms of the residues of the function at the singularities in the domain:

$$\int_l f(z)\,dz = 2\pi i \sum_{s=1}^{m} a_{-1}^{(s)}. \tag{114}$$

THEOREM OF RESIDUES. *If a function is regular in a closed domain save for a finite number of points (poles or essential singularities) then the value of the integral of this function round the contour of the domain is equal to the product of $2\pi i$ and the sum of residues at the above singularities.*

In future we shall find countless applications for this theorem. At the moment, however, we shall only establish some theoretical consequences which are necessary for our further treatment. To begin

with we shall give the practical rules for calculating residues without using the expansion of the function into a Laurent's series.

As the first example consider the following function:

$$f(z) = \frac{\varphi(z)}{\psi(z)}, \tag{115}$$

where $\varphi(z)$ and $\psi(z)$ are regular at the point b and $\psi(b) = 0$, so that the function (115), generally speaking, has a pole at the point b. Suppose, furthermore, that the point $z = b$ is a simple zero of $\psi(z)$, i.e. the expansion of the function $\psi(z)$ into a Taylor's series begins with a first degree term:

$$\psi(z) = c_1(z-b) + c_2(z-b)^2 + \dots \quad (c_1 \neq 0).$$

In this case the function (115) has a simple pole (of multiplicity one) in the neighbourhood of $z = b$:

$$f(z) = \frac{\varphi(b) + \frac{\varphi'(b)}{1!}(z-b) + \dots}{(z-b)[c_1 + c_2(z-b) + \dots]}.$$

It follows from the last formula that we can write for the residue a_{-1}:

$$a_{-1} = f(z)(z-b)\big|_{z=b} = \frac{\varphi(b)}{c_1},$$

or, taking into account that c_1 is equal to $\psi'(b)$:

$$a_{-1} = \frac{\varphi(b)}{\psi'(b)}. \tag{116}$$

As a second example, consider the case when the function $f(z)$ has a pole of an arbitrary order m at the point b:

$$f(z) = \sum_{k=-m}^{\infty} a_k (z-b)^k.$$

The product $f(z)(z-b)^m$ is a regular function at the point b, and the coefficient a_{-1} is the coefficient of $(z-b)^{m-1}$ in this product, whence recalling the expression for the coefficients in Taylor's series, we have the following formula for the residue of our function:

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[f(z)(z-b)^m]\Big|_{z=b}. \tag{117}$$

Let us consider one more example. Suppose that $f(z)$ has a zero of the mth order at the point b, i.e. Taylor's series with centre at b begins

with a term containing $(z-b)^m$. In this case our function has the following form in the neighbourhood of the point b:

$$f(z) = (z-b)^m \varphi(z) \qquad (\varphi(b) \neq 0), \tag{118}$$

where $\varphi(z)$ is regular at b and is not equal to zero. Let us construct *the logarithmic derivative of our function*:

$$\frac{f'(z)}{f(z)} = \frac{m}{z-b} + \frac{\varphi'(z)}{\varphi(z)}. \tag{119}$$

It can easily be seen that the point b is a simple pole of the logarithmic derivative, with a residue equal to the order of the zero of the function $f(z)$. If, instead of having a zero, our function has a pole of multiplicity m at the point b, formula (118) still holds, except that m must be replaced by $(-m)$; also, all the subsequent working is the same, i.e. if at a given point the function has a pole of multiplicity n that its logarithmic derivative has a simple pole with a residue $(-n)$ at this point.

22. Theorem on the number of zeros. Assume that $f(z)$ is regular in the closed domain B with the contour l and that it does not vanish on the contour. Assume that it has zeros $b_1, \ldots, b_m$ in the domain of orders $k_1, \ldots, k_m$. Its logarithmic derivative has simple poles at these points b_s with residues k_s. From the theorem of residues we have:

$$\frac{1}{2\pi i}\int_l \frac{f'(z)}{f(z)}\,dz = k_1 + k_2 + \ldots + k_m. \tag{120}$$

If every multiple zero is counted as many times as there are units in its multiplicity, the number on the right-hand side gives the number of zeros of our function in the domain, i.e. given our assumptions with regard to the function $f(z)$, *the integral on the left-hand side gives the number of zeros of the function included within the contour l.*

The integrand here evidently has the primitive $\log f(z)$ and we obtain the value of the integral by determining the increment that this primitive receives on describing a circuit round the contour l. Here we have to consider the single-valued branch of the function, which means, as we know already, that when describing a circuit round the contour l we must preserve a continuous change of $\arg f(z)$; here,

$$\log f(z) = \log|f(z)| + i \arg f(z).$$

After the circuit is completed $\log |f(z)|$ returns to its former value and receives no increment; consequently, the total increment received by our primitive is equal to the product of i and the increment of $\arg f(z)$. In accordance with formula (120), we must also divide the increment of the primitive function by $2\pi i$, when we finally obtain the following result:

CAUCHY'S THEOREM. *If the function $f(z)$ is regular in the closed domain B and does not vanish on the contour of this domain, the number of zeros of the function in the domain is equal to the change in the amplitude of the function on describing the contour, divided by 2π, or, in other words, it is equal to the change in the amplitude expressed as parts of 2π.*

The above theorem is evident in the case of polynomials. Take, for example, a polynomial of the third degree and represent it as a product of first degree factors:

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 = a_3 (z - b_1)(z - b_2)(z - b_3).$$

Suppose that the zeros b_1 and b_2 lie within the contour l and that the zero b_3 lies outside the contour l. Every difference $(z - b_k)$ corresponds to a vector drawn from b_k to z. When the point z describes a circuit round the contour l the amplitudes of the vectors $(z - b_1)$ and $(z - b_2)$ evidently receive an increment of 2π, but the amplitudes of $(z - b_3)$ remains unaltered. Hence the total increment of the function is equal to 4π (the amplitude of a product is equal to the sum of the amplitudes of the factors) or, expressed as parts of 2π, this increment is equal to 2, i.e. it is equal to the number of zeros inside l.

Let us establish a further theorem on the number of zeros of a regular function; this is a direct result of Cauchy's theorem. Assume, as before, that $f(z)$ is regular in a closed domain and does not vanish on the contour. Assume that we have another function $\varphi(z)$, which is regular in a closed domain, and the modulus of which on the contour l is less than that of $f(z)$, i.e.

$$|\varphi(z)| < |f(z)| \text{ on } l. \tag{121}$$

Notice that, when this condition applies, $f(z)$ evidently cannot vanish on l. Consider the two functions:

$$f(z) \text{ and } f(z) + \varphi(z). \tag{122}$$

Both functions satisfy the conditions of Cauchy's theorem. We have shown this to be so for the first function and the second function cannot vanish on the contour because of the condition (121). We shall

show that the second function has the same number of zeros within the contour as the first function. For this purpose consider the amplitude of this function on the contour, remembering that on the contour $f(z) \neq 0$:

$$\arg [f(z) + \varphi(z)] = \arg f(z) + \arg\left[1 + \frac{\varphi(z)}{f(z)}\right].$$

To prove our assertion it is sufficient to show that, on completing a circuit round the contour l, the change in the amplitude

$$\arg\left[1 + \frac{\varphi(z)}{f(z)}\right] \tag{123}$$

is equal to zero. In accordance with the condition (121), the modulus of the fraction $\varphi(z)/f(z)$ is less than unity and, consequently, on describing a circuit round the contour l the variable point

$$z' = 1 + \frac{\varphi(z)}{f(z)}$$

will always lie inside a circle C, centre $z' = 1$ and unit radius. This variable point describes a closed curve which lies within the circle C and which evidently does not encircle the origin. We can thus see that the variation in the amplitude of the expression (123) is in fact equal to zero.

ROUCHE'S THEOREM. *If $f(z)$ is regular in a closed domain with a contour l and $\varphi(z)$ is also regular in a closed domain and satisfies on the contour l the condition* (121) *then the functions $f(z)$ and $f(z) + \varphi(z)$ have the same number of zeros within the domain.*

Note that from Rouche's theorem follows the *basic theorem of algebra*, viz. any polynomial of the nth degree:

$$a_0 + a_1 z + \ldots + a_n z^n \qquad (a_n \neq 0) \tag{124}$$

has exactly n zeros in the plane. For suppose we put here $f(z) = a_n z^n$ and $\varphi(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}$. On any circle, centre the origin and a sufficiently large radius, we evidently have $|\varphi(z)| < |f(z)|$, since the degree of the polynomial $\varphi(z)$ is lower than the degree of the polynomial $f(z)$. In accordance with Rouche's theorem, the polynomial (124) has the same number of zeros inside the circle as the polynomial $f(z) = a_n z^n$, and the latter polynomial has a zero of multiplicity n at the origin.

We shall also point out a consequence of Cauchy's theorem which is of great importance in the theory of conformal transformation. Suppose that the function

$$w = f(z) \tag{125}$$

is regular in a closed domain and, when the point z describes a circuit round the contour l, the point w describes a simple closed circuit round the contour l_1, which does not cut itself (Fig. 20). We shall show that the function (125) transforms in this case the initial region B into the domain B_1, bounded by the contour l_1. Take a point w_1 inside the contour l_1 and a point w_2 outside the contour l_1. We have to prove that the function

$$F_1(z) = f(z) - w_1$$

has one zero in the domain B and that the function

$$F_2(z) = f(z) - w_2 \cdot$$

has no zeros. When the point z describes a circuit round the contour l the difference $f(z) - w_1 = w - w_1$ will correspond to a vector drawn from the point w_1 to the variable point w on the contour l_1. Two cases are conceivable: the point w can describe the circuit round the contour l_1 in the counter-clockwise direction or the circuit can be described in the clockwise direction, assuming that the point z describes the circuit in the positive direction, i.e. in the counter-clockwise direction. In the first case the variation in the amplitude of the function $F_1(z)$ is evidently equal to 2π, and consequently, this function has one zero within l. In the second case we obtain a negative number (-2π) as the variation in the amplitude of the function $F_1(z)$ and it appears that the function $F_1(z)$ has minus one zero in the domain which does not make sense, since the number of zeros must either be equal to zero or be a positive integer. Hence the second case is impossible, and when the point z describes a circuit round the contour l in the positive sense then the corresponding point w must describe a circuit round l_1 also in the positive sense. Let us now turn to the function $F_2(z)$. The amplitude of the corresponding vector from w_2 to the variable point w on the contour l_1 receives no increment on describing a circuit round the contour l_1 and, consequently, the function $F_2(z)$ has no zeros within l. We thus arrive at the following theorem: if $f(z)$ is regular in a closed domain B with a contour l and transforms l into a simple closed contour l_1 which does not cut itself, then on describing a circuit in the positive sense round the contour l, a circuit in the positive sense is also described round the contour l_1, and *the*

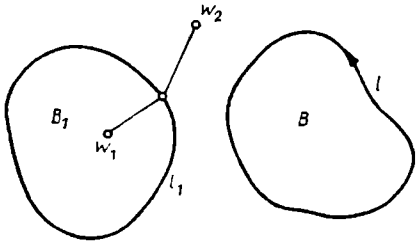

FIG. 20

function $f(z)$ transforms the domain B into the part of the plane bounded by the contour l.

We obtained Cauchy's theorem by considering the integral on the left-hand side of formula (120) and assuming that $f(z)$ is regular in a closed domain and does not vanish on the contour. Let us now assume that $f(z)$ has a finite number of poles in the domain but that it is otherwise regular and that it is also regular on the contour, on which it does not vanish. In this case, as we have seen, the integrand has simple poles in the domain at the zeros of the function $f(z)$, with a residue equal to the order of the zero, and at the poles of the function $f(z)$ with a residue equal to minus the order of the pole. Applying the fundamental theorem of residues to this integral, we obtain in the case under consideration the following formula instead of the formula (120):

$$\frac{1}{2\pi i}\int_l \frac{f'(z)}{f(z)}\,dz = m - n\,, \tag{126}$$

where m is the total number of zeros and n the total number of poles of our function in the domain. Assume that these zeros are situated at the points $b_1, \ldots, b_m$, and the poles at the points $c_1, c_2, \ldots, c_n$, where multiple poles and zeros are counted several times. It is easy to prove the following formula by using the fundamental theorem of residues:

$$\frac{1}{2\pi i}\int_l z\frac{f'(z)}{f(z)}\,dz = (b_1 + b_2 + \ldots + b_m) - (c_1 + c_2 + \ldots + c_n)\,, \tag{127}$$

i.e. the integral on the left-hand side expresses the difference between the sum of the coordinates of the zeros and the sum of the coordinates of the poles. In fact, in the case of a zero b of multiplicity k we have the following expansion in the neighbourhood of that point:

$$z\frac{f'(z)}{f(z)} = [b + (z-b)]\left[\frac{k}{z-b} + a_0 + a_1(z-b) + \ldots\right],$$

whence it follows that the residue at this point is equal to kb. The same argument also applies for a pole.

In conclusion we shall make an addition to the above theorem about the conformal transformation of one domain into another domain. It is given that $f(z)$ has one simple pole in the domain B, i.e. in formula (126) $n = 1$, and that $f(z)$ transforms the contour l into a simple closed contour which does not cut itself, but when describing a circuit

round l in the positive sense a circuit in the negative sense will be described round the contour l_1. Let us consider once again the functions $F_1(z)$ and $F_2(z)$. They both have the same simple pole in the domain as the function $f(z)$. For the first of these functions the variation in the amplitude, expressed as parts of 2π, is equal to minus unity but, on the other hand, in accordance with formula (126), this change in the amplitude must give the difference between the number of zeros and the number of poles; thus by hypothesis, the function has one pole. It therefore follows that the function $F_1(z)$ has no zeros. Conversely, the variation in the amplitude of the function $F_2(z)$ when the point z describes a circuit round l is equal to zero i.e. the difference between the number of zeros and poles is, for this function, equal to zero. But this function has one pole and, consequently, it must also have one zero. Thus in the case under consideration the function $f(z)$ transforms the part of the plane inside the contour l into the part of the plane outside the contour l_1, while the pole is transformed into the point at infinity.

23. The inversion of a power series. We shall now apply Rouche's theorem to the investigation of a function which is the inverse of a power series:

$$w = a_0 + a_1 (z - b) + a_2 (z - b)^2 + \ldots = F(z). \tag{128}$$

To begin with we assume that the coefficient a_1 is not equal to zero, i.e. $F'(b) \neq 0$. For values of z which are close to b we obtain values of w which are close to a_0. We shall show that in the case under consideration a neighbourhood of the point b will be transformed into a one-sheeted neighbourhood of the point a_0 which contains this point. It will then follow that the inverse function to (128) will be single-valued and regular in the neighbourhood of the point a_0 and therefore can be expanded into a Taylor's series in powers of $(w - a_0)$.

The function

$$f(z) = a_1 (z - b) + a_2 (z - b)^2 + \ldots$$

has a simple zero at the point b and it will certainly not vanish in the neighbourhood of this point [18]. Let K be a circle, centre at b, in which the function $f(z)$ is regular and where it has a single zero $z = b$. On the circumference C of this circle $| f(z) |$ does not vanish; a positive uumber m exists such that on the circumference $| f(z) | > m$. Assume further that K_1 is a circle in the w-plane, centre at a_0 and radius ϱ, which is smaller than m. Take a fixed point w_0 in this circle.

We consequently have $|a_0 - w_0| \leqslant \varrho < m$, i.e. on the circumference C of the circle K we have $|a_0 - w_0| < |f(z)|$ since $|f(z)| > m$ on C. In accordance with Rouche's theorem the function

$$a_0 - w_0 + f(z) = a_0 + f(z) - w_0 = F(z) - w_0$$

has the same number of zeros in the circle K as the function $f(z)$, i.e. one zero. In other words, the values of $w = F(z)$ provide a one-sheeted covering of the circle K_1 when z varies in the neighbourhood of the point $z = b$, i.e. the one-sheeted circle K_1 in the w-plane corresponds to non-circular neighbourhood of the point $z = b$ in the z-plane (which contains the point $z = b$). Our assertion has thus been proved, i.e. *if in the series* (128) *the coefficient* $a_1 \neq 0$, *then the neighbourhood of the point* $z = b$ *will be transformed into a one-sheeted neighbourhood of the point* $w = a_0$ *and the inversion of the series* (128), *when the values of* w *are close to* $w = a_0$, *will have the form:*

$$z = b + \sum_{n=1}^{\infty} c_n (w - a_0)^n. \tag{129}$$

We shall now consider the case when the first few coefficients vanish in the series (128):

$$w - a_0 = a_m (z - b)^m + a_{m+1} (z - b)^{m+1} + \\ + a_{m+2} (z - b)^{m+2} + \ldots \qquad (a_m \neq 0), \tag{130}$$

i.e.

$$w - a_0 = a_m (z - b)^m \left[1 + \frac{a_{m+1}}{a_m} (z - b) + \frac{a_{m+2}}{a_m} (z - b)^2 + \ldots\right].$$

This formula can be rewritten in the form:

$$\sqrt[m]{w - a_0} = \sqrt[m]{a_m}\,(z - b) \left\{1 + \left[\frac{a_{m+1}}{a_m} (z - b) + \frac{a_{m+2}}{a_m} (z - b)^2 + \ldots\right]\right\}^{\frac{1}{m}}, \tag{131}$$

where for $\sqrt[m]{a_m}$ we take a definite value of the radical, whilst the expression $\{1 + [\ldots]\}^{1/m}$ has m different values for z close to b, these values being all obtained from any one of them by multiplying by the values of the mth root of unity [I, 175]. The latter equation is equivalent to (130). On the right-hand side the sum in the square brackets is close to zero for values of z close to b, and we can apply Newton's binomial formula to this square bracket [16]:

$$\{1 + [\ldots]\}^{\frac{1}{m}} = 1 + \frac{1}{m} [\ldots] + \frac{\frac{1}{m}\left(\frac{1}{m} - 1\right)}{2!} [\ldots]^2 + \ldots$$

A circle which can be as small as we please, centre at $z = b$, can be found in which the square bracket is a regular function, the modulus of which does not exceed a number q, which is smaller than unity. In this circle the above series converges absolutely and uniformly. The terms of the series in this circle form power series and by using the Weierstrass theorem as applied to power series, we obtain an expansion of the figured bracket on the right-hand side of formula (131) in the form of a power series in the above circle:

$$\{1 + [\ldots]\}^{\frac{1}{m}} = 1 + c_1(z-b) + c_2(z-b)^2 + \ldots,$$

and this formula (131) can be rewritten in the form:

$$\sqrt[m]{w - a_0} = d_1(z-b) + d_2(z-b)^2 + \ldots, \qquad (131_1)$$

where $d_1 = \sqrt[m]{a_m} \neq 0$. When applying Newton's binomial formula, we took a definite value of the radical on the right-hand side of formula (130), and formula (131_1) gives us the same value of the radical on the left-hand side of (131_1). Denote this value of $\sqrt[m]{w-a_0}$ by w':

$$w' = \sqrt[m]{w - a_0} = d_1(z-b) + d_2(z-b)^2 + \ldots \qquad (132)$$

From what was proved above ($d_1 \neq 0$), the one-sheeted neighbourhood of the point $z = b$ will be transformed into the one-sheeted neighbourhood of $w' = 0$ and owing to the fact that $w - a_0 = w'^m$, the one-sheeted neighbourhood of $w' = 0$ will be transformed into the m-sheeted neighbourhood of the point $w = a_0$ [19], if we take all the values of $\sqrt[m]{w-a_0}$ i.e. *in the case* (130) *the one-sheeted neighbourhood of the point* $z = b$ *will be transformed into the* m*-sheeted neighbourhood of the point* $w = a_0$.

Furthermore the derivative of the function (132) does not vanish when $z = b$, and consequently, under transformation by this function, angles at the point $z = b$ remain unaltered [3]. The further transformation $w - a_0 = w'^m$ magnifies angles at the point $w' = 0$ m times since, raising to the mth power, the amplitude of the complex number w' is multiplied by m, i.e. *under transformation by the function* (130), *angles at the point* $z = b$ *are magnified* m *times.*

Finally, in accordance with what has been proved, the inversion of the power series (132) has the form:

$$z = b + \sum_{n=1}^{\infty} e_n w'^n,$$

or, returning to the variable w, we obtain the inversion of the power series in the form:

$$z = b + \sum_{n=1}^{\infty} e_n (\sqrt[m]{w - b})^n. \tag{133}$$

As mentioned above, the formula $w' = \sqrt[m]{w - b}$ transforms the m-sheeted neighbourhood $w = b$ into a one-sheeted neighbourhood $w' = 0$ if we take all the possible values of the radical; in the expansion (133) we must also take all the values of the radical appearing on the right-hand side. Only then do we obtain aone-sheeted neighbourhood of the point $z = b$.

Above we considered the case when the point b and the corresponding point a_0 lie at a finite distance apart. Exactly the same results are obtained when one or both these points lie at infinity. Suppose, for example, that $b = \infty$ and that a_0 is finite. In this case we have the following expansion instead of the expansion (130):

$$w - a_0 = a_m \frac{1}{z^m} + a_{m+1} \frac{1}{z^{m+1}} + \ldots \quad (m > 0\,;\ a_m \neq 0). \tag{134}$$

When $m = 1$, the one-sheeted neighbourhood of $z = \infty$ is transformed into a one-sheeted neighbourhood of $w = a_0$. When $a_0 = \infty$ and b is finite our function has a pole at the point $z = b$. If this pole is simple, i.e. when the expansion has the form:

$$w = \frac{a_{-1}}{z - b} + a_0 + a_1 (z - b) + \ldots, \tag{135}$$

then the one-sheeted neighbourhood of the point$z = b$ is transformed into the one-sheeted neighbourhood of the point $w = \infty$. Finally, when $b = a_0 = \infty$ our function is defined at infinity and has a pole at that point. If this pole is simple then the expansion has the following form:

$$w = az + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots, \tag{136}$$

and the one-sheeted neighbourhood of the point $z = \infty$ is transformed into a one-sheeted neighbourhood of the point $w = \infty$. The function inversa to (136) has an expansion of the same form:

$$z = \frac{1}{a} w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \ldots \tag{137}$$

24. The principle of symmetry. In [18] we described the analytic continuation of the domain B_1 into a new domain B_2 in the event when this new region overlaps the initial domain; but at that time we did not give a practical method for performing the process of analytic continuation in the general case. We shall now describe one

method for performing analytic continuation in a special case, when the new domain does not overlap the initial domain but merely touches it along a contour. However, to begin with, we must prove an auxiliary theorem.

RIEMANN'S THEOREM. *If $f_1(z)$ is regular on one side of an arc of the curve L and also on the curve while $f_2(z)$ has the same property on the other side of the curve, and the values of these functions coincide on the arc L, then these two functions jointly define a single regular function in a domain which contains the given arc or, in other words, $f_2(z)$ is the analytic continuation of $f_1(z)$.*

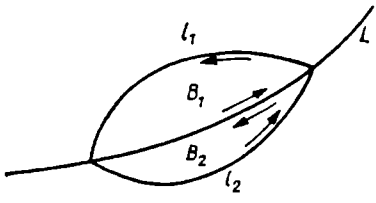

FIG. 21

Draw the contours l_1 and l_2 which have common ends on L; one contour lies in the domain where the function $f_1(z)$ is regular and the other in the domain where the function $f_2(z)$ is regular, so that our functions $f_1(z)$ and $f_2(z)$ are regular in the domains B_1 and B_2, bounded by the closed contours l_1 and L and l_2 and L respectively (Fig. 21). Take a point z in B_1. This point lies outside B_2 and we can therefore write [7]:

$$f_1(z) = \frac{1}{2\pi i} \int_{l_1+L} \frac{f_1(z')}{z'-z}\,dz';$$

$$0 = \frac{1}{2\pi i} \int_{l_2+L} \frac{f_2(z')}{z'-z}\,dz'.$$

If we add these two equations, on the right-hand side we have to integrate twice along the arc L in opposite directions, while the integrands are the same in both cases, since, by hypothesis, the values of $f_1(z')$ and $f_2(z')$ coincide on L. The two integrals thus cancel each other and only integrals along the arcs l_1 and l_2 remain. For the sake of simplicity, denote by $f(z')$ the function which is equal to $f_1(z')$ on the arc l_1 and equal to $f_2(z')$ on the arc l_2. Adding the above equations, we obtain:

$$f_1(z) = \frac{1}{2\pi i} \int_{l_1+l_2} \frac{f(z')}{z'-z}\,dz'.$$

Similarly, by taking the point z in the domain B_2, we obtain:

$$f_2(z) = \frac{1}{2\pi i} \int_{l_1+l_2} \frac{f(z')}{z'-z}\,dz',$$

i.e. our functions $f_1(z)$ and $f_2(z)$ are expressed by the same integral of Cauchy's type round the closed contour (l_1+l_2). Consequently, the first of these functions can be analytically continued from the domain B_1 in the domain B_2, while the second function can be continued from the domain B_2 into the domain B_1 and, as a result of the analytic continuation, a single analytic function is obtained, which proves Riemann's theorem.

Note that we used Cauchy's formula in the above proof; this formula also applies when the function is not regular on the contour but when it is continuous in a closed domain and regular in that domain. We are thus under no obligation to assume that the two given functions $f_1(z)$ and $f_2(z)$ are regular on the arc itself, as is given in the conditions of Riemann's theorem. It is sufficient that $f_1(z)$ is regular on one side of the arc L and that it is continuous as far as the arc; the same applies to $f_2(z)$ on the other side of the arc; also, the values of these functions must coincide on the arc L. Riemann's theorem thus proves that each function can be analytically continued across the arc and that one of these functions is the analytic continuation of the other.

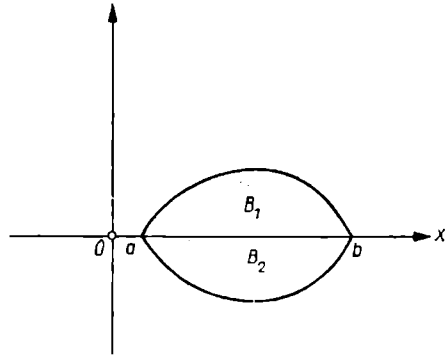

FIG. 22

Let us now consider the principle of symmetry.

THE PRINCIPLE OF SYMMETRY. *If $f_1(z)$ is regular on one side of a segment (a, b) of the real axis and if it is also continuous up to this line, while on the line itself its values are real, then this function can be analytically continued across this line, and at points symmetrical with respect to the real axis, this function has complex conjugate values.*

Assume, for convenience, that our function $f_1(z)$ is regular in a domain B_1, which touches the line (a, b) and lies above it (Fig. 22). Construct the domain B_2 so as to be symmetrical with B_1 with respect to the real axis and proceed to define the function $f_2(z)$ in that domain according to the following rule: assume that at every point A_2 of the domain B_2 the function $f_2(z)$ has a complex value, which is conjugate with the value of the function $f_1(z)$ at the point A_1, symmetrical with respect to the real axis. These symmetrical points evidently have complex conjugate coordinates and denoting, as usual, the complex number which is

conjugate with a by $\bar{a}$, we can define our function $f_2(z)$ in the domain B_2 as follows:

$$f_2(z) = \overline{f_1(\bar{z})}.$$

This newly constructed function will be regular in the domain B_2, for the increments Δz of the independent variable and Δw of the new function are complex conjugates to the analogous magnitudes for the function $f_1(z)$ at the symmetrical point. The same can be said with regard to the ratio of their increments. Consequently, this ratio tends for the function $f_2(z)$ to a definite limit, equal to the complex conjugate to the analogous limit for $f_1(z)$, i.e. it tends to a limit equal to $\overline{f'(z)}$; therefore the function $f_2(z)$ will be regular in the domain B_2. The values of $f_2(z)$ coincide with the values of $f_1(z)$ on the line (a, b), since on this line the values of $f_1(z)$ are real. In accordance with Riemann's theorem, $f_2(z)$ is the analytic continuation of $f_1(z)$ across the line, which proves the principle of symmetry.

The principle of symmetry can be formulated geometrically as follows: if $f_1(z)$ is regular on one side of the segment (a, b) of the real axis and if it transforms this segment into another segment on the real axis, then this function can be analytically continued across this segment; now, points symmetrical with respect to the real axis will be transformed into other points, symmetrical with respect to the real axis. The principle of symmetry can be formulated in a more general way, by introducing the concept of *points symmetrical with respect to a circle*, viz. two points are symmetrical with respect to a circle if these points lie on the same radius of the circle (one point must lie on the radius itself and the other on its continuation) and the product of their distances to the centre of the circle is equal to the square of the radius of the circle (Fig. 23).

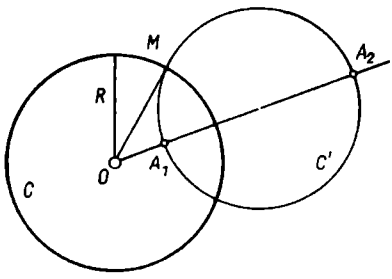

FIG. 23

Let A_1 and A_2 be two points symmetrical with respect to the circle C. Draw a circle C' through these points and let M be one of the points of intersection of this circle and the circle C.

Taking into account the fact that the product of $\overline{OA_2}$ and its outer part $\overline{OA_1}$ must be equal to the square of the tangent and, on the other

hand, that by definition this product must also be equal to the square of the radius $\overline{OM^2}$, we can assert that $\overline{OM}$ is a tangent of the circle C', i.e. the circle C' is orthogonal to the circle C. It is thus easy to see that two points A_1 and A_2, symmetrical with respect to the circle C, have this characteristic property: any circle drawn

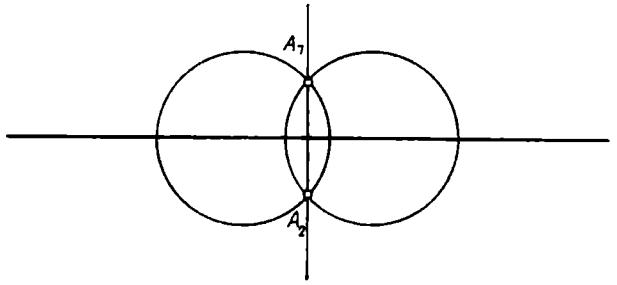

FIG. 24

through these points is orthogonal to C, or in other words, *a family of circles through points, symmetrical with respect to the circle C, consists of circles orthogonal to C*. Two points, symmetrical with respect to a straight line, have the same characteristic property, viz. *a family of circles drawn through these two points consists of circles orthogonal to the straight line* (Fig. 24).

In this general form the principle of symmetry reads as follows: if $f_1(z)$ is regular on one side of an arc (a, b) of the circle C_1, if it is continuous as far as the arc and transforms this arc into an arc of another circle C_2, then $f_1(z)$ can be analytically continued beyond this arc (a, b) and points, symmetrical with respect to the circle C_1, are transformed into points, symmetrical with respect to the circle C_2. In this definition of the principle of symmetry we can understand by the term "circle" a circle in its ordinary sense or a straight line.

The proof of this general definition of the principle of symmetry is given at the beginning of the following chapter.

25. Taylor's series on the circumference of the circle of convergence. Consider Taylor's series

$$\sum_{k=0}^{\infty} a_k (z-b)^k \tag{138}$$

with radius of convergence R. Putting $z - b = \varrho\, e^{i\varphi}$, we can write the series in the form:

$$\sum_{k=0}^{\infty} a_k \varrho^k e^{ik\varphi} \tag{139}$$

or

$$\sum_{k=0}^{\infty} a_k (\cos k\varphi + i \sin k\varphi)\, \varrho^k .$$

It is given that this series converges when $\varrho < R$. When $\varrho = R$, i.e. when we are on the circumference of the circle of convergence, we can say nothing definite about convergence. If we take, for example, the series

$$1 + z + z^2 + \dots \tag{140}$$

with the radius of convergence $R = 1$, then on the circumference of the circle of convergence, i.e. when $|z| = 1$, the moduli of all terms of the series are equal to unity, and the series, evidently, diverges on the entire circumference of the circle of convergence. Consider the following series as the reverse example:

$$1 + \frac{z}{1^2} + \frac{z^2}{2^2} + \dots \tag{141}$$

In this series the ratio of the modulus of one term to the modulus of the preceding term is

$$\left| \frac{z^{n+1}}{(n+1)^2} \right| : \left| \frac{z^n}{n^2} \right| = \left(\frac{n}{n+1} \right)^2 |z| ,$$

and this ratio tends to $|z|$; therefore, in accordance with d'Alambert's test, the radius of convergence of this series is also equal to unity. By substituting $z = e^{i\varphi}$ we obtain a series, the moduli of the terms of which are equal to positive numbers $1/n^2$, forming a convergent series, i.e. the series (141) converges absolutely and uniformly not only inside the circle of convergence but also throughout the closed domain including the circumference. We can thus see that the conditions relevant to the convergence of a power series on the circumference of the circle of convergence can be very varied.

We saw earlier that differentiation and integration of a power series does not alter the circle of convergence. However, these processes can have a definite effect upon the convergence of a series on the

circumference of the circle of convergence. Thus, for example, by integrating the series (140) twice we obtain the series

$$\frac{z^2}{1\cdot 2}+\frac{z^3}{2\cdot 3}+\frac{z^4}{3\cdot 4}+\dots,$$

which, like the series (141), converges absolutely and uniformly throughout the closed circle.

We shall now describe a theorem which deals with the sum of a power series, when the latter converges on the circumference of the the circle of convergence. We proved an analogous theorem earlier for the real variable [I, 149] and we shall therefore not prove this theorem again: below we just formulate the result.

ABEL'S SECOND THEOREM. *If the power series* (138) *converges at a point* $z = b = Re^{i\varphi_0}$ *on the circumference of the circle of convergence, then it will converge uniformly along the whole length of the radius* $\arg(z-b)=\varphi_0$. It follows that the sum of the series is a continuous function along the whole length of the radius, i.e. the value of the sum of the series at a point $Re^{i\varphi_0}$ on the circumference is equal to the limit to which the interior value of the sum of the series tends when approaching the point $Re^{i\varphi_0}$ along the radius from the interior. A simple evaluation of the sums of certain trigonometric series is based on this theorem.

Consider one example. In the expansion

$$\log(1+z)=\frac{z}{1}-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\dots,$$

replace z by $(-z)$ and subtract the series thus obtained from the one above. We thus obtain the expansion

$$\log\frac{1+z}{1-z}=2\left(\frac{z}{1}+\frac{z^3}{3}+\frac{z^5}{5}+\dots\right) \tag{142}$$

in the circle of convergence $|z|<1$. In this expansion, put $z=e^{i\varphi}$ and separate the real and imaginary parts:

$$2\left(\frac{\cos\varphi}{1}+\frac{\cos 3\varphi}{3}+\frac{\cos 5\varphi}{5}+\dots\right)+i2\left(\frac{\sin\varphi}{1}+\frac{\sin 3\varphi}{3}+\frac{\sin 5\varphi}{5}+\dots\right).$$

It can be shown, though we are not going to do so here, that both these trigonometric series converge when φ differs from $k\pi$ ($k=0,\pm1,\pm2,\dots$). Let us determine the sums of these series. Separating the real and imaginary parts of the function, we have:

$$\log\frac{1+z}{1-z}=\log\frac{|1+z|}{|1-z|}+i\arg\frac{1+z}{1-z}.$$

From Fig. 25, when $z = e^{i\varphi}$ we obtain directly:

$$|1 + z| = 2\left|\cos\frac{\varphi}{2}\right| \quad (0 < \varphi < 2\pi),$$

$$|1 - 2| = 2\sin\frac{\varphi}{2}.$$

The amplitude of the fraction $(1 + z)/(1 - z)$ is equal to the angle between the vectors $AM'(-z - 1)$ and $AM(z - 1)$; when $z = 0$ the sum of the series (142) is equal to zero, and in this case the angle must also be equal to zero.

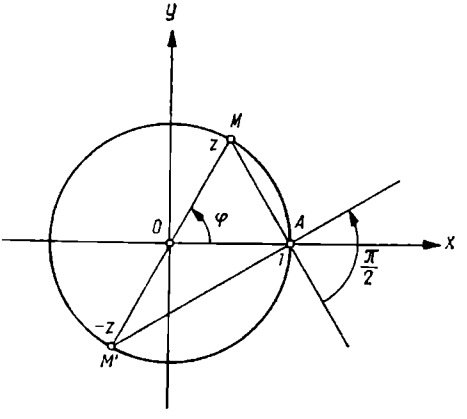

FIG. 25

When the point z coincides with the point $e^{i\varphi}$ this angle rests on the diameter and is evidently equal to $(\pm \pi/2.)$ We have thus found the sum of the above trigonometric series:

$$\log\cot\frac{\varphi}{2} = 2\left(\frac{\cos\varphi}{1} + \frac{\cos 3\varphi}{3} + \dots\right) \quad (0 < \varphi < \pi)$$

$$\frac{\pi}{2} = 2\left(\frac{\sin\varphi}{1} + \frac{\sin 3\varphi}{3} + \dots\right).$$

Note one other circumstance, connected with the representation of the trigonometric series in the form (139). Separate the real and imaginary parts in the coefficient a_k, viz. $a_k = \alpha_k - i\beta_k$. Substituting in formula (139) and separating the real and imaginary parts in the sum, we obtain the following formula:

$$f(z) = \sum_{k=0}^{\infty}(\alpha_k\cos k\varphi + \beta_k\sin k\varphi)\,\varrho^k + \\ + i\sum_{k=0}^{\infty}(-\beta_k\cos k\varphi + \alpha_k\sin k\varphi)\,\varrho^k. \qquad (143)$$

The second trigonometric series differs from the first series only by the fact that the coefficients of $\cos k\varphi$ and $\sin k\varphi$ are interchanged and the coefficient of $\sin k\varphi$ has its sign reversed. The second trigonometric series is usually said to be *conjugate with the first series*. Note that we introduced the minus sign in the coefficients a_k for greater simplicity in subsequent formulae. This is of no fundamental importance, since the real number β_k can be both positive and negative.

26. The principal value of an integral. We shall now consider the limiting values of Cauchy's integrals. To begin with let us introduce a new concept in connection with integrals of discontinuous functions. Let $x = c$ be a point in a finite interval (a, b) and $f(x)$ a function which is defined in this interval. Assume further that the integrals

$$\int_a^{c-\varepsilon} f(x)\,dx \quad \text{and} \quad \int_{c+\varepsilon}^{b} f(x)\,dx \tag{144}$$

exist for any $\varepsilon > 0$. Assume, for example, that $f(x)$ is continuous in the whole interval (a, b) except at the point $x = c$ and that it becomes infinite when x tends to c. The improper integral of $f(x)$ over the interval (a, b) can be defined as follows: if when $\varepsilon \to +0$, the integrals (144) tend to finite limits then the sum of these limits is, in fact, equal to the integral of $f(x)$ in the interval (a, b) [1, 97]. If the integrals taken separately have no limits but the sum of the integrals, when $\varepsilon \to +0$, tends to a finite limit, then this limit

$$\lim_{\varepsilon \to +0} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx \right]$$

is known as the principal value of the integral over the interval (a, b):

$$\text{v. p.} \int_a^b f(x)\,dx = \lim_{\varepsilon \to +0} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx \right]. \tag{145}$$

where v. and p. are the first letters of the French words *valeur principale* which, in English means "principal value".

In future, for the sake of compactness, we shall not write the letters v. p. in front of the integral. The main feature of definition (145) is that the limits of integration on the right-hand side of the formula contain the same number, which tends to $(+0)$.

The principal value of an integral can be defined similarly when $f(x)$ has several discontinuities in the interval. If the usual improper integral of the function $f(x)$ exists over the whole interval (a, b) [I, 97] then the principal value (145) of the integral evidently coincides with this improper integral. It follows from the definition (145) that a common factor can be taken outside the integral and that the integral of a finite number of terms is equal to the sum of the integrals of the individual terms, on the assumption that the integrals of the terms exist only in the principal value sense.

We shall now give some simple examples of the principal value of an integral. Consider the integral:

$$\int_a^b \frac{dt}{(t-x)^p}, \tag{146}$$

where $a < x < b$ and where p is a positive number. When $p > 1$ we have:

$$\int_a^{x-\varepsilon} \frac{dt}{(t-x)^p} + \int_{x+\varepsilon}^b \frac{dt}{(t-x)^p} = -\frac{1}{p-1}\left\{\frac{1}{(b-x)^{p-1}} - \frac{1}{(a-x)^{p-1}} + \right.$$
$$\left. + \left[(-1)^{p-1} - 1\right]\frac{1}{\varepsilon^{p-1}}\right\}.$$

When p is even the last term on the right-hand side is $(-2):\varepsilon^{p-1}$, and the right-hand side increases indefinitely when $\varepsilon \to (+0)$, while the integral (146) does not exist. However, when p is odd the right-hand side of the above formula does not contain ε and we have:

$$\int_a^b \frac{dt}{(t-x)^p} = \frac{1}{1-p}\left[\frac{1}{(b-x)^{p-1}} - \frac{1}{(a-x)^{p-1}}\right] \quad (p \text{ is odd}).$$

When $p = 1$, we obtain:

$$\int_a^{x-\varepsilon} \frac{dt}{t-x} + \int_{x+\varepsilon}^b \frac{dt}{t-x} = \log(x-t)\Big|_{t=a}^{t=x-\varepsilon} + \log(t-x)\Big|_{t=x+\varepsilon}^{b} = \log\frac{b-x}{x-a},$$

i.e.

$$\int_a^b \frac{dt}{t-x} = \log\frac{b-x}{x-a}.$$

We say that the *function* $\omega(x)$ *satisfies a Lipschitz condition of order* α *in the interval* (a, b), *where* $0 < \alpha \leqslant 1$, provided that for arbitrary values of x_1 and x_2 in the interval, the conditions shown below are satisfied:

$$|\omega(x_2) - \omega(x_1)| < k\,|x_2 - x_1|^{\alpha}, \tag{147}$$

where k is a constant. We introduced the same condition earlier, when $\alpha = 1$, and we saw that it is satisfied when $\omega(x)$ has a bounded derivative in the interval [II, 51]. Consider the integral:

$$f(x) = \int_a^b \frac{\omega(t)}{t-x}\,dt, \tag{148}$$

which can be rewritten in the form:

$$\int_a^b \frac{\omega(t)}{t-x}\,dt = \int_a^b \frac{\omega(t)-\omega(x)}{t-x}\,dt + \omega(x)\int_a^b \frac{dt}{t-x}.$$

Using the condition (147) we obtain the following inequality for the integrand of the first integral in the neighbourhood of the point $t = x$:

$$\left| \frac{\omega(t) - \omega(x)}{t - x} \right| \leqslant \frac{k}{|t - x|^{1-\alpha}}, \tag{149}$$

and, consequently, this integral is absolutely convergent in the usual sense [II, 82]. The second integral is equal to:

$$\omega(x) \log \frac{b - x}{x - a}.$$

Hence the integral (148) has a meaning for any x in (a, b) provided that $\omega(t)$ satisfies a Lipschitz condition (147). The function $f(x)$, which is defined by the equation (148), is defined for all values of x in (a, b). Construct the expression:

$$\int_a^{x-\varepsilon} \frac{\omega(t)}{t - x}\,dt + \int_{x+\varepsilon}^{b} \frac{\omega(t)}{t - x}\,dt. \tag{150}$$

When ε is positive the integrand is a continuous function of t and x, provided that x belongs to an arbitrary closed interval within the interval (a, b), and that t belongs to the interval $(a, x - \varepsilon)$ or $(x + \varepsilon, b)$; consequently, the expression (150) is a continuous function of x [II, 80]. Using the identity:

$$\frac{\omega(t)}{t - x} = \frac{\omega(t) - \omega(x)}{t - x} + \omega(x) \frac{1}{t - x}$$

and the condition (147), it is easy to show that when $\varepsilon \to (+0)$, the expression (150) tends uniformly to the limit $f(x)$ with respect to x and, consequently, the function $f(x)$, defined by the formula (148), is a continuous function in any closed interval contained in (a, b); in other words, the function $f(x)$ is a continuous function in the interval (a, b). Later we shall prove the more precise result viz. when $\omega(t)$ satisfies a Lipschitz condition of order $\alpha < 1$ then the function $f(x)$ will also satisfy a Lipschitz condition of the same order α in any interval within (a, b). When $\alpha = 1$ in the condition (147), $f(x)$ satisfies a Lipschitz condition of an order less than unity.

The continuity of the function $\omega(x)$ evidently follows from the condition (147). On the contrary, it does not follow from the continuity of the function that it satisfies a Lipschitz condition, i.e. a Lipschitz condition is a stronger condition than mere continuity. Note, that if the integral (148) is to exist at a point x it is sufficient that $\omega(t)$ satisfies a Lipschitz condition in a neighbourhood of the point x and that in the remaining part of the interval (a, b) it is continuous or merely integrable for the integral (148) exists when the inequality (149) applies to all values of t sufficiently close to x. If every point x in the interval (a, b) belongs to the interval in which a Lipschitz condition (147) is satisfied with a given α and k, then the integral (148) exists for all values of x in (a, b). Here, the constants α and k can be different in different intervals contained in (a, b).

We shall now investigate the possibility of a change of variables in the integral (148). To begin with, let us prove the lemma: *if* $\eta_1(\varepsilon)$ *and* $\eta_2(\varepsilon)$ *are such that the ratios* $\eta_1(\varepsilon):\varepsilon$ *and* $\eta_2(\varepsilon):\varepsilon$ *tend to zero when* $\varepsilon \to (+0)$, *then*

$$\int_a^b \frac{\omega(t)}{t-x}\,dx = \lim_{\varepsilon\to+0}\left[\int_a^{x-\varepsilon+\eta_1(\varepsilon)} \frac{\omega(t)}{t-x}\,dt + \int_{x+\varepsilon+\eta_2(\varepsilon)}^b \frac{\omega(t)}{t-x}\,dt\right].$$

To prove the lemma it is sufficient to prove that

$$\lim_{\varepsilon\to+0}\int_{x-\varepsilon}^{x-\varepsilon+\eta_1(\varepsilon)} \frac{\omega(t)}{t-x}\,dt = 0 \quad \text{and} \quad \lim_{\varepsilon\to+0}\int_{x+\varepsilon}^{x+\varepsilon+\eta_2(\varepsilon)} \frac{\omega(t)}{t-x}\,dt = 0.$$

We shall prove, for example, the first of these equations. Assume that $\eta_1(\varepsilon) > 0$, then $|t-x| > \varepsilon - \eta_1(\varepsilon)$ when $x-\varepsilon < t < x-\varepsilon+\eta_1(\varepsilon)$ and, consequently:

$$\left|\int_{x-\varepsilon}^{x-\varepsilon+\eta_1(\varepsilon)} \frac{\omega(t)}{t-x}\,dt\right| < \frac{m\cdot\eta_1(\varepsilon)}{\varepsilon-\eta_1(\varepsilon)} = \frac{m}{1-\dfrac{\eta_1(\varepsilon)}{\varepsilon}}\cdot\frac{\eta_1(\varepsilon)}{\varepsilon} \to 0,$$

where m is the maximum value of $|\omega(t)|$. When $\eta_1(\varepsilon) < 0$, we can write:

$$\left|\int_{x-\varepsilon}^{x-\varepsilon+\eta_1(\varepsilon)} \frac{\omega(t)}{t-x}\,dt\right| < \frac{m\cdot|\eta_1(\varepsilon)|}{\varepsilon} \to 0,$$

and the lemma is thus proved.

By using this lemma it is easy to prove the formula for the change of variables in the integral (148).

THEOREM. *Let* $t = \mu(\tau)$ *be a monotonically increasing function of* τ, *varying in the interval* (a, b) *when* $\alpha < \tau < \beta$, *while* $\mu(\tau)$ *has continuous derivatives up to the second order in the interval* (α, β) *and* $\mu'(\tau) \neq 0$ *in* (α, β). *In this case the formula for the change of variables applies:*

$$\int_a^b \frac{\omega(t)}{t-x}\,dt = \int_\alpha^\beta \frac{\omega[\mu(\tau)]\,\mu'(\tau)}{\mu(\tau)-\mu(\xi)}\,d\tau, \tag{151}$$

where $x = \mu(\xi)$ *and the integral on the right-hand side is to be understood in the principal value sense.*

In accordance with the definition of the principal value of an integral, we form the sum:

$$\int_\alpha^{\xi-\varepsilon} \frac{\omega[\mu(\tau)]\,\mu'(\tau)}{\mu(\tau)-\mu(\xi)}\,d\varepsilon + \int_{\xi+\varepsilon}^{b} \frac{\omega[\mu(\tau)]\,\mu'(\tau)}{\mu(\tau)-\mu(\xi)}\,d\tau. \tag{152}$$

Denote: $\mu(\xi-\varepsilon) = x-\varepsilon'$ and $\mu(\xi+\varepsilon) = x+\varepsilon'+\eta$. According to Taylor's formula:

$$\mu(\xi+h) = \mu(\xi) + h\mu'(\xi) + \frac{h^2}{2}\mu''(\xi+\theta h) \quad (0<\theta<1).$$

Putting $h = -\varepsilon$ and later, $h = +\varepsilon$, we obtain:

$$x - \varepsilon' = x - \varepsilon\mu'(\xi) + \frac{\varepsilon^2}{2}\mu''(\xi - \theta_1\varepsilon);$$

$$x + \varepsilon' + \eta = x + \varepsilon\mu'(\xi) + \frac{\varepsilon^2}{2}\mu''(\xi + \theta_2\varepsilon) \quad (0 < \theta_1 \text{ and } \theta_2 < 1),$$

whence:

$$\varepsilon' = \varepsilon\left[\mu'(\xi) - \frac{\varepsilon}{2}\mu''(\xi - \theta_1\varepsilon)\right]; \qquad \eta = \frac{\varepsilon^2}{2}\left[\mu''(\xi + \theta_2\varepsilon) + \mu''(\xi - \theta_1\varepsilon)\right],$$

and, consequently, the ratio $\eta : \varepsilon'$ tends to zero when $\varepsilon' \to 0$. Transforming the integrals in the sum (152) to the variable t, we can write this sum as follows:

$$\int_a^{x-\varepsilon'} \frac{\omega(t)}{t-x}\,dt + \int_{x+\varepsilon'+\eta}^{b} \frac{\omega(t)}{t-x}\,dt,$$

and by using the lemma proved above we can assert that the sum (152) gives in the limit the integral on the left-hand side of (151), which proves the above formula. In the conditions of the theorem the "monotonically increasing function $\mu(\tau)$" can evidently be changed to "monotonically decreasing".

27. The principal value of an integral (continuation). The concept of the principal value of an integral can also be applied to a line integral. We shall only consider integrals of Cauchy's type:

$$f(\xi) = \int_L \frac{\omega(\tau)}{\tau - \xi}\,d\tau, \tag{153}$$

where L is a closed or open contour in the plane of the complex variable τ and ξ is a point on that contour, which does not coincide with its end when the contour L is open. Let s be the length of the arc L, measured from a certain point. In future we shall assume that the functions $x(s)$ and $y(s)$ in the parametric equation of the contour $\tau(s) = x(s) + y(s)i$ have continuous derivatives up to the second order. Assume that the point $\tau = \xi$ corresponds to the point $s = s_0$. The principal value of the integral (153) can be defined as the principal value of the integral with respect to the real variable s:

$$\int_0^l \frac{\omega[\tau(s)]}{\tau(s) - \tau(s_0)}\,\tau'(s)\,ds, \tag{154}$$

where l is the length of the contour L, and we can assume that s_0 lies in the interval of integration. Precisely as in [26], it can be shown that the integral (153) exists if the function $\omega(\tau)$ satisfies a Lipschitz condition on L:

$$|\omega(\tau_2) - \omega(\tau_1)| < k|\tau_2 - \tau_1|^{\alpha} \qquad (0 < \alpha \leqslant 1). \tag{155}$$

Using the theorem for the change of variables proved in [26], it is easy to show that if, in the parametric equation of the contour $\tau(t) = x(t) + y(t)i$, the functions have continuous derivatives up to the second order and $\tau'(t) \neq 0$, then the principal value of the integral (153) reduces to the principal value of the integral:

$$\int_{\alpha}^{\beta} \frac{\omega[\tau(t)]}{\tau(t) - \tau(t_0)} \tau'(t)\, dt,$$

where (α, β) is the interval of variation of the parameter t and $t = t_0$ corresponds to the point $\tau = \xi$. If $\omega(\tau)$ is identically unity then we have the primitive $\log(\tau - \tau_0)$ for the integral (153) and we obtain the following result when the contour is closed:

$$\int_{L} \frac{d\tau}{\tau - \xi} = \pi i. \tag{156}$$

We must always remember that we integrate round a closed contour in the counter-clockwise direction. Similarly, as in the case of a straight line, we can assert that, when the condition (155) applies, then formula (153) defines a function $f(\xi)$ which is continuous at all interior points of L, when the curve is open, or at all points of L, when the curve is closed. In this case, as in the case of the straight line, the more exact theorem, proved by I. I. Privalov (*Dokl. Akad. Nauk, S.S.S.R.*, XXIII, No. 9, 1939) applies:

If the condition (155) *applies then the function* $f(\xi)$ *satisfies on a closed contour* L *a Lipschitz condition for the same* a *when* $a < 1$, *or of any order, less than unity, when* $a = 1$. *If* L *is an open curve then the same condition applies for* $f(\xi)$ *on any closed arc of the curve within* L.

We shall prove this theorem for a section of a straight line. The proof is analogous for contour integrals. As a preliminary let us make some remarks with regard to a Lipschitz condition. It is easy to see that a Lipschitz condition:

$$|f(\xi + \Delta\xi) - f(\xi)| \leqslant k\,|\Delta\xi|^{a} \tag{157}$$

can be tested for sufficiently small values of $|\Delta\xi|$. In fact, let (157) be proved for $|\Delta\xi| < m$, where m is a positive constant. When $|\Delta\xi| \geqslant m$ then the relationship:

$$\frac{|f(\xi + \Delta\xi) - f(\xi)|}{|\Delta\xi|^{a}}$$

remains bounded, i.e.

$$|f(\xi + \Delta\xi) - f(\xi)| \leqslant k_1\,|\Delta\xi|^{a} \qquad (|\Delta\xi| \geqslant m),$$

where k_1 is a constant. Choosing the greater of the two constants k and k_1 we obtain a Lipschitz condition for all permissible values of $\Delta\xi$. Suppose also that $\beta < a \leqslant 1$. For values of $\Delta\xi$, with modulus less than unity we have $|\Delta\xi|^{\beta} > |\Delta\xi|^{a}$, and therefore, if $f(\xi)$ satisfies a Lipschitz condition of order a, then it will also satisfy this condition of order β. Suppose that the two functions $f_1(\xi)$ and $f_2(\xi)$ satisfy a Lipschitz condition of the same order a. It is easy to see that their sum and their product also satisfy this condition of

the same order. In the case of the sum this is directly due to the fact that the modulus of a sum is less than or equal to the sum of the moduli, and in the case of the product we can write:

$$f_1(\xi+\Delta\xi)\,f_2(\xi+\Delta\xi)-f_1(\xi)\,f_2(\xi)=f_2(\xi+\Delta\xi)\left[f_1(\xi+\Delta\xi)-f_1(\xi)\right]+ \\ +f_1(\xi\left[f_2(\xi+\Delta\xi)-f_2(\xi)\right],$$

from which our remark about the product follows directly.

We shall now prove the above theorem. We have:

$$f(\xi)=\int_a^b \frac{\omega(t)}{t-\xi}\,\mathrm{d}t,$$

or

$$f(\xi)=\int_a^b \frac{\omega(t)-\omega(\xi)}{t-\xi}\,\mathrm{d}t+\omega(\xi)\log\frac{b-\xi}{\xi-a},$$

where $\omega(t)$ satisfies a Lipschitz condition of order α. Assume that ξ belongs to an interval l, in (a, b). In the second term on the right-hand side the factor $\omega(\xi)$ satisfies a Lipschitz condition of order α and the second factor has a bounded derivative which satisfies the same Lipschitz condition of the first order. Thus the whole product satisfies a Lipschitz condition of order α and it is sufficient to prove the theorem for the function:

$$\psi(\xi)=\int_a^b \frac{\omega(t)-\omega(\xi)}{t-\xi}\,\mathrm{d}t,$$

expressed by the usual bounded integral. We have to find the upper bound of the modulus of the difference:

$$\psi(\xi+\Delta\xi)-\psi(\xi)=\int_a^b\left[\frac{\omega(t)-\omega(\xi+\Delta\xi)}{t-\xi-\Delta\xi}-\frac{\omega(t)-\omega(\xi)}{t-\xi}\right]\mathrm{d}t, \qquad (158)$$

where $|\Delta\xi|$ is sufficiently small. On separating the interval $(\xi-\varepsilon, \xi+\varepsilon)$ from the interval of integration, where $\varepsilon=2|\Delta\xi|$, we take the upper bound of the integral (158) in this part of the interval. By using the condition (155) we obtain:

$$k\int_{\xi-\varepsilon}^{\xi+\varepsilon}\left(|t-\xi-\Delta\xi|^{\alpha-1}+|t-\xi|^{\alpha-1}\right)\mathrm{d}t.$$

The integral of the second term can be represented in the form:

$$\int_{\xi-\varepsilon}^{\xi}(\xi-t)^{\alpha-1}\,\mathrm{d}t+\int_{\xi}^{\xi+\varepsilon}(t-\xi)^{\alpha-1}\,\mathrm{d}t=\frac{1}{\alpha}\left(2^{\alpha}|\Delta\xi|^{\alpha}+2^{\alpha}|\Delta\xi|^{\alpha}\right).$$

We can proceed similarly with the integral of the first term and the modulus of the integral (158) in the interval $(\xi - \varepsilon, \xi + \varepsilon)$ has an upper bound: $k_1 \mid \Delta\xi \mid a$, where k_1 is a constant. Upper bounds of the integrals in the added intervals $(a, \xi - \varepsilon)$ and $(\xi + \varepsilon, b)$, remain to be found. For this purpose we shall represent the integrand in the form:

$$\left[\omega(t) - \omega(\xi + \Delta\xi)\right] \frac{\Delta\xi}{(t-\xi)(t-\xi-\Delta\xi)} - \left[\omega(\xi + \Delta\xi) - \omega(\xi)\right] \frac{1}{t-\xi}. \qquad (159)$$

Using (155) we obtain the following inequality for the modulus of the integral of the second term:

$$k \mid \Delta\xi \mid^a \left| \int_a^{\xi-\varepsilon} \frac{dt}{t-\xi} + \int_{\xi+\varepsilon}^{b} \frac{dt}{t-\xi} \right| = k \left| \log \frac{b-\xi}{\xi-a} \right| \mid \Delta\xi \mid^a < k_2 \mid \Delta\xi \mid^a,$$

where k_2 is a constant. We recall that the modulus of the above logarithm remains finite during variations of ξ in the interval l. The upper bound of the integral of the first term in the expression (159) remains to be found in the added intervals $(a, \xi - \varepsilon)$ and $(\xi + \varepsilon, b)$. Let us find the upper bound of the integral in the first interval. The inequality in the second interval will be exactly the same. From (155) we have the following inequality for the first term of the expression (159):

$$\left| \left[\omega(t) - \omega(\xi + \Delta\xi)\right] \frac{\Delta\xi}{(t-\xi)(t-\xi-\Delta\xi)} \right| <$$

$$< k \frac{\mid \Delta\xi \mid}{\mid t-\xi \mid \mid t-\xi-\Delta\xi \mid^{1-a}} = \frac{k \mid \Delta\xi \mid}{\mid t-\xi \mid^{2-a} \left| 1 - \dfrac{\Delta\xi}{t-\xi} \right|^{1-a}},$$

When t varies in the interval $(a, \xi - \varepsilon)$ then $(\xi - t) > \varepsilon$, i.e. $(\xi - t) > 2 \mid \Delta\xi \mid$, and, consequently $\mid \Delta\xi \mid : \mid t - \xi \mid < 1/2$; hence:

$$\left| 1 - \frac{\Delta\xi}{t-\xi} \right| > \frac{1}{2}.$$

Thus during the variation of t in the first interval $(a, \xi - \varepsilon)$ the modulus of the first term of the expression (159) does not exceed

$$\frac{2^{1-a} k \mid \Delta\xi \mid}{(\xi - t)^{2-a}} \qquad (\xi - t > 0),$$

and the modulus of the integral of the above first term can be written as follows:

$$2^{1-a} k \mid \Delta\xi \mid \int_a^{\xi-\varepsilon} \frac{dt}{(\xi-t)^{2-a}}. \qquad (159_1)$$

When $a < 1$, then the inequality is as follows:

$$\frac{2^{1-a} k}{1-a} \mid \Delta\xi \mid \left[- \frac{1}{(\xi-a)^{1-a}} + \frac{1}{2^{1-a} \mid \Delta\xi \mid^{1-a}} \right] < \frac{k}{1-a} \mid \Delta\xi \mid^a.$$

We thus obtain the required upper bound for the difference (158) when $\alpha < 1$. When $\alpha = 1$ the upper bound (159_1) takes the following form:

$$k\,|\,\Delta\xi\,|\,[\log(\xi - a) - \log(2\,|\,\Delta\xi\,|)],$$

the difference (158) when $\alpha = 1$ can be written as follows:

$$k_3\,|\,\Delta\xi\,| + k_4\,|\,\Delta\xi\,|\log\frac{1}{|\,\Delta\xi\,|},$$

where k_3 and k_4 are constants. Bearing in mind that as $\Delta\xi \to 0$, $\log(1|\Delta\xi|)$, tends to infinity more slowly than any negative power of $|\,\Delta\xi\,|$ we can write:

$$k_3\,|\,\Delta\xi\,| + k_4\,|\,\Delta\xi\,|\log\frac{1}{|\,\Delta\xi\,|} \leqslant k_5\,|\,\Delta\xi\,|^{\beta},$$

where β is any number which satisfies the condition $0 < \beta < 1$; the theorem has thus been proved for the case when $\alpha = 1$.

We shall now investigate the behaviour of the function $f(\xi)$ when the point ξ aproaches the ends of the line, for example, when it approaches the end $t = a$. We are assuming, as we did above, that $\omega(t)$ satisfies a Lipschitz condition of order α on the whole closed section (a, b). To start with we assume that $\omega(a) = 0$. In doing so we can extend this to saying that the function is zero when $t < a$, i.e. we can assume that $\omega(t) = 0$ when $t < a$. In this case $\omega(t)$ is defined on a section (a_1, b) where $a_1 < a$ and the Lipschitz condition is not affected by the above extension. The integral

$$\int_{a_1}^{b}\frac{\omega(t)}{t-\xi}\,\mathrm{d}t = \int_{a}^{b}\frac{\omega(t)}{t-\xi}\,\mathrm{d}t$$

gives the former function $f(\xi)$, and bearing in mind that the point $t = a$ lies within the line (a_1, b) we can maintain on the strength of what was proved above, that $f(\xi)$ satisfies a Lipschitz condition of order α (we assume that $\alpha < 1$) and on any line (a, b_1) where $b_1 < b$. Suppose now that $\omega(a) \neq 0$.

We can write:

$$f(\xi) = \int_{a}^{b}\frac{\omega(t) - \omega(a)}{t-\xi}\,\mathrm{d}t + \omega(a)\int_{a}^{b}\frac{\mathrm{d}t}{t-\xi}.$$

The numerator of the first integrand vanishes when $t = a$ and this integral gives a function which satisfies a Lipschitz condition of order α up to the point $\xi = a$. As we saw in [26] the second term on the right-hand side is equal to:

$$\omega(a)\log(b - \xi) - \omega(a)\log(\xi - a).$$

The minuend in this difference satisfies a Lipschitz condition of order one up to $\xi = a$.

Thus in the neighbourhood of the point $\xi = a$ the function $f(\xi)$ represents a sum:

$$-\omega(a)\log(\xi - a) + f_1(\xi),$$

where $f_1(\xi)$ satisfies a Lipschitz condition of order a up to the point $\xi = a$. When the end $\xi = b$ is considered, we obtain the analogous result:

$$\omega(b) \log(b - \xi) + f_2(\xi),$$

where $f_2(\xi)$ satisfies a Lipschitz condition up to the point $\xi = b$.

The behaviour of the function $f(\xi)$ near the ends of the line was also considered when more general assumptions were made with regard to $\omega(t)$. We only quote the result here, but the proof can be found in Muskhelishvili's book *Singular Integral Equations*, which contains the results of the first investigations into integrals of Cauchy's type.

THEOREM. Let $\omega(t)$ satisfy a Lipschitz condition (147) of order a on any closed line (a', b') within (a, b); the constant k depends on the choice of the line (a', b') (k grows indefinitely when $a' \to a$ or when $b' \to b$). Assume further that near the ends of the line, the function $\omega(t)$ can be represented in the form:

$$\omega(t) = \frac{\omega^*|t|}{(t-c)^\gamma}, \tag{160}$$

where c denotes either a or b, $\gamma = \gamma_1 + \gamma_2 i$ ($\gamma \neq 0$) when $0 \leqslant \gamma_1 < 1$, and $\omega^*(t)$ satisfies a Lipschitz condition up to $t = c$. At the same time $f(\xi)$ satisfies a Lipschitz condition of order a if $a < 1$, or of any order less than unity, when $a = 1$, on any closed line within (a, b); in the neighbourhood of $\xi = c$ it has the form:

$$f(\xi) = \pm \pi \cot \gamma\pi \frac{\omega^*(c)}{(\xi - c)^\gamma} + f_1(\xi),$$

and when $\gamma_1 = 0$ then $f_1(\xi)$ satisfies a Lipschitz condition up to $\xi = c$; if $\gamma_1 \neq 0$ then

$$f_1(\xi) = \frac{f^*(\xi)}{|\xi - c|^{\gamma_0}},$$

where $f^{(*)}(\xi)$ satisfies a Lipschitz condition up to $\xi = c$, and $\gamma_0 < \gamma$; the sign $(+)$ refers to the case when $c = a$ and the sign $(-)$ to the case when $c = b$. All this applies when the line is replaced by a sufficiently smooth arc with ends $t = a$ and $t = b$, and when we integrate with respect to the complex variable t.

Note that when $\gamma = 0$ the result shown above applies:

$$f(\xi) = \pm \omega(c) \log \frac{1}{\xi - c} + f_1(\xi),$$

where $f(\xi)$ satisfies a Lipschitz condition up to $\xi = c$.

28. Integrals of Cauchy's type. Consider an integral of Cauchy's type (8):

$$F(z) = \frac{1}{2\pi i} \int_L \frac{w(\tau)}{\tau - z} \, d\tau, \tag{161}$$

where z does not lie on L. If L is a closed contour, then this integral defines two different regular functions: one within L and the other outside L. If the

contour is not closed then $F(z)$ is regular outside L. In either case $F(\infty) = 0$. When $z = \xi$ lies on the contour we have the principal value of the integral and we can rewrite it in the form:

$$\frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau-\xi}\,d\tau = \frac{\omega(\xi)}{2\pi i}\int_L \frac{d\tau}{\tau-\xi} + \frac{1}{2\pi i}\int_L \frac{\omega(\tau)-\omega(\xi)}{\tau-\xi}\,d\tau,$$

i.e. from (156)

$$\frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau-\xi}\,d\tau = \frac{1}{2}\omega(\xi) + \frac{1}{2\pi i}\int_L \frac{\omega(\tau)-w(\xi)}{\tau-\xi}\,d\tau. \tag{162}$$

To begin with we assume the contour to be closed. We shall prove the theorem: *if z tends towards a point ξ on L then the integral* (161) *has the limit:*

$$\pm\frac{1}{2}\omega(\xi) + \frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau-\xi}\,d\tau, \tag{163}$$

where the $(+)$ *sign is taken when* $z \to \xi$ *from inside* L *and the* $(-)$ *sign when* $z \to \xi$ *from outside* L. Consider the first case. The integral (161) can be rewritten in the form:

$$\frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau-z}\,d\tau = \frac{\omega(\xi)}{2\pi i}\int_L \frac{d\tau}{\tau-z} + \frac{1}{2\pi i}\int_L \frac{\omega(\tau)-\omega(\xi)}{\tau-z}\,d\tau$$

or

$$\frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau-z}\,d\tau = \omega(\xi) + \frac{1}{2\pi i}\int_L \frac{\omega(\tau)-\omega(\xi)}{\tau-z}\,d\tau. \tag{164}$$

Consider the difference:

$$\frac{1}{2\pi i}\int_L \frac{\omega(\tau)-\omega(\xi)}{\tau-z}\,d\tau - \frac{1}{2\pi i}\int_L \frac{\omega(\tau)-\omega(\xi)}{\tau-\xi}\,d\tau =$$

$$= \frac{1}{2\pi i}\int_L \frac{\omega(\tau)-\omega(\xi)}{\tau-\xi}\cdot\frac{z-\xi}{\tau-z}\,d\tau. \tag{165}$$

On either side of the point ξ mark small arcs η. Denote the part of the contour thus formed by L_1 and the remaining part by L_2. Denoting the difference (165) by a single letter Δ we can write:

$$\Delta = \frac{1}{2\pi i}\int_{L_1} \frac{\omega(t)-\omega(\xi)}{\tau-\xi}\cdot\frac{z-\xi}{\tau-z}\,d\tau + \frac{1}{2\pi i}\int_{L_2} \frac{\omega(\tau)-\omega(\xi)}{\tau-\xi}\cdot\frac{z-\xi}{\tau-z}\,d\tau. \tag{166}$$

Suppose that z tends to ξ along the normal to the contour L. In this case the distance from z to ξ is less than the distance from z to other points on the contour, i.e. $|z-\xi| < |\tau-z|$. Also $d\tau = |x'(s)+y'(s)\,i|\,ds$, where

$| x'(s) + y'(s)\, i | = 1$. By finding in the usual way an upper bound of the first integral in the formula (166) we obtain:

$$\left| \frac{1}{2\pi i} \int_{L_1} \frac{\omega(\tau) - \omega(\xi)}{\tau - \xi} \cdot \frac{z - \xi}{\tau - z} \, d\tau \right| < \frac{k}{2\pi} \int_{s_0 - \eta}^{s_0 + \eta} \frac{ds}{| \tau(s) - \tau(s_0) |^{1-a}},$$

where $s = s_0$ corresponds to the point $\tau = \xi$. Bearing in mind that the ratio of the length of the chord $| \tau(s) - \tau(s_0) |$ to the length of the arc $| s - s_0 |$ tends to unity we can assert that the latter integral converges and consequently, for any given positive ε we can choose η so small that the modulus of the integral along L_1 is smaller than $\varepsilon/2$. Having thus fixed η we obtain the usual integral along L_2 in which $| \tau - \xi |$ and $| \tau - z |$ remain greater than a given positive number and therefore the modulus of the integral along L_2 will be smaller than $\varepsilon/2$ for all z's which are sufficiently close to ξ. Owing to the lack of restriction on ε we can maintain that the difference (166) tends to zero when $z \to \xi$ along the normal, i.e.

$$\lim_{z \to \xi} \frac{1}{2\pi i} \int_L \frac{\omega(\tau) - \omega(\xi)}{\tau - z} \, d\tau = \frac{1}{2\pi i} \int_L \frac{\omega(\tau) - \omega(\xi)}{\tau - \xi} \, d\tau$$

or, from (156)

$$\lim_{z \to \xi} \frac{1}{2\pi i} \int_L \frac{\omega(\tau) - \omega(\xi)}{\tau - z} \, d\tau = \frac{1}{2\pi i} \int \frac{\omega(\tau)}{\tau - \xi} \, d\tau - \frac{1}{2} \omega(\xi),$$

and formula (164) gives the required result:

$$\lim_{z \to \xi} \frac{1}{2\pi i} \int_L \frac{\omega(\tau)}{\tau - z} \, d\tau = \frac{1}{2} \omega(\xi) + 2\pi i \int_L \frac{\omega(\tau)}{\tau - \xi} \, d\tau. \tag{167}$$

In the case when the point tends towards the contour from the outside the proof is exactly the same but we must remember that:

$$\frac{1}{2\pi i} \int \frac{d\tau}{\tau - z} = \begin{cases} 1, & \text{when } z \text{ lies inside } L \\ 0, & \text{when } z \text{ lies outside } L. \end{cases} \tag{168}$$

Until now we assumed that $z \to \xi$ along the normal. It can be shown that formula (167) still applies when z tends to ξ in an arbitrary way. In order to do so it is sufficient to show that the limiting tendency of the point towards the contour is along the normal; the integral (161) tends uniformly to the limit (163) for all values of ξ on the contour L. We shall only consider the circle $| z | = 1$. To begin with we suppose that $z \to \xi$ along the normal. In this case $\tau = e^{i\varphi}$; $\xi = e^{i\varphi_0}$ and $ds = d\varphi$. It is easy to show that when $0 < x < \pi/2$ then $\sin x > 2x/\pi$. Using this we can write:

$$| \tau - \xi | = 2 \sin \frac{| \varphi - \varphi_0 |}{2} > \frac{2}{\pi} | \varphi - \varphi_0 | \qquad (| \varphi - \varphi_0 | < \pi),$$

and the modulus of the integral along L will be smaller:

$$\frac{k}{2\pi}\int_{\varphi_0-\eta}^{\varphi_0+\eta}\frac{\pi^{1-a}\,d\varphi}{2^{1-a}\,|\,\varphi-\varphi_0\,|^{1-a}}=\frac{k}{2^{1-a}\,\pi^a}\int_{\varphi_0}^{\varphi_0+\eta}\frac{d\varphi}{(\varphi-\varphi_0)^{1-a}}=\frac{k}{2^{1-a}\,\pi^a\,a}\,\eta^a.$$

On the contour L_2, provided z lies sufficiently close to ξ we have::

$$|\,\tau-\xi\,|>\frac{1}{2}\sin\eta\,;\quad |\,\tau-z\,|>\frac{1}{2}\sin\eta\,;\quad |\,\omega(\tau)-\omega(\xi)\,|\leqslant 2M\,,$$

where M is the maximum value of $|\,\omega(\tau)\,|$ on L. Assuming that $\delta=|\,z-\xi\,|$ we obtain

$$\left|\frac{1}{2\pi i}\int_{L_1}\frac{\omega(\tau)-\omega(\xi)}{\tau-\xi}\cdot\frac{z-\xi}{\tau-z}\,d\tau\right|<\frac{1}{2\pi}\,\frac{8M\delta}{\sin^2\eta}\,(2\pi-2\eta)<\frac{8M\delta}{\sin^2\eta}\,,$$

and finally:

$$|\,\Delta\,|<\frac{k}{2^{1-a}\,\pi^a\,a}\,\eta^a+\frac{8M}{\sin^2\eta}\,\delta.$$

To begin with η is so chosen that the first term is less than $\varepsilon/2$; as a result of this fixed η the second term will be less than $\varepsilon/2$ provided $\delta>(\varepsilon\sin^2\eta)$: $(16M)$. These inequalities do not contain ξ and therefore the difference (166) tends to zero uniformly with respect to ξ as z tends towards the circumference along the radius. Consequently this tending to a limit in formula (167) also takes place uniformly with respect to ξ. One result of this is that the right-hand side of formula (167) and the integral (161) represent a continuous function of ξ [1, 145]. In [26] we mentioned the fact that this function satisfies a Lipschitz condition.

Denote the right-hand side of formula (167) by $\omega_1(\xi)$ and assume that $z\to\xi$ in any manner. Let ξ' be a variable point on the circumference which lies on the same radius as z. Evidently $\xi'\to\xi$ and $|\,z-\xi'\,|\to 0$. Using the fact proved above that the tending to a limit in (167) proceeds uniformly when z tends towards ξ along the radius we can maintain that for any given positive ε we have for all z's sufficiently close to ξ:

$$\left|\frac{1}{2\pi i}\int_{L}\frac{\omega(\tau)}{\tau-z}\,d\tau-\omega_1(\xi')\right|<\frac{\varepsilon}{2}\,.$$

On the other hand, as a result of the continuity of $\omega_1(\xi)$, the modulus $|\,\omega_1(\xi)-\omega_1(\xi')\,|<\varepsilon/2$ holds for all z's sufficiently close to ξ and therefore:

$$\left|\frac{1}{2\pi i}\int_{L}\frac{\omega(\tau)}{\tau-z}\,d\tau-\omega_1(\xi)\right|<\varepsilon$$

for all z's sufficiently close to ξ. Owing to the lack of restriction on ξ this shows that the tend to a limit in formula (167) takes place when z tends towards ξ from the inside uniformly and in any manner with respect to ξ. In other words we can maintain that the function $F(z)$ which is defined in the circle by the

integral (161) is continuous up to the circumference and its limiting values on the circumference are given by the formula (167). The same condition applies when the limit is approached from outside.

This property of integrals of Cauchy's type can be proved for any closed contour L when the assumptions given in [27] with regard to $x(s)$ and $y(s)$ are made. We can also assume that L has a finite number of angular points. Let M be an angular point on L and suppose that when describing a circuit round L in the counter-clockwise direction, the direction of the tangent at M revolves through an angle $\pi\theta$, where $-1 < \theta < +1$. From this it is easy to see that on the right-hand side of formula (156) we have $(1 - \theta)\,\pi i$ instead of πi, and that the expression (163) at the point M should be replaced by the following expression (I. I. Privaloff, *Dokl. Akad. Nauk, SSSR.*, XXIII, No. 9, 1939)

$$\pm \frac{1 \pm \theta}{2}\,\omega(\xi) + \frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau - \xi}\,d\tau,$$

where the positive or negative signs should be taken simultaneously.

If we denote by $F_i(\xi)$ and $F_e(\xi)$ the limiting values of the function (161) on the contour, defined within and outside L, then the theorem which we proved above can be rewritten in the form:

$$F_i(\xi) = \frac{1}{2}\,\omega(\xi) + \frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau - \xi}\,d\tau,$$

$$F_e(\xi) = -\frac{1}{2}\,\omega(\xi) + \frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau - \xi}\,d\tau. \tag{169}$$

This theorem can be proved similarly for an open contour. Let us consider a finite interval (a, b) of the real axis:

$$F(z) = \frac{1}{2\pi i}\int_a^b \frac{\omega(t)}{t - z}\,dt. \tag{170}$$

If $\omega(t)$ is identically unity then, instead of formula (168), we can write:

$$\frac{1}{2\pi i}\int_a^b \frac{dt}{t - z} = \frac{1}{2\pi i}\log\frac{b - z}{a - z}, \tag{171}$$

where we must take those values of the logarithm which vanish when $z = \infty$. If ξ lies in the interval (a, b), then instead of formula (156) we have:

$$\frac{1}{2\pi i}\int_a^b \frac{dt}{t - \xi} = \frac{1}{2\pi i}\log\frac{b - \xi}{\xi - a},$$

where real values of the logarithm are taken. Repeating the previous arguments word for word we obtain:

$$\lim_{z\to\xi}\frac{1}{2\pi i}\int_a^b\frac{\omega(t)}{t-z}\,dt=\frac{\omega(\xi)}{2\pi i}\left[\log\frac{b-z}{a-z}\Big|_{z\to\xi}-\log\frac{b-\xi}{\xi-a}\right]+\frac{1}{2\pi i}\int_a^b\frac{\omega(t)}{t-\xi}\,dt.$$

The function (171) has different limits when z tends to ξ from above or below (a, b), viz.,

$$\log\frac{b-z}{z-a}\Big|_{z\to\xi}=\log\frac{b-\xi}{\xi-a}\pm\pi i,$$

where the positive sign refers to the case when z tends to ξ from above, i.e. to values with a positive imaginary part, and the negative sign to the case when z tends to ξ from below. When integrating from a to b the upper half-plane lies to the left and therefore, the tendency of z to ξ from above is analogous with the tendency of z to ξ from inside a closed curve. Similarly, the tendency from below is analogous with the tendency from outside a closed curve. Denoting by $F_i(\xi)$ and $F_e(\xi)$ the limiting values of the function (170) when the tendency of z to ξ is from above or below, we obtain formulae which are analogous with the formulae (169):

$$\begin{aligned}F_i(\xi)&=\frac{1}{2}\omega(\xi)+\frac{1}{2\pi i}\int_a^b\frac{\omega(t)}{t-\xi}\,dt,\\ F_e(\xi)&=-\frac{1}{2}\omega(\xi)+\frac{1}{2\pi i}\int_a^b\frac{\omega(t)}{t-\xi}\,dt.\end{aligned}\tag{172}$$

If $\omega(t)$ satisfies the conditions given at the end of [27] in the interval and if near the ends of the interval it has the form (160), then for points z near the ends of the interval, the following statements apply:

1. If $\gamma=0$, then

$$F(z)=\pm\frac{\omega(c)}{2\pi i}\log\frac{1}{z-c}+F_0(z),$$

where the $(+)$ sign refers to the case when $c=a$ and the $(-)$ sign to the case when $c=b$, whilst $F(z)$ is a bounded function which has a definite limit when $z\to c$. For $\log(z-c)$ we can take any branch, which is single-valued in the neighbourhood of $z=c$ in a plane with the cut (a, b).

2. If $\gamma=\gamma_1+\gamma_2 i\neq 0$ then

$$F(z)=\pm\frac{e^{\pm\gamma\pi i}}{2i\sin\gamma\pi}\cdot\frac{\omega^*(c)}{(z-c)^\gamma}+F_0(z),$$

where the signs are chosen in the way described above and $(z-c)^\gamma$ denotes a single-valued branch in the neighbourhood of the point $z=c$ in the plane with the cut (a, b); the value of $(z-c)^\gamma$ on the upper (left) edge of the cut is equal to that value of $(t-c)^\gamma$, which enters the formula (160). Furthermore $F_0(z)$

has the following properties: if $\gamma_1 = 0$ then $F_0(z)$ is bounded and has a definite limit when $z \to c$; if $\gamma_1 > 0$, then

$$| F_0(z) | < \frac{c}{| z - c |^{\gamma_0}},$$

where c and γ_0 are constants and $\gamma_0 < 1$. Using the concept of Leber's integral, the values of integrals of Cauchy's type can be investigated for any integrable function $\omega(t)$ and for a great variety of contours (see Privaloff, *Integrals of Cauchy's type*, 1918).

Let us notice one particular case. If $\omega(\tau)$ are the limiting values on L of a function, which is regular within the closed contour L and which is continuous up to L, while $\omega(\tau)$ satisfies a Lipschitz condition, then $F_i(\xi) = \omega(\xi)$, and the first of the formulae (169) shows that $\omega(\tau)$ is the solution of a homogeneous integral equation of the second kind:

$$\omega(\xi) = \frac{1}{\pi i} \int_L \frac{\omega(\tau)}{\tau - \xi} \, d\tau. \tag{173}$$

Let L be a simple closed contour, as described above. The principal value of the integral

$$\frac{1}{2\pi i} \int_L \frac{\omega(\tau)}{\tau - \xi} \, d\tau \tag{174}$$

transforms any function $\omega(\tau)$ which is given on L and which satisfies a Lipschitz condition into another function $\omega_1(\xi)$, which is also defined on L and which also satisfies a Lipschitz condition. In other words, the integral (174) is an operator for the function $\omega(\tau)$. To the function thus obtained we can again apply an operator with Cauchy's nucleus. In this case the following formula applies:

$$\frac{1}{2\pi i} \int_L \frac{1}{\xi - \eta} \left[\frac{1}{2\pi i} \int_L \frac{\omega(\tau)}{\tau - \xi} \, d\tau \right] d\xi = \frac{1}{4} \omega(\eta). \tag{175}$$

In other words, as a result of the two transformations by Cauchy's nucleus, we obtain the initial function multiplied by 1/4. To prove (175) we rewrite the first of the formula (169) in the following form:

$$\frac{1}{2\pi i} \int_L \frac{\omega(\tau)}{\tau - \xi} \, d\tau = F_i(\xi) - \frac{1}{2} \omega(\xi). \tag{176}$$

The right-hand side gives the result of the linear transformation of the function $\omega(\tau)$ by Cauchy's nucleus. This operation can again be applied to the right-hand side:

$$\frac{1}{2\pi i} \int_L \frac{F_i(\xi) - \frac{1}{2} \omega(\xi)}{\xi - \eta} \, d\xi, \tag{177}$$

where η lies on L and the integral is in the principal value sense as before. Owing to the fact that $F_l(\xi)$ gives the limiting values on L of a function which is regular within L, from (173) we have the following:

$$\frac{1}{2\pi i}\int_L \frac{F_l(\xi)}{\xi-\eta}\,d\xi = \frac{1}{2}F_l(\eta).$$

On the other hand, from (176):

$$\frac{1}{2\pi i}\int_L \frac{\frac{1}{2}\omega(\xi)}{\xi-\eta}\,d\xi = \frac{1}{2}F_l(\eta) - \frac{1}{4}\omega(\eta),$$

and, finally, the integral (177) appears to be equal to $\omega(\eta) : 4$, i.e. we obtain the formula (175).

CHAPTER II

CONFORMAL TRANSFORMATION AND THE TWO-DIMENSIONAL FIELD

29. Conformal transformation. In this chapter we shall consider some applications of the theory of functions of a complex variable to problems of two-dimensional hydrodynamics, electrostatics and the theory of elasticity. Conformal transformation plays an important part in these applications and we shall therefore begin this chapter with a detailed account of conformal transformation. We explained the basic characteristics of the transformation of a regular function in [3] and later in [22]. We considered in greater detail transformations at points where the derivative did not vanish and at other points, where it was zero. At points of the first kind all angles remain unchanged, while at points of the second kind all angles are magnified, as we described in [23]. Let

$$w = f(z) \tag{1}$$

be a regular function which conformally transforms the domain B into the domain B_1. If $f'(z)$ does not vanish anywhere in the domain B, then the domain B_1 has no branch-points, but it can still have several sheets, i.e. it can overlap itself. Consider in the domain B a curve l, a function $\varphi(s)$ given on this curve and a line integral

$$\int_l \varphi(s)\, ds,$$

where ds is an element of the arc of the curve l. As a result of the transformation (1) the curve l is transformed into another curve l_1 in the domain B_1, and an element of arc of the new curve is expressed by the product $ds_1 = |f'(z)|\, ds$ since $|f'(z)|$ gives the change in linear dimensions [3].

Introducing the function

$$z = F(w), \tag{2}$$

which is the inverse of (1), we evidently have $F'(w) = 1/f'(z)$ and, consequently, we can write $ds = |F'(w)|\,ds_1$. The integral obtained after the transformation can be written in the form:

$$\int_l \varphi(s)\,ds = \int_{l_1} \varphi(s_1)\,|F'(w)|\,ds_1. \tag{3}$$

Similarly, bearing in mind that $|f'(z)|^2$ is multiplied by the change in surface area at the given point we obtain the following formula for the conformal transformation of the double integral:

$$\iint_B \varphi(z)\,d\sigma = \iint_{B_1} \varphi_1(w)\,|F'(w)|^2\,d\sigma_1, \tag{4}$$

and the following formula applies to an element of area:

$$d\sigma_1 = |f'(z)|^2\,d\sigma. \tag{5}$$

When the real and imaginary parts are separated

$$w = f(z) = u(x, y) + iv(x, y), \tag{6}$$

it can easily be seen that $|f'(z)|^2$ is equal to the functional determinant of the functions $u(x, y)$ and $v(x, y)$ of the variables x and y. In fact, this functional determinant is expressed by the following formula:

$$\frac{D(u, v)}{D(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x},$$

or, from the Cauchy–Riemann equations, by the formula:

$$\frac{D(u, v)}{D(x, y)} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2,$$

and this is, in fact, the square of the modulus of the derivative

$$|f'(z)|^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2.$$

Consider in the $z = x + iy$ plane two families of lines:

$$u(x, y) = C_1; \quad v(x, y) = C_2, \tag{7}$$

where C_1 and C_2 are arbitrary constants. In the $w = u + iv$ plane they correspond to the lines $u = C_1$ and $v = C_2$, which are parallel to the axes of coordinates; therefore the lines (7) are obtained from the net of straight lines parallel to the axes, as a result of the transformation (2). One of the consequences of this is that the lines (7), which belong to different families, are orthogonal except at the points where $f'(z)$ vanishes. Conversely, if we take the equations

$$u = u(x, y); \quad v = v(x, y)$$

and assume that $x = C_1$ or $y = C_2$ on the right-hand side of these equations, where C_1 and C_2 are arbitrary constants, we obtain a net of lines in the $w = u + iv$ plane, consistsing of two orthogonal families of lines. This net is obtained as a result of the transformation effected by the function (1) from a net of straight lines, parallel to the axes of coordinates in the z-plane. These two nets of lines which are of great importance in what follows, are usually known as *isothermic*

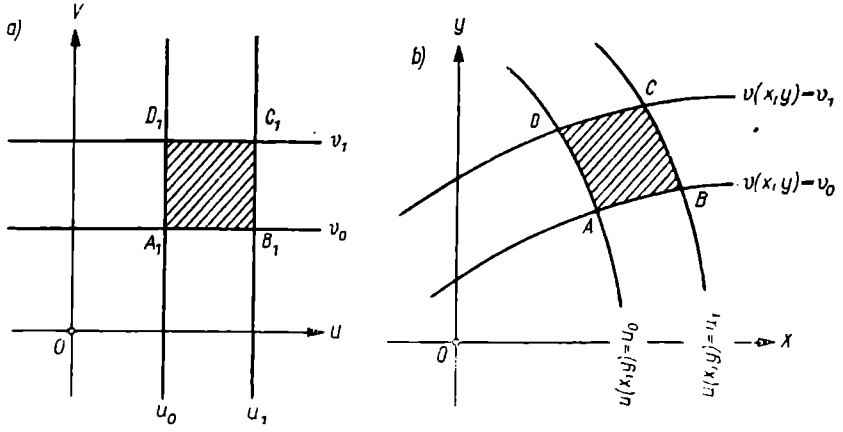

FIG. 26

nets. We shall explain the meaning of this term. The real part $u(x, y)$ (or the imaginary part) of a regular function should satisfy the Laplace equation [2]:

$$\frac{\partial^2 u\ (x, y)}{\partial x^2} + \frac{\partial^2 u\ (x, y)}{\partial y^2} = 0.$$

This equation is satisfied by the temperature of an established heat current [II, 117] and we suppose that this case is two-dimensional, i.e. the temperature u is independent of one of the coordinates. When interpreting the function $u(x, y)$ as the temperature of an established heat current, the lines of the first family (7) will be the isothermic lines and this is how the name "isothermic net" is derived. In the case under consideration, lines of the second family (7), orthogonal with the first family of lines, serve as vectorial lines for the vectors which we considered in [II, 117] and which we called "the vectors of the heat current".

Two lines $u(x, y) = u_0$ and $u(x, y) = u_1$ are transformed by (1) into the lines $u = u_0$ and $u = u_1$ which are parallel to the $u = 0$ axis

and part of the domain B bounded by the above lines, is transformed into part of a strip, bounded by the lines parallel to the $u = 0$ axis. The curved rectangle, bounded by four lines of the isothermic net, is transformed by (1) into a rectangle, bounded by straight lines, parallel to the axes (Fig. 26)

$$u = u_0;\ u = u_1;\ v = v_0;\ v = v_1.$$

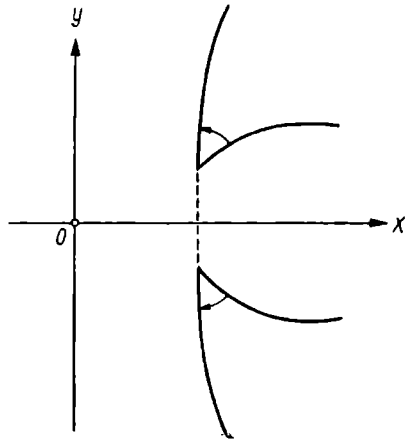

FIG. 27

Let us make one further addition to the fundamental principles of conformal transformation before considering any examples. We saw that the transformation of a regular function $f(z)$ conserved the angles both in magnitude and in sign at points where the derivative did not vanish. Sometimes the transformation of a plane is considered where the magnitude of the angles is conserved but their sign is reversed. This transformation is sometimes called a *conformal transformation of the second class*. For example, the symmetrical transformation of the real axis is, evidently, a conformal transformation of the second class (Fig. 27). This transformation is given by the formula $w = \bar{z}$. Generally speaking, if $f(z)$ is a regular function in the domain B then the formula

$$w = f(\bar{z}) \tag{8}$$

gives a conformal transformation of the second class, defined in the domain B', which with, the domain B is symmetrical about the real axis. In fact, the transition from z to $\bar{z}$ transforms B' into B while

conserving the magnitude of angles but not their sign. The subsequent transition from $\bar{z}$ to $f(\bar{z})$, in accordance with formula (8), will neither alter the magnitude of angles nor their sign; thus in the final transformation from z to w we have a conformal transformation of the second class.

30. Linear transformation. As the first example of conformal transformation we shall consider the simplest linear function

$$w = az + b \quad (a \neq 0), \tag{9}$$

from where

$$z = \frac{1}{a} w - \frac{b}{a}.$$

This function transforms the whole plane, including the point at infinity, into itself, the point at infinity remaining in its former position. In particular, when $a = 1$, the function $w = z + b$ gives the parallel transition of the plane along a vector corresponding to the complex number b. In the case $b = 0$ and $a = e^{i\psi}$ (where ψ is a real number), the number ψ must be added to the amplitude of z, and so the transformation $w = e^{i\psi} z$, will involve the rotation of the plane about the origin by an angle ψ. In general, the movement of a plane as a whole is obtained by a combined rotation and parallel transition:

$$w = e^{i\psi} z + b. \tag{10}$$

If $a = e^{i\psi} \neq 1$, i.e. the transition is not purely parallel, the coordinates of the stationary point of the transformation i.e. of the point which remains in its former position during the transformation, can easily be determined by the formula (10). The coordinates of this point are determined from the equation

$$z_0 = e^{i\psi} z_0 + b, \text{ whence } z_0 = \frac{b}{1 - e^{i\psi}}.$$

It can easily be shown that the transformation (10) can be written in the form

$$w - z_0 = e^{i\psi} (z - z_0),$$

i.e. in general the transformation (10) can be regarded as the rotation of the plane about the point z_0 through an angle ψ. Note that there will be a second stationary point at infinity for the transformation (10).

We shall now consider the case when the modulus of the coefficient a in the linear transformation (9) is not unity. Introducing the modulus

and amplitude of a, we shall consider the transformation when $b = 0$:

$$w = \varrho e^{i\psi} z$$

In this case the length of the vector from the origin to the point z must be multiplied by ϱ and the plane rotated about the origin by an angle ψ. This transformation is known as an *identity transformation with the origin as the centre of similarity and with a coefficient of similarity* ϱ.

We shall now consider the general case of the linear transformation (9) when $a \neq 1$. Introducing the stationary point of the transformation:

$$z_0 = az_0 + b, \text{ i.e. } z_0 = \frac{b}{1-a},$$

we can rewrite formula (9), in the form:

$$w - z_0 = a(z - z_0,)$$

and we evidently have here an identity transformation with the centre not at the origin but at a point z_0. We leave the reader to show that the isothermic net will, in this case, consist of two families of parallel straight lines; this is obvious and purely geometrical.

31. Bilinear transformation. A bilinear transformation is a transformation which can be expressed by a quotient of two linear functions

$$w = \frac{az + b}{cz + d}. \tag{11}$$

Here $ad - bc \neq 0$ since otherwise the fraction in the equation (11) could be simplified and would simply be a constant. Solving the equation (11) with respect to z we obtain a formula for the transformation of the inverse of (11), which will also be a bilinear transformation

$$z = \frac{-dw + b}{cw - a}. \tag{12}$$

Every point in the z-plane has a corresponding point in the w-plane and vice versa, i.e. the transformation (11) transforms the whole plane, including the point at infinity, into itself.

If in formula (11) $c = 0$, then the transformation is simply a linear transformation. Otherwise the point $z = \infty$ is transformed into the point a/c and the point $z = -d/c$ gives the point $w = \infty$, i.e. in general the point at infinity does not remain stationary during the bilinear transformation.

We shall now give one characteristic property of the bilinear transformation, viz. that it transforms a circle into a circle, where by a "circle" we shall now and in future understand not only a circle in the usual sense but also a straight line. This property is quite obvious for a linear transformation where the plane moves as a whole, or in an identity transformation; a linear transformation transforms a straight line into a straight line and a circle into a circle, in the usual sense of the word. Before proving this property of the bilinear transformation we shall describe it in a slightly different way. Let us suppose that $c \neq 0$ and divide the numerator by the denominator; formula (11) can then be written in the form:

$$w = e + \frac{f}{z + \frac{d}{c}} \quad \text{where } e = \frac{a}{c} \text{ and } f = \frac{bc - ad}{c^2}.$$

Our transformation thus involves a parallel transition $w_1 = z + d/c$, a transformation of the form $w_2 = f/w_1$ and another parallel transition $w = w_2 + e$. It is therefore sufficient to consider the simple transformation:

$$w = \frac{\gamma}{z} \tag{13}$$

and prove that it transforms a circle into a circle. An equation of a circle can be written as follows:

$$A(x^2 + y^2) + 2Bx + 2Cy + D = 0,$$

where $A = 0$ for a straight line. The equation can then be written as follows:

$$Az\bar{z} + \delta z + \delta\bar{z} + D = 0, \text{ where } \delta = B - iC, \tag{14}$$

where the line above the letters shows that complex conjugate numbers were taken. We now suppose that we have a circle l in the z-plane. To obtain the equation of the transformed circle in the w-plane it is sufficient to determine z from the equation (13) and substitute the expression obtained in the equation (14). We then obtain a curve l_1 in the w-plane which is given by the equation:

$$A\gamma\bar{\gamma} + \delta\gamma\bar{w} + \bar{\delta}\bar{\gamma}w + Dw\bar{w} = 0.$$

This equation is of the same type as the equation (14) i.e. it corresponds to a circle (or to a straight line). Thus *every transformation of the type* (11) *transforms a circle into a circle (a straight line is a circle which passes through the point at infinity).*

Suppose that the transformation (11) transforms a circle l into a circle l_1 and that both circles are circles in the usual sense of the word. Bearing in mind what was said in [22] we can maintain that if the completion of a circuit in the positive direction round l corresponds to the completion of a circuit in the positive direction round l_1, then the interior of l is transformed into the interior of l_1. However, if the circuits round l and l_1 are described in opposite directions, then the interior of l is transformed into the exterior of l_1 and vice versa. If one of the above circles is a straight line or, if both are straight lines, then in order to determine the domains of the plane which are transformed into each other, we have to take the corresponding direction of circuits along both lines, when parts of the plane which lie to one side of the moving observer, for example, to the left, are transformed into each other.

Let us consider two points A_1 and A_2 which are symmetrical with respect to the circle l. Suppose that after the transformation they become the points B_1 and B_2. We shall show that these points are also symmetrical with respect to the transformed circle l_1. In fact, a family of circles through the points A_1 and A_2 will, as we said in [24], consist of circles orthogonal to l. After the transformation we evidently obtain a family of circles through the points B_1 and B_2 and, as a result of conformity of circles belonging to a family, these circles will be orthogonal with the circle l_1; this, as we know, is one of the characteristics of symmetry. Thus if the circle l is transformed by (11) into the circle l_1 then points, symmetrical with respect to the circle l, are transformed into points, symmetrical with respect to the circle l_1. Notice that the point at infinity is symmetrical with the centre of the circle. In the case under consideration a family of circles which passes through these two points is transformed into a family of lines which passes through the centre of this circle and these lines are, obviously, orthogonal to the circle itself.

If a and c are both non-zero then the transformation (11) can be written in the following form:

$$w = k\frac{z-a}{z-\beta} \quad \left(k = \frac{a}{c}\right). \tag{15}$$

The numbers a and β have a simple geometric meaning, viz. the point $z = a$ is transformed into the origin $w = 0$ and the point $z = \beta$ into the point at infinity.

Let us consider a family of concentric circles, centre the origin, in the w-plane. The equation of these circles is $|w| = C$ and the

points mentioned above, viz. $w = 0$ and $w = \infty$, are symmetrical with respect to these circles. It follows that these circles correspond to circles in the z-plane for which the points $z = a$ and $z = \beta$ are symmetrical; the equation of this family of circles has the form:

$$\left|\frac{z-a}{z-\beta}\right| = C, \tag{16}$$

where C is an arbitrary constant. Thus *the equation* (16) *corresponds to a family of circles with respect to which the points* a *and* β *are symmetrical* (Fig. 28). The straight line which is perpendicular to the

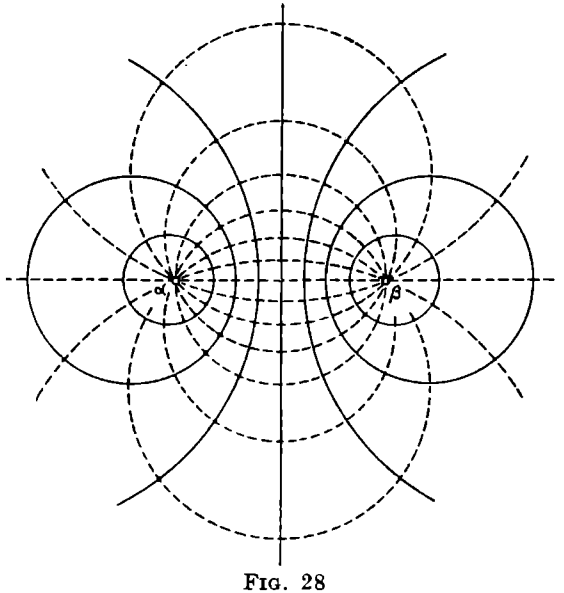

FIG. 28

line from a to β at its midpoint also belongs to this family. Let us now consider a family of straight lines in the w-plane which passes through the origin or, in other words, a family of circles which passes through the points $w = 0$ and $w = \infty$. This family of circles is evidently given by the equation $\arg w = C$. It corresponds to a family of circles in the z-plane which passes through the points a and β and the equation of this family is (since the amplitude of k is a constant):

$$\arg\frac{z-a}{z-\beta} = C_1. \tag{17}$$

Hence *the equation* (17) *describes a family of circles in the z-plane*

which passes through the points a and β. The circles of the family (17) evidently cut the circles of the family (16) at right angles (Fig. 28).

Let us now define the isothermic net in the z-plane. It corresponds to two families of straight lines in the w-plane which are parallel to the axes. Each family can be regarded as a family of circles which touch at infinity. In the z-plane each family corresponds to a family of circles which touch at the point $z = -d/c$. Hence *the required isothermic net consists of two families of circles, the circles of each family touching at the point* $z = -d/c$*; circles belonging to the two different families intersect at this point at right angles* (Fig. 29).

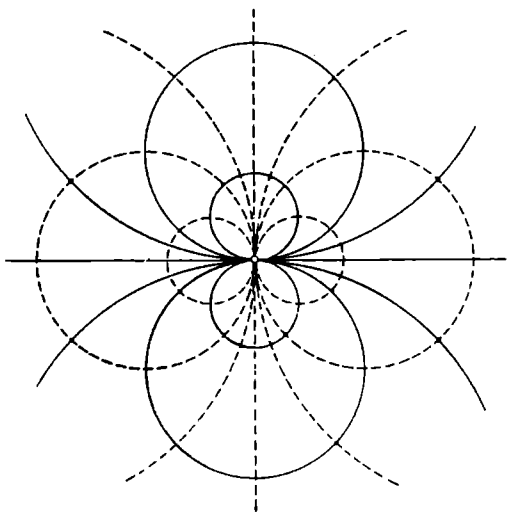

FIG. 29

The accurate definition of one family, and consequently, also of the other family, depends on the values of the complex coefficient of transformation (11).

The transformation (11) contains three arbitrary complex parameters, viz. the relationship of three of the coefficients a, b, c, d, to the fourth. Hence the transformation (11) can be defined if a corresponding number of auxiliary conditions is given. We can, for example, assume that three given points z_1, z_2, z_3 in the z-plane should be transformed into three given points w_1, w_2, w_3 in the w-plane. It is easy to write down the bilinear transformation which takes these conditions into account. It has the form:

$$\frac{w - w_1}{w - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}. \tag{18}$$

For, by solving this equation with respect to w we obtain a bilinear transformation of the type (11). Also, when substituting $z = z_1$ and $w = w_1$, we obtain zero on both sides of formula (18). When substituting the second pair of points $z = z_3$ and $w = w_3$ we obtain unity on both sides of the equation, and when substituting the third pair of points, we obtain infinity on both sides of the equation. It can be seen from this that the bilinear transformation, as given by formula (18), does in fact, satisfy the required conditions. It is also easy to show that these conditions define the bilinear transformation uniquely. Obviously the constructed transformation transforms a circle, defined by the three points z_1, z_2, z_3 into a circle, defined by the three points w_1, w_2, w_3. If both sets of points are taken on the same circle, then the bilinear transformation transforms that circle into the same circle. If, in addition, the sequence of the points z_k on this circle gives the same direction in which the circuit is to be described as the sequence of the points w_k, then the interior of the constructed circle is transformed into itself.

Consider, for example, the upper half-plane bounded by the real axis where interior points are defined by the condition that the coefficients of the imaginary parts of the coordinates are positive. In the case under consideration any transformation which transforms the upper half-plane into itself should also transform the real axis into itself, i.e. real z's should have corresponding real w's; consequently we can take it that all four coefficients are real in formula (11). However, this is not sufficient; in addition it is also necessary that w should increase as z increases along the positive part of the real axis. Otherwise the upper half plane z will be transformed into the lower half-plane w.

Substituting $z = x + iy$ in formula (11) we obtain:

$$w = \frac{(ax + b) + iay}{(cx + d) + icy},$$

or, separating the real and imaginary parts:

$$w = u + iv = \frac{(ax + b)(cx + d) + acy^2}{(cx + d)^2 + c^2 y^2} + i \frac{(ad - bc) y}{(cx + d)^2 + c^2 y^2}.$$

It follows that when $y > 0$ the coefficient of the imaginary part of w will also be positive, provided the following condition is satisfied:

$$ad - bc > 0. \tag{19}$$

Hence *the general form of the bilinear transformation which transforms the upper half-plane into itself is* (11), *where the real coefficients can be arbitrary provided they satisfy the condition* (19).

The transformation of a unit circle into itself takes place similarly, i.e. a unit circle is a circle with unit radius and centre the origin, the equation of which can be written as follows: $|z| \leqslant 1$. To begin with we shall explain certain properties of points symmetrical with respect to the circumference C of this circle.

Let A_1 and A_2 be two such points and M a point on the circumference C. We have $\overline{OA_1} \cdot \overline{OA_2} = \overline{OM}^2$, which can be written as a proportion (Fig. 30):

$$\frac{\overline{OA_1}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OA_2}} .$$

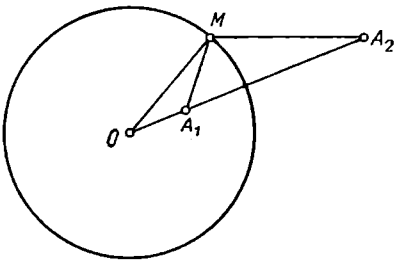

FIG. 30

It follows that the triangles OA_1M and OA_2M, which have a common angle A_1OM and proportional sides adjacent to this angle, are similar; from this similarity we obtain the following proportion:

$$\frac{\overline{MA_1}}{\overline{MA_2}} = \frac{\overline{OA_1}}{\overline{OM}} . \tag{20}$$

Denote by a the complex coordinate of A_1 and let $a = \varrho e^{i\varphi}$. For the symmetrical point A_2 we must have $\beta = e^{i\varphi}/\varrho$ or, in other words, the complex coordinate of this symmetrical point can be expressed by the fraction $\beta = 1/\bar{a}$. We shall construct a bilinear transformation so as to transform the unit circle into itself and the point a into the origin. This transformation should transform the symmetrical point β into the point at infinity, i.e. it should have the form:

$$w = k \frac{z - a}{z - \beta} \tag{21}$$

or substituting $\beta = 1/\bar{a}$,

$$w = k \frac{\bar{a}(z - a)}{\bar{a}z - 1} , \tag{22}$$

where k is a factor, the form of which is still to be determined from the condition that the right-hand side of the formula (21) for any z on the circumference C should have a modulus equal to

unity, i.e.

$$|k| \frac{|z-a|}{|z-\beta|} = 1 \quad \text{when} \quad |z| = 1.$$

But from (20):

$$\frac{|z-a|}{|z-\beta|} = \frac{|a|}{1},$$

hence $|ka| = 1$. We can see that the product $\overline{ka}$ must have a unit modulus, i.e. it must have the form $\overline{ka} = e^{i\psi}$, where ψ can take every real value. Thus returning to the formula (22) we obtain the following formula for the required transformation:

$$w = e^{i\psi} \frac{z-a}{\bar{a}z-1}, \tag{23}$$

in which the point a and the real parameter ψ can have any value we please. In particular when $a = 0$, i.e. when the origin is transformed into itself, we have the simple transformation $w = e^{i(\psi+\pi)} z$, i.e. the unit circle revolves about the origin by an angle $(\psi + \pi)$. The general transformation (23) can be separated into two parts, viz. into the transformation

$$w = \frac{z-a}{\bar{a}z-1}, \tag{24}$$

which transforms a unit circle into itself and which transforms the point a into the origin, and, subsequently, into the rotation about the origin by an angle ψ.

We can also construct a countless number of transformations which transform a circle K_1 into another circle K_2. For this purpose it is sufficient to construct one of these transformations to transform K_1 into K_2 and, subsequently, to apply any bilinear transformation to the result so obtained, in order to transform the circle K_2 into itself. It is important to note that two bilinear transformations applied in succession also produce a bilinear transformation. In fact, suppose that we have a bilinear transformation of the variable z to the variable w of the form (11) and, subsequently, the following bilinear transformation

$$w_1 = \frac{a_1 w + b_1}{c_1 w + d_1} \tag{25}$$

of the variable w to the variable w_1. Substituting the expression (11) in the above formula we obtain, after elementary transpositions, the final bilinear transformation of the variable z to the variable w_1:

$$w_1 = \frac{(a_1 a + b_1 c) z + (a_1 b + b_1 d)}{(c_1 a + d_1 c) z + (c_1 b + d_1 d)}.$$

This is usually known as *the product of the bilinear transformations* (11) *and* (25) and, generally speaking, this product depends on the order of factors, i.e. on the order in which the bilinear transformations (11) and (25) are carried out.

Let us now return to the unit circle and the upper half-plane; we shall construct a bilinear transformation to transform a half-plane into a unit circle. We take for this purpose the following transformation:

$$w = \frac{z - i}{z + i}. \tag{26}$$

It is easy to see that the point $z = i$ in the upper half-plane will be transformed into the origin and that real values of z correspond to values of w with modulus equal to unity. In fact

$$|w| = \frac{|z - i|}{|z + i|},$$

where the numerator and the denominator of the fraction are respectively equal to the distances from the point z to the points i and $(-i)$; when the point z lies on the real axis then these distances are equal and, consequently $|w| = 1$. If an arbitrary bilinear transformation which transforms a unit circle into itself, is applied to the variable w, we obtain the general form of the transformation which transforms the upper half-plane into a unit circle.

In conclusion we shall prove *the principle of symmetry in the general form* as formulated in [24]. Let the function $f(z)$ be regular on one side of an arc AB of the circle C, let it be continuous as far as the arc AB and let it transform it into another arc A_1B_1 of the circle C_1. Subjecting z to a bilinear transformation which transforms C into the real axis we obtain:

$$z_1 = \frac{az + b}{cz + d},$$

and performing a similar bilinear transformation on the function itself we transform the circle C_1 into the real axis. We thus obtain a new function $f_1(z_1)$ of a new independent variable z_1:

$$f_1(z_1) = \frac{a' f(z) + b'}{c' f(z) + d'}.$$

This new function $f_1(z_1)$ is regular on one side of the real axis and continuous as far as the line which it transforms into part of the real axis. In accordance with our earlier definition of the principle of symmetry given in [24] this function can be analytically continued beyond the above line; at points symmetrical with respect to the

real axis its values are also symmetrical with respect to the real axis. Bearing in mind that the two bilinear transformations mentioned above transform symmetrical points into other symmetrical points, we can maintain that the initial function $f(z)$ can be analytically continued beyond the arc AB of the circle C, and points symmetrical with respect to this circle are transformed into points, symmetrical with respect to the circle C_1.

Bilinear transformations, as we shall see later, are of great importance in the theory of a complex variable. They are often used in the same way as the transformation of coordinates is used in analytical geometry viz. before starting to solve a problem, the plane of the complex variable in the problem, is subjected to the bilinear transformation so as to obtain the simplest possible conditions. Thus, for example, by using the bilinear transformation we reduced the general case of the principle of symmetry to the particular case considered above.

Let us call the reflection in a circle or in a straight line the transformation of a plane where every point A is transformed into a point A_1, symmetrical with it with respect to C. Let z be the complex coordinate of A and w the complex coordinate of A_1. Let us assume that C is a circle, centre $B(z = a)$ and radius R. The vectors $\overline{BA}$ and $\overline{BA}_1$ should have the same amplitudes and the product of their lengths should be equal to R^2. It is easy to see that this leads to the following formula which expresses w in terms of $\bar{z}$:

$$w - a = \frac{R^2}{\bar{z} - \bar{a}}, \tag{27}$$

i.e. the reflection of the circle is expressed by a bilinear function of $\bar{z}$:

$$w = \frac{a\bar{z} + (R^2 - a\bar{a})}{\bar{z} - \bar{a}}$$

and, consequently, it is a conformal transformation of the second class. Let us now consider the reflection in a straight line. Assume that the straight line passes through the origin and makes an angle ψ with the positive direction of the real axis (Fig. 31). In this case the point z is transformed into the point w which has the same modulus $|w| = |z|$ and amplitude $\arg w = 2\psi - \arg z$, i.e. the transformation can, in this case, be written as:

$$w = e^{i2\psi}\bar{z}, \tag{28}$$

where it is expressed as a simple linear function of $\bar{z}$. It is clear that the same result is obtained when reflection takes place in any straight line.

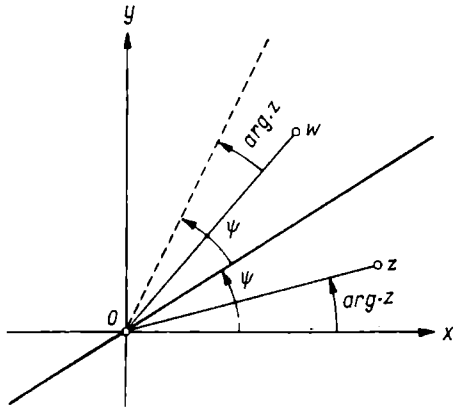

FIG. 31

If we produce two successive reflections in two different circles or straight lines, we obtain a bilinear transformation. Let us consider in greater detail the case when two successive reflections are produced in two intersecting straight lines. It can always be assumed that the point of intersection lies at the origin. Let ψ_1 and ψ_2 be the angles made by these straight lines with the positive direction of the real axis. When producing these successive reflections we move from the point z to the point w_1 and from the point w_1 to the point w, according to the formulae:

$$w_1 = e^{i2\psi_1}\bar{z}; \quad w = e^{i2\psi_2}\bar{w}_1.$$

Substituting the expression for w_1 on the right-hand side of the second formula we obtain the final transformation for z to w in the form

$$w = e^{i2(\psi_2-\psi_1)}z,$$

this involves the rotation about the origin by an angle $2(\psi_2 - \psi_1)$, i.e. two successive reflections in intersecting straight lines involve the rotation of the plane about the point of intersection by an angle equal to twice the angle between these straight lines. It is also easy to show that two successive reflections in parallel straight lines give parallel transformation of the plane.

32. The function $w = z^2$. Using a different notation we have already investigated the function

$$w = z^2 \tag{29}$$

and we saw that it transforms the z-plane into a two-sheeted Riemann surface in the w-plane with branch-points of the first order at $w = 0$ and $w = \infty$. We shall now establish the form of the isothermic nets in the z- and w-planes. On separating the real and imaginary parts we obtain:

$$w = u(x, y) + iv(x, y) = (x + iy)^2 = (x^2 - y^2) + i2xy.$$

The isothermic net in the z-plane consists of two families of rectangular hyperbolae (Fig. 32):

$$x^2 - y^2 = C_1; \quad 2xy = C_2.$$

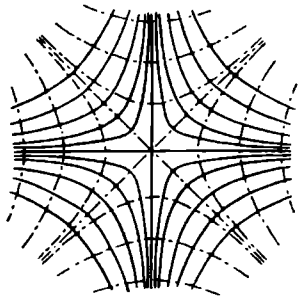

FIG. 32

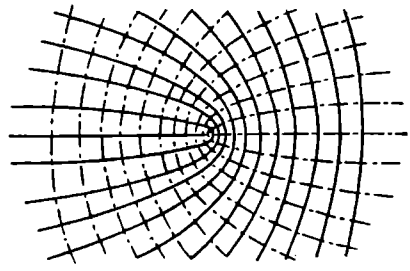

FIG. 33

Let us now consider the isothermic net in the w-plane. In the formulae:

$$u = x^2 - y^2; \quad v = 2xy, \quad x = C_1$$

let us assume that $x = C_1$ and eliminate y, then if we assume that $y = C_2$ and eliminate x, we obtain two families of parabolae (Fig. 33):

$$v^2 = 4C_1^2(C_1^2 - u); \quad v^2 = 4C_2^2(C_2^2 + u),$$

which have been obtained as a result of the transformation of the straight lines $x = C_1$ and $y = C_2$ from the z-plane.

We can evidently regard the isothermic net formed by these parabolae as an isothermic net in the w-plane of the function $w = \sqrt{z}$ which is the inverse of the function (29).

Let use consider in Fig. 32 any rectangular hyperbola represented by the dotted line for which OX is the major axis. Its equation is $x^2 - y^2 = C_1$, where C_1 is a positive constant. Let us consider its right branch. If in the equation $x^2 - y^2 = C$ we alter C from C_1 to $(+\infty)$, we obtain one of the hyperbolae which are shown by the dotted lines; its

right branch will lie further to the right than the right branch of the hyperbola $x^2 - y^2 = C_1$; it follows that the function (29) conformally transforms that part of the z-plane within the right branch of the hyperbola into the half-plane $u \geqslant C_1$ of the w-plane. Similarly the function (29) conformally transforms part of the z-plane within the left branch of the hyperbola $x^2 - y^2 = C_1$ into the half-plane $u \geqslant C_1$.

Let us now consider in Fig. 33 any parabola shown by a dotted line. Its equation has the form $v^2 = 4C_2^2(C_2^2 + u)$ and there will be a corresponding straight line $y = C_2$ in the z-plane, where the constant C_2 can be regarded as positive since only C_2^2 enters the equation of the parabola. If in the equation $v^2 = 4C^2(C^2 + u)$, C is altered from C_2^2 to $(+\infty)$ then parabola represented by dotted lines will be obtained which lies further to the left than the parabolae $v^2 = C_2^2(C_2^2 + u)$; it follows directly from what was said above that the function $z = \sqrt{w}$ conformally transforms that part of the w-plane outside the parabola $v^2 = C_2^2(C_2^2 + u)$ into the half-plane $y \geqslant C_2$ of the z-plane.

33. The function $w = k(z + 1/z)/2$. Let us consider the transformation of a plane by the function

$$w = \frac{k}{2}\left(z + \frac{1}{z}\right), \tag{30}$$

where k is a given positive number. Let us consider the form into which the net of polar coordinates in the z-plane will be transformed, i.e. consider the form into which the circles $|z| = \varrho$, centre the origin, and the family of straight lines $\arg z = \varphi$, which passes through the origin will be transformed. Substituting in formula (30) $z = \varrho e^{i\varphi}$ and separating the real and imaginary parts we obtain the equations:

$$u = \frac{k}{2}\left(\varrho + \frac{1}{\varrho}\right)\cos\varphi; \qquad v = \frac{k}{2}\left(\varrho - \frac{1}{\varrho}\right)\sin\varphi. \tag{31}$$

Consider the circle $\varrho = \varrho_0$. On eliminating φ from the equation (31) the following equation is obtained:

$$\frac{u^2}{\frac{k^2}{4}\left(\varrho_0 + \frac{1}{\varrho_0}\right)^2} + \frac{v^2}{\frac{k^2}{4}\left(\varrho_0 - \frac{1}{\varrho_0}\right)^2} = 1. \tag{32}$$

i.e. in the w-plane the circle is transformed into an ellipse, the semi-axes of which are

$$a = \frac{k}{2}\left(\varrho_0 + \frac{1}{\varrho_0}\right); \qquad b = \frac{k}{2}\left|\varrho_0 - \frac{1}{\varrho_0}\right|,$$

where we take the positive value for b, since the difference can be either positive or negative. When $\varrho = \varrho_0$, equation (31) gives the parametric equation for this ellipse. In the case of a unit circle, when $\varrho = 1$, the equation (31) gives $u = k \cos \varphi$ and $v = 0$, i.e. the ellipse degenerates into a line $(-k, +k)$ on the real axis described twice, or, as we shall say in future, it degenerates into a double line. When ϱ decreases from unity to zero the ellipses grow indefinitely until they cover the whole plane, i.e. the whole w-plane with a cut $(-k, +k)$ corresponds to the interior of the unit circle. Similarly, when ϱ increases from unity to infinity we also obtain indefinitely growing ellipses, i.e. the whole w-plane with the cut $(-k, +k)$ corresponds to that part of the z-plane outside the unit circle. The whole z-plane is transformed into a two-sheeted Riemann surface in the w-plane with branch-points at $w = -k$ and $w = +k$. It follows that the function, which is the inverse of (30):

$$z = \frac{w \pm \sqrt{w^2 - k^2}}{k}, \tag{30$_1$}$$

is two-valued and has the same branch-points. Let us investigate the ellipses (31) more closely. The foci of these ellipses lie on the real axis and their abscissae c are determined, as usual, by means of the semi-axes a and b, according to the formula: $c = \pm\sqrt{a^2 - b^2}$. In this case we have:

$$c = \pm \sqrt{\frac{k^2}{4}\left(\varrho_0 + \frac{1}{\varrho_0}\right)^2 - \frac{k^2}{4}\left(\varrho_0 - \frac{1}{\varrho_0}\right)^2} = \pm k,$$

i.e. for every value ϱ_0 the foci lie on the ends of the line $(-k, +k)$ or, in other words, the ellipses (32) have coinciding foci.

Let us now consider the straight lines $\varrho = \varphi_0$ under the transformation. Eliminating the variable ϱ from the equations (31) we have:

$$\frac{u^2}{k^2 \cos^2 \varphi_0} - \frac{v^2}{k^2 \sin^2 \varphi_0} = 1, \tag{33}$$

i.e. we obtain a family of hyperbolae with semi-axes $a = k \mid \cos \varphi_0 \mid$ and $b = k \mid \sin \varphi_0 \mid$. We shall show that the foci of these hyperbolae coincide with the foci of the above ellipses. We know that the foci of the hyperbolae (33) lie on the real axis and that the abscissae of the foci are found from the formula $c = \pm\sqrt{a^2 + b^2}$. In this case $c = \pm k$, i.e. the ellipses and the hyperbolae do, in fact, have coinciding foci. The hyperbolae, which correspond to

the axes of coordinates in the z-plane

$$\left(\varphi = 0, \frac{\pi}{2}, \ \pi \text{ and } \frac{3\pi}{2}\right),$$

degenerate into the $u = 0$ axis and into the lines $(-\infty, -k)$ and $(k, +\infty)$ on the real axis. Hence we can finally say that the net of polar coordinates of the z-plane is transformed by (30) into a net of ellipses and hyperbolae with foci at the points $\pm k$ (Fig. 34).

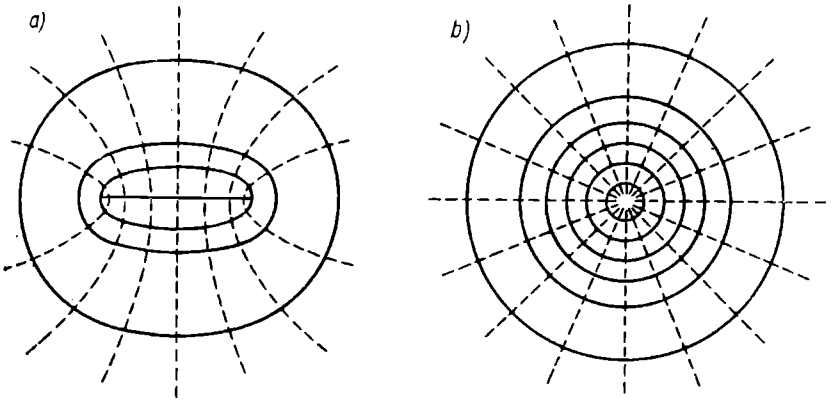

FIG. 34

It is easy to construct a function for which the net of ellipses and hyperbolae with coinciding foci serves as an isothermic net. In order to do this we recall what we know already about an exponential function [19]:

$$w = e^z,$$

with a period of $2\pi i$. From the formula

$$w = e^x e^{yi}$$

it follows that the lines $x = x_0$ are transformed into circles, centre the origin and radii e^{x_0}, whilst the lines $y = y_0$ are transformed into the straight lines $\varphi = y_0$, which pass through the origin, i.e. the function e^z transforms the net of Cartesian coordinates in the z-plane into a net of polar coordinates in the w-plane.

Consider the function:

$$w_1 = e^{iz} = e^{ix} e^{-y}, \tag{34}$$

with period 2π. It follows from the above formula that the net of Cartesian coordinates of this function will be transformed into a net of polar coordinates for the lines $y = y_0$ will be transformed into circles, whilst the lines $x = x_0$ will be transformed into straight lines.

Let us now consider the function

$$w = \frac{k}{2}\left(w_1 + \frac{1}{w_1}\right) = k\left(\frac{e^{iz} + e^{-iz}}{2}\right) = k\cos z. \tag{35}$$

As a result of the transformation (34) the net of Cartesian coordinates will be transformed into a net of polar coordinates and subsequently, as a result of the transformation (35), the net of polar coordinates will be transformed into a net of ellipses and hyperbolae with coinciding foci as mentioned above. The application of these two transformations viz. from z to w_1 and from w_1 to w gives, as the final result, the transformation $w = k\cos z$; thus the function $k\cos z$ transforms the net of Cartesian coordinates into a net of ellipses and hyperbolae with coinciding foci, i.e. this latter net is the isothermic net for the function $w = k\cos z$ in the w-plane. If we were considering the inverse function $w = \text{arc cos}\,(z/k)$, then the net of ellipses and hyperbolae with coinciding foci would be the isothermic net of this function in the z-plane.

Proceeding as in the previous paragraph, we find that one value of the function (30_1) transforms the w-plane with the cut $(-k, +k)$ into the interior of a unit circle in the z-plane. The same function, for any fixed ϱ_0, transforms the part of the plane outside the ellipse (32) into the interior part of a circle, centre the origin and radius ϱ_0, where $\varrho_0 < 1$. If we take the other value of the function (30_1) we obtain part of the plane outside the above circle, provided $\varrho_0 > 1$. Similarly, one value of the function (30_1) conformally transforms part of the w-plane between the two branches of the hyperbola (33), into a corner of the z-plane, defined by the inequalities: $\varrho_0 \leqslant \arg z \leqslant \pi - \varphi_0$, where $0 < \varphi_0 < \pi/2$.

A detailed investigation of conformal transformations of curves of the second order can be found in Privalloff's book *The Introduction Of The Complex Variable Into The Theory Of Functions.*

34. The bi-angular figure and the strip. Let us consider the bi-angular figure formed by two arcs of the circles C_1 and C_2 (Fig. 35). Let ψ be the angle of this bi-angular figure and a_1 and a_2 the coordinates

of its vertices. On effecting the bilinear transformation

$$w_1 = \frac{z - a_1}{z - a_2},$$

we transform the points a_1 and a_2 into $w_1 = 0$ and $w_1 = \infty$, so that the arcs of this bi-angular figure are transformed into straight lines which pass from the origin to infinity, the angle between these straight lines being ψ. If we subsequently effect the transformation

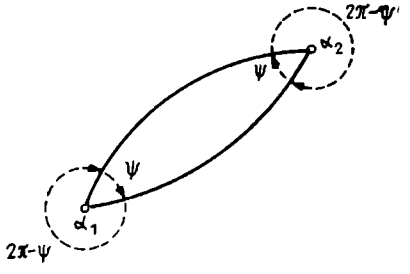

FIG. 35

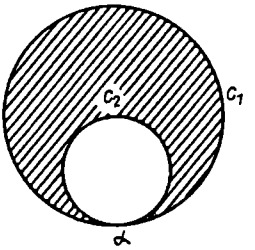

FIG. 36

$w_2 = w_1^{\pi/\psi}$, then this angle will be equal to π and the bi-angular figure will be transformed into a half-plane. By multiplying w_2 by the factor $e^{i\varphi_0}$, we can make this half-plane into the upper half-plane, bounded by the real axis. Grouping together all these transformations we finally obtain a formula for the transformation of our bi-angular figure into the upper half-plane:

$$w = e^{i\varphi_0}\left(\frac{z - a_1}{z - a_2}\right)^{\frac{\pi}{\psi}}. \tag{36}$$

Here φ_0 is a real number which depends on the position of our bi-angular figure. Performing the bilinear transformation of w as shown in [31] we can transform our bi-angular figure into a unit circle.

Notice that we are considering a bi-angular figure confined inside the contour formed by the arcs of two circles. Figure 35 shows part of the plane outside the closed contour which can be regarded as a bi-angular figure defined by the arcs of two circles. However, the angle of this bi-angular figure will no longer be ψ but $(2\pi - \psi)$.

We assumed above that the angle of this bi-angular figure is not zero. Let us now consider the case when this angle is zero. Assume that the two circles C_1 and C_2 touch from inside (Fig. 36). In this case

that part of the plane confined within the closed contour, is a bi-angular figure with zero angle. Similarly, when two circles touch from outside (Fig. 37), that part of the plane outside those circles also is a bi-angular figure with a zero angle. If a is the coordinate of the point of contact, then by performing the bilinear transformation

$$w_1 = \frac{1}{z - a},$$

we transform the circles into parallel straight lines, while the bi-angular figure itself is transformed into a strip, bounded by the two parallel lines. If subsequently we perform an identity transformation and also a parallel transition and rotation, i.e. a linear transformation, we can always cause the given strip to be bounded between two given parallel lines, for example, between the lines

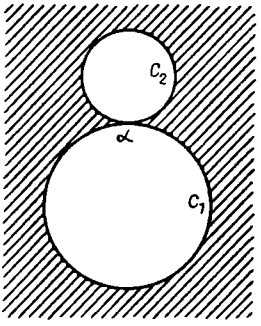

FIG. 37

$$y = 0 \text{ and } y = 2\pi.$$

Let us now try to find a regular function which would transform this strip into the upper half-plane. We know that the function $w = e^z$ transforms our strip into the w-plane with the cut $(0, +\infty)$ along the positive part of the real axis. Performing subsequently the transformation $\sqrt{w}$, we obtain the upper half-plane, i.e. the final function which transforms our strip into the upper half-plane will be

$$w = e^{z/2}.$$

It follows from above that the function e^z transforms a strip, bounded by the straight lines $y = 0$ and $y = \pi$ into the upper half-plane. Performing the bilinear transformation of the variable e^z, which transforms the upper half-plane into a unit circle [31], we obtain the function

$$w = \frac{e^z - i}{e^z + i}, \tag{37}$$

which transforms the strip, bounded by the straight lines $y = 0$ and $y = \pi$ into a unit circle.

We shall consider in greater detail one case of the bi-angular figure, viz. the upper half-circle, constructed on the line $(-1, +1)$ of the

real axis as its diameter. The function

$$w = \left(\frac{z+1}{z-1}\right)^2 \tag{38}$$

transforms the vertices of this bi-angular figure, viz. $z = -1$ and $z = +1$, into the points $w = 0$ and $w = +\infty$, while the diameter and the semi-circle are transformed into two straight lines; the angle between these lines is equal to twice the corresponding angle of the semi-circle, i.e. it is equal to π.

In other words our two halves of straight lines form a single straight line, viz. the real axis, as can easily be seen; when describing a circuit round the contour in the counter-clockwise direction we move along the real axis from $-\infty$ to $+\infty$, i.e. the function (38) transforms the semi-circle into the upper half-plane. On performing, in addition, the bilinear transformation (26) we obtain the function

$$\frac{(z+1)^2 - i(z-1)^2}{(z+1)^2 + i(z-1)^2},$$

which transforms our semi-circle into a unit circle.

35. The fundamental theorem. We dealt above with the cases of connected domains being conformally transformed into half-planes or into unit circles, and we considered both bounded connected domains (semicircle) and connected domains containing the point at infinity (the outside of an ellipse, the outside of the bi-angular figure). Let us now try to solve the problem of the transformation of any given connected domain in the z-plane into a unit circle or into a half-plane in the w-plane. In doing so we exclude two cases, viz. when the given domain is the whole z-plane, including the point at infinity, and when the given domain is the whole plane except for one point, for example, the point at infinity. In all other cases a regular function $w = f(z)$ exists in the given connected domain B, which transforms this domain into the unit circle $|w| < 1$. However, we can subsequently transform this unit circle into itself by a bilinear transformation and we then obtain a new conformal transformation of the region B into a unit circle. Let us mark within the region a definite point A and assume that during the transformation by our function

$$w = f(z), \tag{39}$$

this point is transformed into the point a, which lies in the unit circle. Similarly, performing a suitably chosen bilinear transformation of

this circle, we can always translate the point a to the origin without changing the unit circle [31]. This new transformation transforms the point A into the origin. Also, by rotating the unit circle about the origin we can cause the linear elements to remain stationary during the transition of the point A to the origin i.e. $f'(z)$ should be positive at the point A. We can thus see that from one conformal transformation of the region B into a unit circle we can construct a countless number of similar transformations among which there is one transformation which transforms any given point A in B, into the centre of the unit circle, without altering the direction at that point. It can be shown that with this additional condition the function effecting the conformal transformation is defined uniquely, viz. the following fundamental theorem in the theory of conformal transformation applies:

RIEMANN'S THEOREM. *If B is a given connected domain in the z-plane and z_0 is a point in B, then a function $f(z)$ exists, (except for the two cases mentioned above) which is regular in B; this function transforms the domain B into a unit circle so that z_0 is transformed into the origin and the value of the derivative $f'(z)$ is positive.*

We shall accept this theorem without proof. Note, that the function mentioned in this theorem can only in exceptional cases and for the simplest domains be expressed as an elementary function. The usual proof of Riemann's theorem establishes the existence of this function but it is of little use for even the approximate construction of the function. We shall deal later with the practical problem of the approximate construction of the function which performs the conformal transformation.

Let us now make one important addition to Riemann's theorem. If the contour of the domain is a simple closed curve and has the properties mentioned in [4], then the function $f(z)$ is continuous up to the contour of the domain B and transforms this contour into the circumference of a unit circle.

The inverse function is, in this case, not only regular in the unit circle but it is also continuous in the closed circle.

As we said above, the function which performs the conformal transformation of the given domain B into a unit circle can only be fully defined when the additional condition mentioned in the above formulation of Riemann's theorem is given. We can replace this additional condition by another condition, but we must still assume that the function effecting the conformal transformation is

continuous up to the contour. In doing so we can use the bilinear transformation of a unit circle into itself, so that three given points on the contour of the domain B are transformed into three given points on the circumference of the unit circle. In this case the function effecting the conformal transformation is fully defined. The former condition can also be formulated differently, viz. in the first place it is necessary that the given point z_0 in B should be transformed into the origin. We can then rotate the unit circle about the origin so that a given point on the contour of the domain B is transformed into a given point on the circumference of the unit circle, whence it can be shown that the function is fully defined.

Thus when fulfilling the conditions which guarantee the continuity of the function performing the conformal transformation as far as the contour of the domain B *we can fully define this function by freely choosing three points on the contour of the domain B which must correspond to three points on the circumference of the unit circle, or by freely choosing one interior point and one point on the contour of the domain B, which must correspond to two similar points of the unit circle.* If we have two connected domain B_1 and B_2 in the z-plane, then according to Riemann's theorem, we have two regular functions:

$$w_1 = f_1(z_1) \quad \text{and} \quad w_1 = f_2(z_2), \tag{40}$$

which transform these regions into a unit circle $|w_1| < 1$. Eliminating, in theory, the variable w_1 in the above equations, we obtain the regular function $z_2 = \varphi(z_1)$, which is transforms B_1 into B_2.

In this case every point z_1 corresponds to a point z_2, so that one and the same w_1 can be defined by both z_1 and z_2, according to (40). We can thus achieve the conformal transformation of any two connected domains (except for the two cases mentioned above) into each other. Simultaneously we can, of course, make the same additional conditions which are mentioned above in connection with the transformation of a region into a circle.

Notice one important property of the function $f(z)$ which transforms a connected domain into a circle or into another connected domain.

We take it that our domains are one-sheeted domains or, more strictly, that they can overlap but do not contain any branch-points. Also the derivative $f'(z)$ cannot vanish in the domain, since the vanishing of a derivative indicates the formation of a branch-point in the transformed domain [23]. We also notice that the functions $\log f'(z)$ and $\sqrt{f'(z)}$, can have no singularities in the course of the analytic

continuation within our connected domain B, and they will, therefore, be single-valued [18] and regular functions in this domain.

If we have not a connected domain but a domain with two boundaries in the z-plane, e.g. an annulus confined between two closed curves then it is obviously impossible to transform it conformally into a connected domain, so that every point of the annulus should correspond to a definite point of the connected domain and vice versa.

In the case of a multiply-connected domain there is one circumstance which makes this case different from the case of the simply-connected domain, viz. not every two domain with the same number of boundaries can be conformally transformed into each other. Thus, for example, two annulae confined between concentric circles can be conformally transformed into each other only when the proportion of the radii of the circles which confine these annulae, is the same in both cases.

In the case of a multiply-connected domain, one domain can still be transformed into another domain of a definite type, viz. any n-bounded domain can be transformed into a plane with n cuts which have the appearance of parallel sections of straight lines, and some of these cuts can degenerate into points.

Before considering the approximate methods for constructing the functions effecting a conformal transformation we shall develop an analytic expression for a function effecting the conformal transformation of a unit circle or of the upper half-plane into a domain confined by a broken line, i.e. a polygon. This formula frequently occurs in various applications.

36. Christoffel's formula. Let us suppose that we have a polygon $A_1, A_2, \ldots, A_n$ (Fig. 38) in the z-plane and let the angles of this polygon be $a_1\pi, a_2\pi, \ldots, a_n\pi$. Consider the function

$$z = f(t), \tag{41}$$

which conformally transforms the upper half-plane t into a polygon. We have to construct an analytic expression for this function. Assume that the following points A_k which lie on the real axis, correspond to the vertices of this polygon

$$t = a_k \quad (k = 1, 2, \ldots, n),$$

and take it that all these points lie at a finite distance; this can always be achieved by a bilinear transformation of the t-plane. Let a_1 be the extreme point on the left and a_n the extreme point on the right. We shall now consider the problem of

analytic continuation of the function $f(t)$ across the real axis. Take a certain part $a_k\, a_{k+1}$ of the real axis which corresponds to the side $A_k\, A_{k+1}$ of the polygon. In accordance with the principle of symmetry we can analytically continue the function $f(t)$ across the line $a_k\, a_{k+1}$ and we obtain as a result a new polygon in the lower half-plane which is produced by the reflection of the initial side in the side $A_k\, A_{k+1}$. We can then analytically continue the new function so obtained from the lower half-plane into the upper half-plane, across the line $a_l\, a_{l+1}$ on the real axis. As a result of the principle of symmetry the new values of $f(t)$ again give a polygon in the upper half-plane; this polygon is obtained from the second polygon by its reflection in that side of the second polygon which corresponds to the line $a_l\, a_{l+1}$ of the real axis, etc. We can thus see that it is possible to continue our function $f(t)$ without difficulty across the real axis and in doing so the function transforms any plane into a polygon, which is obtained from the initial polygon, after several reflections in sides which correspond to those parts of the real axis across which the analytic continuation was performed. Note that the side $A_n\, A_1$ corresponds to a line on the real axis from a_n to ∞ and from ∞ to a_1, so that the point at infinity of the t-plane corresponds to a point on the side $A_n\, A_1$ of the polygon. The points a_k will, in general, be singularities of the function $f(t)$. Let us investigate the character of these singularities. Take a point a_2 and describe a circuit around this point by starting from and returning to the upper half-plane. In doing this we have to cross from the upper half-plane to the lower half-plane by cutting across the line $a_1 a_2$ and, subsequently, we return from the lower half-plane to the upper half-plane across the line $a_2\, a_3$. It follows from what was said above that the values of $f(t)$ in the lower half-plane form a polygon $A_1 A_2 A_3 \ldots A_n$, which

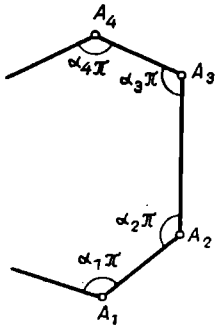

FIG. 38

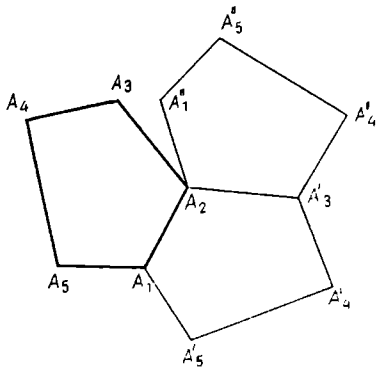

FIG. 39

is obtained from the initial polygon as a result of the reflection in the side A_1A_2; the return to the upper half-plane involves a reflection in the side A_2A_3 of this new polygon (Fig. 39).

Thus the above circuit round the point a_2, corresponds to the reflection in the straight lines A_2A_1 and A_2A_3', i.e. it follows from [31] that a linear transformation of the form $z' - b_2 = e^{i\varphi}(z - b_2)$ is involved, where b_2 is the coordinate of the point A_2.

It follows from this that

$$\overset{*}{f}(t) = e^{i\varphi} f(t) + \gamma,$$

where γ is a constant ($\gamma = b_2 - e^{i\varphi} b_2$) and $f^*(t)$ is a new branch of $f(t)$ in the upper half-plane.

It also follows that

$$\frac{\overset{*}{f}''(t)}{\overset{*}{f}'(t)} = \frac{f''(t)}{f'(t)},$$

i.e. the function

$$\frac{f''(t)}{f'(t)} \tag{42}$$

is regular and single-valued in the neighbourhood of the point a_2 this point can either be a pole or an essential singularity of the function (42). We shall show that this point is a simple pole with a residue $(a_2 - 1)$. In fact we can replace z by a new complex variable z':

$$z' = (z - b_2)^{\frac{1}{a_2}},$$

where b_2 is the coordinate of the vertex A_2. The value $z' = 0$ corresponds to this vertex and the sides A_2A_1 and A_2A_3, which cut at an angle $a_2\pi$ are transformed into two straight lines, which cut at an angle π i.e. the above two straight lines are transformed into two parts of one and the same line l in the z'-plane, which go from the origin in opposite directions. If we now return to the t-plane, we can see that the neighbourhood of the point a_2, situated above the real axis is transformed into the neighbourhood of the point $z' = 0$ in the z'-plane which lies to one side of the straight line l. It follows from the principle of symmetry that the same condition applies in the neighbourhood of the points $t = a_2$ and $z' = 0$ which lie on the other side of the above lines. Hence the neighbourhood of the point $t = a_2$ is transformed into a one-sheeted neighbourhood of the point $z' = 0$ and we have an expansion of the form:

$$z' = (z - b_2)^{\frac{1}{a_2}} = c_1(t - a_2) + c_2(t - a_2)^2 + \ldots \qquad (c_1 \neq 0).$$

It follows that:

$$z = b_2 + c_1^{a_2}(t-a_2)^{a_2}\left\{1 + \frac{c_2}{c_1}(t-a_2) + \frac{c_3}{c_1}(t-a_2)^2 + \ldots\right\}^{a_2}$$

or, by applying the binomial expansion [cf. 23]:

$$f(t) = b_2 + (t-a_2)^{a_2} f_1(t),$$

where $f_1(t)$ is regular and does not vanish at the point $t = a_2$. Hence:

$$f'(t) = a_2(t-a_2)^{a_2-1} f_1(t) + (t-a_2)^{a_2} f_1'(t),$$

$$f''(t) = a_2(a_2-1)(t-a_2)^{a_2-2} f_1(t) + 2a_2(t-a_2)^{a_2-1} f_1'(t) + (t-a_2)^{a_2} f_1''(t)$$

and consequently:

$$\frac{f''(t)}{f'(t)} = \frac{1}{t-a_2} \cdot \frac{a_2(a_2-1) f_1(t) + 2a_2(t-a_2) f_1'(t) + (t-a_2)^2 f_1''(t)}{a_2 f_1(t) + (t-a_2) f_1'(t)}.$$

The second factor on the right hand side is a regular function at the point $t = a_2$ where it is equal to $(a_2 - 1)$, i.e. in the neighbourhood of the point $t = a_2$ the following expansion applies:

$$\frac{f''(t)}{f'(t)} = \frac{a_2 - 1}{t-a_2} + P(t-a_2),$$

where $P(t-a_2)$ is a regular function at the point $t = a_2$.

It can similarly be shown that the function (42) has at every point a_k on the real axis a pole of the first order with a residue $(a_k - 1)$. Also, as we know, our function has no other singularities at a finite distance and therefore the difference

$$\frac{f''(t)}{f'(t)} - \sum_{s=1}^{n} \frac{a_s - 1}{t - a_s} \tag{43}$$

is a regular single-valued function in the whole plane. We shall now explain the behaviour of the function (43) at infinity. As we saw above, the function $f(t)$ tends to a definite value at infinity, viz. to the coordinate b_- of that point in the side $A_n A_1$, which corresponds to $t = \infty$; consequently, in the neighbourhood of the point at infinity we have an expansion of the type

$$f(t) = b_\infty + \frac{c_1}{t} + \frac{c_2}{t^2} + \ldots$$

It follows that the function $f''(t)/f'(t)$ can be expanded as follows at infinity:

$$\frac{f''(t)}{f'(t)} = \frac{d_1}{t} + \frac{d_2}{t^2} + \ldots,$$

i.e. the function tends to zero when $t \to \infty$. We thus see that the function (43), which is regular in the whole plane, tends to zero when $t \to \infty$ and, consequently, it is bounded in the whole plane. Liouville's theorem [9], therefore states that the expression (43) must be constant, but from what we have just seen, it tends to zero as $t \to \infty$; it therefore follows that the constant must be equal to zero. We thus have the equation:

$$\frac{f''(t)}{f'(t)} = \frac{a_1 - 1}{t - a_1} + \frac{a_2 - 1}{t - a_2} + \ldots + \frac{a_n - 1}{t - a_n}. \tag{44}$$

Integrating once we obtain:

$$\log f'(t) = (a_1 - 1)\log(t - a_1) + (a_2 - 1)\log(t - a_2) + \ldots + \\ + (a_n - 1)\log(t - a_n) + C$$

or

$$f'(t) = A(t - a_1)^{a_1 - 1}(t - a_2)^{a_2 - 1}\ldots(t - a_n)^{a_n - 1},$$

and, finally, by integrating once again, we get the result:

$$z = f(t) = A\int_0^t (t - a_1)^{a_1 - 1}(t - a_2)^{a_2 - 1}\ldots(t - a_n)^{a_n - 1}\,dt + B, \tag{45}$$

where A and B are constants. Our problem has thus been solved and *the conformal transformation of the upper half-plane t into a polygon with angles $a_k \pi$ is given by the formula* (45), *where a_k are points on the real axis, and A and B are complex constants.*

We will, first of all, explain the importance of these constants. Above we only used the magnitude of the angles of our polygon. Thus when the polygon is subjected to movement or even to an identity transformation we conserve the angles and therefore formula (45) should also apply to the new polygon. The importance of the constants A and B is due to the fact that when altering their magnitude we pass from one polygon to another similar polygon. The part played by the numbers a_k in formula (45) is of much greater importance. The position of these points on the real axis together with the value of the constant A give the lengths of the sides of the polygon. We shall return to this problem later.

In deducing formula (45) we assumed that all the vertices of the polygon have corresponding points on the real axis which lie within a finite distance. We now suppose that one of the vertices, say A_n, corresponds to the point at infinity. This transformation can easily be obtained from

the one above by replacing t by a new variable τ according to the formula

$$t = -\frac{1}{\tau} + a_n,$$

since when $t = a_n$ we have $\tau = \infty$.

Completing the change of variables we obtain:

$$f(\tau) = A \int_{\tau_0}^{\tau} \left(a_n - a_1 - \frac{1}{\tau}\right)^{\alpha_1 - 1} \cdots$$
$$\cdots \left(a_n - a_{n-1} - \frac{1}{\tau}\right)^{\alpha_{n-1}-1} \left(-\frac{1}{\tau}\right)^{\alpha_n - 1} \frac{d\tau}{\tau^2} + B.$$

Notice also that as a result of the well-known property of the sum of angles of a polygon we have:

$$a_1 + a_2 + \ldots + a_n = n - 2. \tag{46}$$

Using this relationship and changing the symbols of the constants, the above formula can be rewritten in the following form:

$$f(\tau) = A' \int_0^{\tau} (\tau - a_1')^{\alpha_1 - 1}(\tau - a_2')^{\alpha_2 - 1} \ldots (\tau - a_{n-1}')^{\alpha_{n-1}-1} d\tau + B'. \tag{47}$$

This formula applies when one vertex of the polygon corresponds to the point at infinity $\tau = \infty$.

From formula (45) it is not difficult to obtain a formula which gives the conformal transformation of a unit circle $|w| < 1$ into our polygon. It is sufficient to apply the bilinear transformation which transforms the upper half-plane t into the unit circle $|w| < 1$. This transformation takes the form:

$$w = \frac{t - i}{t + i}, \quad \text{or} \quad t = \frac{1}{i} \frac{w + 1}{w - 1}.$$

Substituting the expression for t in formula (45) and using (46) we arrive at the following formula:

$$z = A'' \int_0^{w} (w - a_1'')^{\alpha_1 - 1} (w - a_2'')^{\alpha_2 - 1} \ldots (w - a_n'')^{\alpha_n - 1} dw + B'', \tag{48}$$

where the points a_k'' lie on the circumference of a unit circle and are defined with respect to the points a_k by the formula

$$a_k'' = \frac{a_k - i}{a_k + i}.$$

In the formulae (47) and (48) we altered the lower limit of integration; this is of no great significance as it only affects the values of the constants B' and B''.

Let us recall the assumptions made in deducing the formula (45). We assumed that there is a function $f(t)$ which transforms the upper half-plane into our polygon and we then obtained the expression (45) for this function. Let us now investigate the formula (45); assume that a_k are given points on the real axis and α_k are positive numbers which satisfy the condition (46). We shall show that under these circumstances formula (45) transforms the upper half-plane into a domain without branch-points (one-sheeted or many-sheeted), the contour of which is a broken line with angles $\alpha_k \pi$ ($k = 1, 2, \ldots, n$). To begin with note that every factor $(t - a_k)^{\alpha_k - 1}$ of the integrand is a regular and single-valued function in the upper half-plane, and the derivative

$$f'(t) = A(t - a_1)^{\alpha_1 - 1}(t - a_2)^{\alpha_2 - 1} \ldots (t - a_n)^{\alpha_n - 1}$$

does not vanish anywhere in the upper half-plane. Formula (45) thus gives the conformal transformation of the upper half-plane into a domain B in the z-plane which contains no branch-points. Let us now consider the form which the contour of the half-plane, viz. the real axis, acquires as a result of the transformation. Let us suppose that t varies in the interval $a_1 \leqslant t \leqslant a_2$ on the real axis. The corresponding part of the contour of the domain B can be represented by the equation

$$z = A \int_{a_1}^{t} (t - a_1)^{\alpha_1 - 1}(t - a_2)^{\alpha_2 - 1} \ldots (t - a_n)^{\alpha_n - 1}\, dt + C, \qquad (49)$$

where C is a constant, which is given in terms of the former constants by the formula:

$$C = B + A \int_{0}^{a_1} (t - a_1)^{\alpha_1 - 1}(t - a_2)^{\alpha_2 - 1} \ldots (t - a_n)^{\alpha_n - 1}\, dt.$$

When t varies in the above interval each of the differences $t - a_k$ has a constant amplitude which we denote by φ_k. We can, obviously take it that $\varphi_1 = 0$ and $\varphi_k = \pi$ when $k > 1$ ($a_1 < a_2 < \ldots < a_n$). The amplitude of the integrand in the expression (49) always remains constant, viz. it is equal to

$$(\alpha_1 - 1)\varphi_1 + (\alpha_2 - 1)\varphi_2 + \ldots + (\alpha_n - 1)\varphi_n = \varphi.$$

Formula (49) can therefore be written in the form:

$$z = Ae^{i\varphi} \int_{a_1}^{t} |t - a_1|^{\alpha_1 - 1} |t - a_2|^{\alpha_2 - 1} \dots |t - a_n|^{\alpha_n - 1} \, dt + C, \qquad (50)$$

where we integrate along the interval of the real axis and in fact the above integral is real. It can be seen directly from formula (50) that the line $a_1 \leqslant t \leqslant a_2$ on the real axis corresponds to the line A_1A_2 in the half-plane z, which starts at the origin and makes an angle arg $(Ae^{i\varphi})$ with the real axis. In the transition from $a_1 \leqslant t \leqslant a_2$ to $a_2 \leqslant t \leqslant a_3$ we have to translate the point a_2 from the upper half-plane. The amplitude of the difference $(t - a_2)$ then receives an increment $(-\pi)$ and the amplitude of the factor $(t - a_2)^{\alpha_2 - 1}$ receives an increment $-\pi\,(a_2 - 1)$. Thus in the next interval $a_2 \leqslant t \leqslant a_3$ we have a formula analogous with (50) and only the amplitude φ has a new value which differs from the one above by the term $-\pi\,(a_2 - 1)$, i.e. this interval $a_2 \leqslant t \leqslant a_3$ corresponds to the line A_2A_3 in the z-plane so that the angle between the lines A_1A_2 and A_2A_3 is equal to $(\pi - a_2\,\pi)$.

We finally consider the point at infinity of the t-plane. For this purpose we can rewrite the integrand in the formula (45) in the form

$$t^{\alpha_1 + \alpha_2 \dots + \alpha_n - n} \left(1 - \frac{a_1}{t}\right)^{\alpha_1 - 1} \left(1 - \frac{a_2}{t}\right)^{\alpha_2 - 1} \dots \left(1 - \frac{a_n}{t}\right)^{\alpha_n - 1}.$$

Applying Newton's binomial theorem and using the relationship (46) we obtain the following expansion for the integrand in the neighbourhood of the point at infinity:

$$\frac{1}{t^2} + \frac{C_3}{t^3} + \frac{C_4}{t^4} + \dots,$$

and after integration the right-hand side of the formula (45) is as follows:

$$d_0 + \frac{d_1}{t} + \frac{d_2}{t^2} + \dots,$$

i.e. the point $t = \infty$ given by formula (45) is a regular point for the function $f(t)$. Hence when the real axis of the t-plane passes through ∞ we obtain, as for other segments of the real axis, part of a straight line in the z-plane. Note that as a result of the condition $a_k > 0$ the integral (45) has a fully defined finite value at the point $t = a_k$. Hence the above hypothesis with regard to the transformation by the function (45) has been proved. As we have already mentioned

above the polygon obtained can cross itself (Fig. 40). The same applies also to the formulae (47) and (48).

Thus, for example, formula (48) gives the conformal transformation of the unit circle into the domain B bounded by a broken line which contains no branch-points, provided the points a_k'' on the unit circumference and the constants a_k' are chosen arbitrarily to satisfy the condition (46).

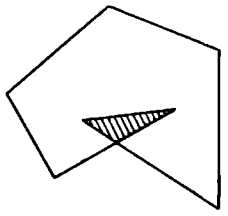

FIG. 40

37. Individual cases. Let us begin with the simplest case, viz. the triangle. Applying the bilinear transformation to the t-plane we can always simplify our problem by making the vertices of the triangle correspond to the points $t = 0$, 1 and ∞. To do this we use the formula (47) and assume that $a_1' = 0$ and $a_2' = 1$, whence we obtain the following formula

$$z = A' \int_0^\tau \tau^{\alpha_1 - 1} (\tau - 1)^{\alpha_2 - 1} \, d\tau + B'. \tag{51}$$

In this case our formula only includes the arbitrary constants A' and B'; these constants are of no significance since they are connected with an identity transformation of a triangle. The comparative simplicity of formula (51) is due to the fact that any two triangles with equal angles are necessarily similar. In the case of a quadrangle this circumstance no longer applies and the integrand in the general formula for a quadrangle with given angles, contains an undefined parameter, which depends on the lengths of the sides of the polygon.

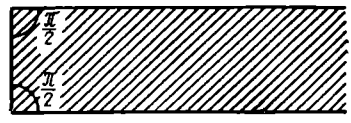

FIG. 41

Formula (51) also applies to an infinite triangle with angles $\pi/2$, $\pi/2$ and 0. This triangle represent a strip, bounded by two parallel halves of straight lines and a perpendicular line (Fig. 41). Assuming that $\alpha_1 = \alpha_2 = 1/2$ we have in formula (51):

$$z = A' \int_0^\tau \frac{d\tau}{\sqrt{\tau(\tau - 1)}} + B'.$$

Let us consider this rectangle in greater detail. Assume that the vertices of the rectangle B have the following coordinates (Fig. 42):

$$-\frac{\omega_1}{2}, \quad \frac{\omega_1}{2}, \quad \frac{\omega_1}{2}+i\omega_2, \quad -\frac{\omega_1}{2}+i\omega_2,$$

where ω_1 and ω_2 are given real positive numbers. Take the right half of this rectangle with the vertices

$$0, \frac{\omega_1}{2}, \frac{\omega_1}{2}+i\omega_2, \quad i\omega_2,$$

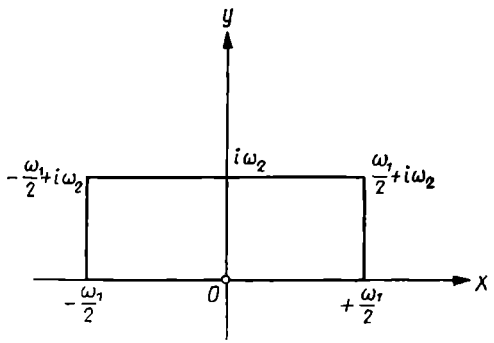

FIG. 42

and assume that it is conformally reflected in the right half of the upper half-plane t, i.e. in that half of the upper half-plane t where the real parts of all points are positive. We can take it that the vertices 0, $\omega_1/2$ and $i\omega_2$ correspond to the points 0, 1 and ∞ on the contour of the above right part of the upper half-plane. The vertex $\omega_1/2+i\omega_2$ corresponds to a point on the real axis, which lies between the points 1 and ∞. Denote this point by $1/k$, where $0<k<1$. As a result of the principles of symmetry the left side of our rectangle corresponds to the left half of the upper half-plane t and the vertices $-\omega_1/2$, $-\omega'/2+i\,\omega_2$ correspond to the points $t=-1$ and $t=-1/k$. It follows from above that we can always conformally transform the upper half-plane into a rectangle B, so that the points $t=-1, 0, 1, \infty$ should correspond to the points $z=\omega_1/2, 0, \omega_1/2, i\,\omega_2$, while the points $t=1/k$ and $t=-1/k$ should correspond to $z=\omega_1/2+i\,\omega_2$ and $z=-\omega_1/2+i\,\omega_2$. We can now apply formula (45), assuming that $a_1=-1/k$; $a_2=-1$; $a_3=1$; $a_4=1/k$ and $\alpha_1=\alpha_2=\alpha_3=\alpha_4=1/2$.

We thus obtain the following formula, bearing in mind that $t = 0$ when $z = 0$:

$$z = A' \int_0^t \frac{\mathrm{d}t}{\sqrt{(1-t^2)\left(\frac{1}{k^2} - t^2\right)}}$$

i.e. the following formula:

$$z = A \int_0^t \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{52}$$

The values of t in the interval $-1 < t < 1$ on the real axis, give the line $(-\omega_1/2, +\omega_1/2)$ on the real axis in the z-plane. It follows that we can assume in formula (52) that A is a positive constant and that the radical is unity when $t = 0$. Other values of the radical in the upper half-plane are unique, since this radical is a regular function without branch-points in this half-plane. Bearing in mind that the vertices $\omega_1/2$ and $\omega_1/2 + i\,\omega_2$ correspond to the values $t = 1$ and $t = 1/k$, we obtain the following formulae:

$$\left.\begin{aligned} \frac{\omega_1}{2} &= A \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}, \\ \omega_2 &= A \int_1^{\frac{1}{k}} \frac{\mathrm{d}t}{\sqrt{(t^2-1)(1-k^2t^2)}}. \end{aligned}\right\} \tag{53}$$

The lengths of the sides of our rectangle are equal to ω_1 and ω_2; we can therefore construct an equation to determine the parameter k, which enters the integrand, from our knowledge of the relationship of the lengths of the sides of the rectangle:

$$\omega_1 : \omega_2 = 2 \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} : \int_1^{\frac{1}{k}} \frac{\mathrm{d}t}{\sqrt{(t^2-1)(1-k^2t^2)}}. \tag{54}$$

Having thus determined k (theoretically speaking) we can proceed to determine A from one of the equations (53).

The integral in formula (52) cannot be expressed by elementary functions and is known as an elliptic integral of the first class in the Legendre form. We shall deal with these integrals later and therefore we shall not investigate the problem of finding k in the equations (54) in greater detail here. The above argument was introduced so

as to explain more clearly the problem of the determination of k in Christoffel's formula.

Let us consider one more case. Suppose that we have a regular n-polygon $A_1A_2 \ldots A_n$ in the z-plane and let $z = 0$ be its centre (Fig. 43 for $n = 6$). Take the conformal transformation of the triangle OA_1A_2 into the sector $O'A_1'A_2'$ of a unit circle with the angle at the centre equal to $2\pi/n$, so that the vertices of the triangle O, A_1 and A_2 should correspond to the centre of the circle O' and to the ends A_1' and A_2' of the arc. According to the principle of symmetry the reflection of

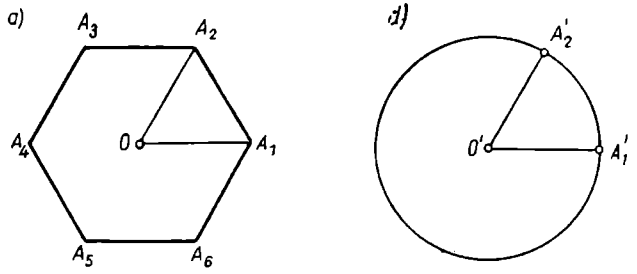

FIG. 43

the triangle in its sides gives the reflection of the sector in the corresponding radii. Thus in the course of analytic continuation the function reflects the whole regular polygon into a unit circle. It follows directly from these considerations that by reflecting a regular polygon into a unit circle, the vertices of the polygon correspond to points, which divide the circumference of a unit circle into equal parts. In this case we must also assume in formula (48) that

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{n-2}{n} = 1 - \frac{2}{n}.$$

Rotating the unit circle about the origin, we can, of course, take it that the vertix A_1 corresponds to any arbitrary point on the circumference, for example to the point $w = 1$. At the same time the points $e^{i2\pi k/n}$ ($k=1, 2, \ldots, n-1$), on the circumference also correspond to vertices of the polygon so that the integrand in formula (48) has the following form:

$$\left[(w-1)\left(w - e^{i\frac{2\pi}{n}}\right)\left(w - e^{i\frac{4\pi}{n}}\right) \ldots \left(w - e^{i(n-1)\frac{2\pi}{n}}\right)\right]^{-\frac{2}{n}}.$$

If we suppose that the origin is the centre of the polygon we obtain the following formula for the transformation of a unit circle into a

regular n-polygon

$$z = A'' \int_0^w \frac{dw}{\sqrt[n]{(w^n - 1)^2}} . \tag{55}$$

The modulus of the constant A'' is determined from the dimensions of the polygon and the amplitude of this constant simply gives the rotation of the polygon about the origin.

38. The exterior of the polygon. Let us now consider that part of the plane outside the broken line (Fig. 44). In this case our domain, which we can also call a polygon, contains the point at infinity. We construct the function $z = f(w)$ which conformally transforms a unit circle into our infinite polygon. In this case the sum of the angles of the polygon is equal to $\pi(n + 2)$; denoting the angles by $\alpha_k \pi$, as before, we obtain instead of (46) the following relationship:

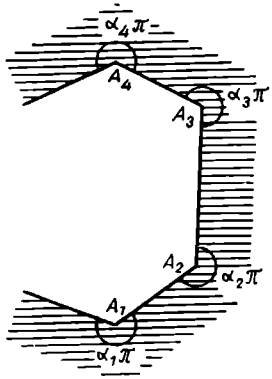

FIG. 44

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = n + 2. \tag{56}$$

We suppose that the origin $w = 0$ is transformed into the point at infinity. The function $f(w)$ will then have a simple pole at the origin and the function $f'(w)$ can be expanded as follows in the neighbourhood of the origin:

$$f'(w) = \frac{c_{-2}}{w^2} + c_0 + c_1 w + \ldots \tag{57}$$

We now suppose that a_k'' are the points on the circumference of the unit circle which correspond to the vertices of our polygon. We construct the function $f''(w)/f'(w)$ as before. By proceeding in the same way as in [36] we can see that this function has a simple pole at every point a_k'' with a residue $\alpha_k - 1$. Also, as a result of (57), we see that it has a simple pole at the origin with a residue (-2) and at all other points it is regular, as before. Let us investigate its behaviour at infinity. The function $f(w)$ is equal to infinity at the origin, and in the course of analytic continuation across any arbitrary arc $a_k'' a_{k+1}''$ of the unit circle it acquires the value infinity, symmetrical with the value of this function at the origin with respect to that side of the polygon which corresponds to the arc $a_k'' a_{k+1}''$ of the unit circle, i.e. the value of the function $f(w)$ at infinity is equal to infinity and our function transforms the neigh-

bourhood of infinity into a one-sheet neighbourhood of infinity. (This is that part of the polygon obtained from the given polygon as the image in the side $A_k A_{k+1}$). Hence the values of $f(w)$ obtained after the above analytic continuation in the neighbourhood of infinity can be expanded as follows:

$$f(w) = d_{-1} w + d_0 + \frac{d_1}{w} + \dots \qquad (d_{-1} \neq 0). \tag{58}$$

The function $f''(w)/f'(w)$ is single-valued and regular in the whole plane except at the above poles. Differentiating formula (58) we obtain the following expansion for the function $f''(w)/f'(w)$ in the neighbourhood of infinity:

$$\frac{f''(w)}{f'(w)} = \frac{h_3}{w^3} + \frac{h_4}{w^4} + \dots, \tag{59}$$

where the direction of analytic continuation is irrelevant owing to the single-valuedness of the function. Thus in the case under consideration, the function $f''(w)/f'(w)$ has the above poles and vanishes at infinity, but otherwise it is regular. Carrying our argument further, as in [36] instead of formula (44) we obtain the following:

$$\frac{f''(w)}{f'(w)} = -\frac{2}{w} + \frac{a_1 - 1}{w - a_1''} + \frac{a_2 - 1}{w - a_1''} + \dots + \frac{a_n - 1}{w - a_n''}, \tag{60}$$

this gives the following formula instead of (45):

$$z = A \int_1^w (w - a_1'')^{a_1 - 1} (w - a_2'')^{a_2 - 1} \dots (w - a_n'')^{a_n - 1} \frac{dw}{w^2} + B. \tag{61}$$

If we apply the transformation $w = 1/\tau$ to the variable w then the interior of the unit circle is transformed into the exterior of the unit circle and, after performing corresponding transpositions in the integral (61), we obtain a formula which transforms the exterior of a unit circle into that part of the plane outside the broken line, where the points at infinity correspond with each other:

$$z = A' \int_1^\tau (\tau - a_1)^{a_1 - 1} (\tau - a_2)^{a_2 - 1} \dots (\tau - a_n)^{a_n - 1} \frac{d\tau}{\tau^2} + B. \tag{62}$$

The form of this formula is the same as that of formula (61).

Let us consider, for example, the transformation of that part of the plane outside a square. Owing to symmetry we obtain points a_k, which divide the circumference of the unit circle into equal parts and, by rotating this circumference, we can take it that these points will be

$$a_1 = 1; \quad a_2 = i; \quad a_3 = -1; \quad a_4 = -i.$$

In this case we have the following relationship for the angles:

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{3}{2},$$

and therefore the final formula has the form:

$$z = A' \int_1^{\tau} \sqrt{\tau^4 - 1}\, \frac{d\tau}{\tau^2} + B. \qquad (63)$$

The values of the constants A' and B depend on the dimensions of the square and its position in the plane.

Note that at the beginning of [36] we considered the transformation of the half-plane into a polygon and in this paragraph we applied the result obtained to the transformation of a circle into a polygon. It can easily be seen that all arguments used in [36] also apply here.

Let us make one remark in connection with formula (62). We expand the integrand in the neighbourhood of the point at infinity. From (56) we can rewrite this function in the form:

$$\left(1 - \frac{a_1}{\tau}\right)^{\alpha_1 - 1} \cdot \left(1 - \frac{a_2}{\tau}\right)^{\alpha_2 - 1} \ldots \left(1 - \frac{a_n}{\tau}\right)^{\alpha_n - 1},$$

or applying Newton's binomial theorem, we obtain the following expansion:

$$1 - \frac{(\alpha_1 - 1)\, a_1 + (\alpha_2 - 1)\, a_2 + \ldots + (\alpha_n - 1)\, a_n}{\tau} + \frac{c_2}{\tau^2} + \frac{c_3}{\tau^3} + \ldots$$

When integrating the term containing $1/\tau$ we obtain the function $\log \tau$ and, consequently, in order that the neighbourhood of the point $\tau = \infty$ should correspond to a one-sheeted domain it is necessary (and sufficient) that the constants a_k should satisfy the relationship:

$$(\alpha_1 - 1)\, a_1 + (\alpha_2 - 1)\, a_2 + \ldots + (\alpha_n - 1)\, a_n = 0. \qquad (64)$$

If this relationship is not satisfied then formula (62) transforms the outside of a unit circle $|\tau| > 1$ into the domain B bounded by a broken line with a branch-point of the logarithmic type at infinity.

Formula (62), as we noticed above, can also be used for the transformation of the interior of the unit circle $|\tau| < 1$ into an infinite polygon, when the point $\tau = 0$ is transformed into the point at infinity. If we expand the integrand in the neighbourhood of the point $\tau = 0$ and cancel the terms containing $1/\tau$, we obtain the condition:

$$(\alpha_1 - 1)\frac{1}{a_1} + (\alpha_2 - 1)\frac{1}{a_2} + \ldots + (\alpha_n - 1)\frac{1}{a_n} = 0.$$

It is identical with (64) since it is given that $|a_k| = 1$; hence it is given that $a_k^{-1} = \bar{a}_k$ and that the numbers a_k, are real (and positive).

39. The minimum property of the transformation into a circle. Consider the function

$$z = f(\tau) = \tau + c_2\tau^2 \ldots. \tag{65}$$

which is regular in the circle $|\tau| < R$. This function transforms the circle into a domain B which can have several sheets and which can contain branch-points. The circle $|\tau| < R_1$, where $R_1 < R$, is transformed by (65) into a part of the domain B, which we denote by B_1. Let us determine the surface area of this domain. As we know it can be expressed by the integral [29]:

$$S_1 = \int\int_{|\tau|<R_1} |f'(\tau)|^2 \, ds,$$

where we integrate round the circle $|\tau| < R_1$. We can rewrite this integral as follows:

$$S_1 = \int_0^{R_1}\int_0^{2\pi} (1 + 2c_2 re^{i\varphi} + 3c_3 r^2 e^{i2\varphi} + \ldots)(1 + 2\bar{c}_2 re^{-i\varphi} + 3\bar{c}_3 r^2 e^{-i2\varphi} + \ldots) r \, dr \, d\varphi.$$

Owing to the absolute and uniform convergence of the series in the circle $|\tau| < R_1$ we can multiply our two series term by term and we can also integrate them term by term. Notice that by integrating the function $e^{ik\varphi}$ through the interval $(0, 2\pi)$, where k is an integer other than zero, we obtain zero. Thus when multiplying the above series it is sufficient to retain only those terms which do not contain factors of the type $e^{ik\varphi}$, and integration with respect to φ will then simply involve multiplication by 2π. We thus obtain:

$$S_1 = 2\pi \int_0^{R_1} (1 + 2^2 |c_2|^2 r^2 + \ldots + n^2 |c_n|^2 r^{2n-2} + \ldots) r \, dr$$

or

$$S_1 = \pi R_1^2 + \pi \sum_{n=2}^{\infty} n |c_n|^2 R_1^{2n}. \tag{66}$$

As R_1 tends to R, the latter increases and tends either to a finite limit or to infinity. In any case this limit, which gives the surface area of the domain B, will be greater than πR^2, i.e. greater than the surface area of the initial circle $|\tau| < R$, provided that in the expansion (65) at least one of the coefficients c_k is other than zero. We thus obtain the following result. *In the transformation of the circle* $|\tau| < R$ *by the function* (65), *which is regular in this circle, the surface area of the domain increases, provided at least one of the coefficients* c_k *is other than zero.*

Having established this preliminary theorem we can now explain one very important property of a function performing conformal transformation. Let B be a connected bounded domain in the z-plane and assume that the origin $z = 0$ lies in this domain. Suppose also that $F_1(z)$ is a function which

conformally transforms B into a unit circle and that the origin $z = 0$ is transformed into the centre of this circle. In the neighbourhood of the point $z = 0$ this function can be expanded as follows:

$$F_1(z) = d_1 z + d_2 z^2 + \ldots,$$

where $d_1 > 0$. Let us now consider a new function

$$F(z) = \frac{1}{d_1} F_1(z).$$

This function transforms B into the unit circle $|\tau| < R$, where $R = 1/d_1$, and its expansion in the neighbourhood of $z = 0$ is:

$$\tau = F(z) = z + a_2 z^2 + a_3 z^3 + \ldots \tag{67}$$

Its inverse function is regular in the circle $|\tau| < R$ where it can be expanded as follows:

$$z = f(\tau) = \tau + c_2 \tau^2 + c_3 \tau^3 + \ldots \tag{68}$$

The double integral

$$\iint_B |F'(z)|^2 \, ds, \tag{69}$$

which gives the surface area of the circle must be equal to πR^2. If instead of the function $F(z)$ we take any other function $\varphi(z)$, which is regular in B and has an expansion of the form (67) in the neighbourhood of the point $z = 0$, then by substituting for z in the expansion (68) we obtain a function τ which is regular in the circle $|\tau| < R$ where it can be expanded as follows:

$$\varphi(z) = \varphi[f(\tau)] = \tau + e_2 \tau^2 + e_3 \tau^3 + \ldots = f_1(\tau). \tag{70}$$

Let us evaluate the double integral (69) for this new function $\varphi(z)$. Changing to the τ-plane and recalling the expression for an element of surface area in the τ-plane in terms of an element in the z-plane [29]:

$$ds_z = |f'(\tau)|^2 \, ds_\tau,$$

we find:

$$\iint_B |\varphi'(z)|^2 \, ds_z = \iint_{|\tau|<R} |\varphi'(z) \cdot f'(\tau)|^2 \, ds_\tau = \iint_{|\tau|<R} |f_1'(\tau)|^2 \, ds_\tau,$$

and in accordance with the above hypothesis this integral will be greater than πR^2, provided that at least one of the coefficients e_k in the expansion (70) is other than zero. If all coefficients are equal to zero, i.e. $\varphi(z) = \tau$ then evidently $\varphi(z) = F(z)$. We thus arrive at the following theorem.

THEOREM. *Among all the functions which are regular in B and have in the neighbourhood of $z = 0$ an expansion of the form* (67) *there is one function which conformally transforms B into a circle, centre the origin, and which gives the integral* (69) *its minimum value.*

This theorem can be used for the construction of an approximate expression for the function $F(z)$ which transforms B into a circle, in the form of a polynomial. Hence $F(z)$ can be approximately represented by a polynomial of the nth degree:

$$F(z) = z + a_2 z^2 + \ldots + a_n z^n, \tag{71}$$

and we shall determine the coefficients a_k of this polynomial from the condition that the polynomial (71) alone among all other polynomials of a similar kind, gives the integral (69) its minimum value. Let us construct an arbitrary polynomial

$$\omega(z) = b_2 z^2 + b_3 z^3 + \ldots + b_n z^n$$

and subsequently construct a new polynomial which has the same form (71) as the polynomial $F(z)$:

$$\Phi(z) = F(z) + \varepsilon\omega(z),$$

where ε is a real parameter. Construct the integral (69) for this new polynomial

$$\iint_B [F'(z) + \varepsilon\omega'(z)]\,[\overline{F'(z)} + \overline{\varepsilon\omega'(z)}]\,ds.$$

This function of ε should reach its minimum value when $\varepsilon = 0$. Equating to zero its derivative with respect to ε when $\varepsilon = 0$ we obtain the following condition:

$$\iint_B [F'(z)\,\overline{\omega'(z)}] + [\overline{F'(z)}\,\omega'(z)]\,ds = 0, \tag{72_1}$$

which should apply for every arbitrary choice of the polynomial $\omega(z)$.

Similarly, substituting $i\varepsilon$ for ε, where ε is real, we obtain instead of (72_1) the condition

$$\iint_B [F'(z)\,\overline{\omega'(z)} - \overline{F'(z)}\,\omega'(z)]\,ds = 0. \tag{72_2}$$

On adding, we obtain the condition

$$\iint_B F'(z)\,\overline{\omega'}(z)\,ds = 0.$$

Assuming that $\omega(z)$ is, in turn, equal to

$$\omega(z) = z^2, z^3, \ldots, z^n$$

and introducing the symbol:

$$p_{lk} = \iint_B \bar{z}^l z^k\,ds, \tag{73}$$

we obtain the following system of equations of the first order for the required coefficients a_k of the polynomial (71):

$$\left.\begin{aligned} p_{10} + 2p_{11}a_2 + 3p_{12}a_3 + \ldots + np_{1,n-1}a_n &= 0,\\ p_{20} + 2p_{21}a_2 + 3p_{22}a_3 + \ldots + np_{2,n-1}a_n &= 0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\ p_{n-1,0} + 2p_{n-1,1}a_2 + 3p_{n-1,2}a_3 + \ldots + np_{n-1,n-1}a_n &= 0. \end{aligned}\right\} \tag{74}$$

Hence this problem must involve the evaluation of integrals of the type (73).

If a simple closed curve, which does not cross itself, serves as the contour of the domain, then it can be shown that the above polynomials tend uniformly as $n \to \infty$, to a function in B, which reflects B into a circle.

In conclusion we shall make a remark about the first theorem proved in this section. The function (65) transforms the circle $|\tau| < R$ into a domain B which can have extremely complicated geometric properties, viz. it can have several sheets and the form of the contour can be very complex. It can be shown that such a domain may not even have a surface area in the usual sense of the word and what we have called the surface area of the domain must be understood as the limit of the surface areas of the domains B_1, which lie in B and which expand in such a way that every point of B which falls within these domains makes them tend to B as their limit. If B has a surface area in the usual sense then the latter evidently coincides with the above limit.

40. The method of conjugate trigonometric series. We shall now give another method for the approximate construction of a function which conformally transforms the connected domain B into a circle. In this case we obtain this approximate representation in the form of a polynomial which does not lie in the z-plane of the domain B, as was the case above, but in the τ-plane of the unit circle. For the sake of simplicity we assume that the centre of the circle is transformed into the origin which lies in B. Let

$$z = a_1\tau + a_2\tau^2 + \dots \tag{75}$$

be a function which transforms the unit circle $C(|\tau| < 1)$ into B. If the contour of B is a simple closed curve, then it can be shown that the series (75) will converge uniformly within and on the circumference of the closed circle C. On the circumference we assume that $\tau = e^{i\varphi}$, where $0 \leqslant \varphi \leqslant 2\pi$; we then obtain the equation for the contour Γ of our domain B:

$$z = x + iy = a_1 e^{i\varphi} + a_2 e^{i2\varphi} + a_3 e^{i3\varphi} + \dots \tag{76}$$

or, separating the real and imaginary parts in the coefficients $a_k = \alpha_k - i\beta_k$, we can write the equation of the contour in the form:

$$x = \sum_{k=1}^{\infty} (\alpha_k \cos k\varphi + \beta_k \sin k\varphi); \qquad y = \sum_{k=1}^{\infty} (-\beta_k \cos k\varphi + \alpha_k \sin k\varphi). \tag{77}$$

In a particular case a_1 can be real, i.e. $\beta_1 = 0$. The equations (77) give the parametric representation of the contour Γ of the domain B in a special form, viz. they give the parametric representation in the form of conjugate trigonometric series [25]. This is known as the normal parametric representation of a curve. In its complex form this can be written in the form (76). Conversely, if we have the normal parametric representation of the contour Γ of the domain in the form (76) or (77) we can construct the function itself by substituting $e^{ik\varphi}$ by τ^k in the series (76). In this case the series (76) must be uniformly convergent. The problem thus involves the finding of the normal parametric representation for the contour Γ of the given domain B.

We assume that we have an equation for the contour Γ in an indefinite form and that this equation is as follows:

$$x^2 + y^2 - 1 + \lambda P(x^2, y^2) = 0, \tag{78}$$

where λ is a constant and $P(x^2, y^2)$ a polynomial containing only even powers of x and y. The equation (78) can be rewritten in complex form. Note that $P(x^2, y^2)$ can be considered to be a polynomial of two arguments:

$$x^2 + y^2 = z\bar{z} \quad \text{and} \quad 2(x^2 - y^2) = z^2 + \bar{z}^2,$$

so that the equation (78) can be rewritten in the form:

$$z\bar{z} - 1 + \lambda \sum_{l=0}^{l_0} \sum_{k=0}^{k_0} A_{kl} (z\bar{z})^k (z^2 + \bar{z}^2)^l = 0, \tag{79}$$

where A_{kl} are given real coefficients. It is given that our curve Γ is symmetrical with respect to the axes of coordinates and by repeating the arguments analogous with those in [37], when we considered a regular polygon it can be shown that in the formulae (77) $\beta_k = 0$ and $a_{2k} = 0$ so that the complex form of the equation of the contour is as follows:

$$z = a_1 e^{i\varphi} + a_3 e^{i3\varphi} + \dots \qquad (a_1 > 0), \tag{80}$$

where a_{2k+1} are real coefficients; consequently:

$$\bar{z} = a_1 e^{-i\varphi} + a_3 e^{-i3\varphi} + \dots \tag{81}$$

By direct multiplication we obtain the expressions:

$$\left.\begin{aligned} z\bar{z} &= \sum_{p=-\infty}^{+\infty} \left[\sum_{j-j'=p} a_{2j+1}\, a_{2j'+1} \right] e^{2ip\varphi}, \\ z^2 + \bar{z}^2 &= \sum_{p=0}^{+\infty} \left[\sum_{j+j'=p} a_{2j+1}\, a_{2j'+1} \right] e^{i(2p+2)\varphi} + \\ &\quad + \sum_{p=0}^{+\infty} \left[\sum_{j+j'=p} a_{2j+1}\, a_{2j'+1} \right] e^{-i(2p+2)\varphi}. \end{aligned}\right\} \tag{82}$$

In each of the above expressions summation with respect to j and j' is done from 0 to $+\infty$ but only those values are taken which satisfy the equations written below the symbol of summation. Substituting the expression (82) in the left-hand side of (79) we should, by multiplying the series once again and collecting terms with identical powers of $e^{i\varphi}$, equate to zero all terms with different powers of $e^{i\varphi}$. Note that in the formulae (82) the coefficients of positive and negative powers will be the same but only positive powers of $e^{i\varphi}$ enter the formulae. The same evidently applies to the expansion of the left-hand side of the equation (79), so that we have only the constant term and the coefficients of $e^{i2p\varphi}$ where $p > 0$ to equate to zero.

Without actually performing all of the evaluations in the general case we find that from the first equation in (82) we obtain a system of equations in the form:

$$\left.\begin{aligned} a_1^2 + a_3^2 + a_5^2 + \dots + \lambda T_0(a_{2j+1}) &= 1 \\ a_1 a_3 + a_3 a_5 + \dots + \lambda T_1(a_{2j+1}) &= 0 \\ a_1 a_5 + a_3 a_7 + \dots + \lambda T_2(a_{2j+1}) &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots & \end{aligned}\right\}, \tag{83}$$

where $T_p(a_{2j+1})$ are definite expressions containing the given coefficients A_{k_1} and the unknown coefficients a_{2j+1}. We shall not write them out for the general case. Let us rewrite the above system of equations leaving on the left-hand side of each equation only the first term, extracting the square root in the first equation and dividing all remaining equations by a_1:

$$
\begin{aligned}
a_1 &= \sqrt{1 - [a_3^2 + a_5^2 + \ldots + \lambda T_0(a_{2j+1})]},\\
a_3 &= -\frac{a_3 a_5}{a_1} - \frac{a_5 a_7}{a_1} - \ldots - \frac{1}{a_1}\lambda T_1(a_{2j+1}),\\
a_5 &= -\frac{a_3 a_7}{a_1} - \frac{a_5 a_9}{a_1} - \ldots - \frac{1}{a_1}\lambda T_2(a_{2j+1}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Expanding the radical by using Newton's binomial theorem we obtain:

$$
\left.
\begin{aligned}
a_1 &= 1 - \frac{1}{2}[a_3^2 + a_5^2 + \ldots + \lambda T_0(a_{2j+1})] +\\
&\quad + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}[a_3^2 + a_5^2 + \ldots + \lambda T_0(a_{2j+1})]^2 + \ldots\\
a_3 &= -\frac{a_3 a_5}{a_1} - \frac{a_5 a_7}{a_1} - \ldots - \frac{1}{a_1}\lambda T_1(a_{2j+1})\\
a_5 &= -\frac{a_3 a_7}{a_1} - \frac{a_5 a_9}{a_1} - \ldots - \frac{1}{a_1}\lambda T_2(a_{2j+1})\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\right\} \qquad (84)
$$

We shall solve this system by the method of successive approximations and by taking the following for the initial values:

$$
a_1^{(0)} = 1; \quad a_3^{(0)} = a_5^{(0)} = \ldots = 0. \qquad (85)
$$

Substituting the expressions (85) in the right-hand sides of the equations (84) and rejecting all terms which contain powers of λ higher than the first degree we obtain the first approximation for the coefficients:

$$
a_{2j+1}^{(0)} + \lambda a_{2j+1}^{(1)}, \qquad (86)
$$

whence, by using the expressions $T_k(a_{2j+1})$, it can be shown that all expressions (86), are equal to zero for sufficiently large values of j.

By substituting again the expressions (86) in the right-hand sides of the equations (84) and rejecting all terms containing powers of λ higher than the second degree we obtain the second approximation for the coefficients in the form:

$$
a_{2j+1}^{(0)} + \lambda a_{2j+1}^{(1)} + \lambda^2 a_{2j+1}^{(2)},
$$

where again all expressions will be zero for large j's etc. It can be shown that the infinite series thus obtained for a_{2j+1} converge for all values of λ which are sufficiently close to zero and that they give the solution of the problem.

Example **1**. To clarify the above method we shall consider an example, viz. we shall find a function which reflects a unit circle into the interior of the following ellipse (87)

$$x^2 + y^2 - 1 - \lambda (x^2 - y^2) = 0. \tag{87}$$

In its complex form this equation can be written as follows:

$$z\bar{z} - \lambda \frac{z^2 + \bar{z}^2}{2} = 1,$$

and by using the formulae (82) we obtain an infinite system of equations in the form:

$$\left.\begin{aligned}
&a_1^2 + a_3^2 + a_5^2 + a_7^2 + a_9^2 + a_{11}^2 + \ldots && = 1\\
&a_1 a_3 + a_3 a_5 + a_5 a_7 + a_7 a_9 + a_9 a_{11} + \ldots && = \lambda \left(\frac{1}{2} a_1^2\right)\\
&a_1 a_5 + a_3 a_7 + a_5 a_9 + a_7 a_{11} + \ldots && = \lambda (a_1 a_3)\\
&a_1 a_7 + a_3 a_9 + a_5 a_{11} + \ldots && = \lambda \left(\frac{1}{2} a_3^2 + a_1 a_5\right)\\
&a_1 a_9 + a_3 a_{11} + \ldots && = \lambda (a_1 a_7 + a_3 a_5)\\
&a_1 a_{11} + a_3 a_{13} + \ldots && = \lambda \left(a_1 a_9 + a_3 a_7 + \frac{1}{2} a_5^2\right)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{88}$$

Let us introduce the new unknown ϱ_k, assuming that

$$\varrho_0 = a_1; \quad \varrho_1 = \frac{a_3}{a_1}; \quad \varrho_2 = \frac{a_5}{a_1}; \quad \ldots \tag{89}$$

In this case the system (88) can be rewritten as follows:

$$\left.\begin{aligned}
&\varrho_0 = (1 + \varrho_1^2 + \varrho_2^2 + \ldots)^{-\frac{1}{2}}\\
&\varrho_1 = \frac{1}{2}\lambda - \varrho_1 \varrho_2 - \varrho_2 \varrho_3 - \varrho_3 \varrho_4 - \varrho_4 \varrho_5 - \ldots\\
&\varrho_2 = \lambda \varrho_1 - \varrho_1 \varrho_3 - \varrho_2 \varrho_4 - \varrho_3 \varrho_5 - \ldots\\
&\varrho_3 = \lambda \left(\frac{1}{2} \varrho_1^2 + \varrho_2\right) - \varrho_1 \varrho_4 - \varrho_2 \varrho_5 - \ldots\\
&\varrho_4 = \lambda (\varrho_1 \varrho_2 + \varrho_3) - \varrho_1 \varrho_5 - \ldots\\
&\varrho_5 = \lambda \left(\varrho_4 + \varrho_1 \varrho_3 + \frac{1}{2} \varrho_2^2\right) - \varrho_1 \varrho_6 - \ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{90}$$

Without paying any attention to the first equation for the present, we can solve the remaining equations by using the above method of successive approximations.

Thus, on going as far as terms containing λ^5 we obtain:

$$\varrho_1 = \frac{1}{2}\lambda - \frac{1}{4}\lambda^3 + \frac{3}{32}\lambda^5; \quad \varrho_2 = \frac{1}{2}\lambda^2 - \frac{9}{16}\lambda^4;$$

$$\varrho_3 = \frac{5}{8}\lambda^3 - \frac{9}{8}\lambda^5; \quad \varrho_4 = \frac{7}{8}\lambda^4; \quad \varrho_5 = \frac{21}{16}\lambda^5.$$

where all remaining ϱ_k are equal to zero. We took as the initial values $\varrho_1^{(0)} = \varrho_2^{(0)} = \ldots = 0$. Substituting the expressions obtained for ϱ_k in the right-hand side of the first equation of the system (90) and using Newton's binomial formula, we obtain an expression for ϱ_0 which is accurate up to terms in λ^5.

$$\varrho_0 = 1 - \frac{1}{8}\lambda^2 + \frac{3}{128}\lambda^4.$$

By knowing the values of ϱ_k, and from (89) we can construct a_k:

$$a_1 = \varrho_0; \quad a_3 = \varrho_0\varrho_1; \quad a_5 = \varrho_0\varrho_2; \quad \ldots$$

Thus the unknown function which reflects the interior of the unit circle into the exterior of the ellipse (87) can be approximately represented by a polynomial of the eleventh degree:

$$z = \left(1 - \frac{1}{8}\lambda^2 + \frac{3}{128}\lambda^4\right)\tau\left[1 + \left(\frac{1}{2}\lambda - \frac{1}{4}\lambda^3 + \frac{3}{32}\lambda^5\right)\tau^2 + \right.$$

$$\left. + \left(\frac{1}{2}\lambda^2 - \frac{9}{16}\lambda^4\right)\tau^4 + \left(\frac{5}{8}\lambda^3 - \frac{9}{8}\lambda^5\right)\tau^6 + \frac{7}{8}\lambda^4\tau^8 + \frac{21}{16}\lambda^5\tau^{10}\right]. \quad (91)$$

2. Consider the conformal transformation of a unit circle into the interior of a square bounded by straight lines $x = \pm 1$ and $y = \pm 1$ parallel to the axes. The equation of this square can be written in the form:

$$(1 - x^2)(y^2 - 1) = 0 \quad \text{or} \quad x^2 + y^2 - 1 - x^2 y^2 = 0.$$

Introducing the parameter λ we obtain a family of lines:

$$x^2 + y^2 - 1 - \lambda x^2 y^2 = 0.$$

In the complex form this can be written as follows:

$$z\bar{z} - 1 + \lambda\left(\frac{z^2 - \bar{z}^2}{4}\right)^2 = 0.$$

In this case the square is symmetrical with respect to the axes of coordinates and the bisectors of the angles between these axes. Bearing this in mind and using the same arguments as in [37] we find that the normal representation of the contour of our square must have the form:

$$z = a_1 e^{i\varphi} + a_5 e^{i5\varphi} + a_9 e^{i9\varphi} + \ldots \qquad (a_1 > 0),$$

where a_{4k+1} are the required real coefficients. The above method gives the following infinite system of equations for these coefficients:

$$\left.\begin{aligned}
&a_1^2 + a_5^2 + a_9^2 + \ldots = \\
&\quad = 1 + \frac{\lambda}{2}\left[\left(\frac{a_1^2}{2}\right)^2 + (a_1 a_5)^2 + \left(a_1 a_9 + \frac{1}{2}a_5^2\right)^2 + (a_1 a_{13} + a_5 a_9)^2 + \ldots\right] \\
&a_1 a_5 + a_5 a_9 + a_9 a_{13} + \ldots = \\
&\quad = \frac{\lambda}{2}\left[-\frac{1}{2}\left(\frac{a_1^2}{2}\right)^2 + (a_1 a_5)\left(\frac{1}{2}a_1^2\right) + \left(a_1 a_9 + \frac{1}{2}a_5^2\right)(a_1 a_5) + \ldots\right] \\
&a_1 a_9 + a_5 a_{13} + \ldots = \\
&\quad = \frac{\lambda}{2}\left[-\left(\frac{a_1^2}{2}\right)(a_1 a_5) + \left(a_1 a_9 + \frac{1}{2}a_5^2\right)\left(\frac{1}{2}a_1^2\right) + \ldots\right] \\
&a_1 a_{13} + \ldots = \\
&\quad = \frac{\lambda}{2}\left[-\left(\frac{a_1^2}{2}\right)\left(a_1 a_9 + \frac{1}{2}a_5^2\right) - \frac{1}{2}(a_1 a_5)^2 + (a_1 a_{13} + a_5 a_9)\left(\frac{1}{2}a_1^2\right) + \ldots\right] \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad (92)$$

In this case we act somewhat differently, viz. we suppose immediately that $\lambda = 1$ and we then solve the system (92) so obtained by the method of successive approximations; the initial values are as follows:

$$a_1 = 1; \quad a_5 = a_9 = \ldots = 0.$$

Substituting this in the system (91) we have:

$$a_1^2 = 1 + \frac{1}{2}\left(\frac{1}{2}\right)^2; \quad a_5 = \frac{1}{2}\cdot\left[-\frac{1}{2}\left(\frac{1}{2}\right)^2\right]; \quad a_9 = a_{13} = \ldots = 0$$

or

$$a_1 = 1.0607; \quad a_5 = -0.0625; \quad a_9 = a_{13} = \ldots = 0.$$

Substituting these approximations in the system (92) we have:

$$a_1^2 + (-0.0625)^2 = 1 + \frac{1}{2}\left[\frac{(1.0607)^2}{4} + 1.0607)^2\cdot(-0.0625)^2 + \frac{1}{4}(-0.0625)^4\right];$$

$$1.0607a_5 = \frac{1}{2}\left[-\frac{1}{2}\frac{(1.0607)^4}{4} + \frac{1}{2}(1.0607)^3\cdot(-0.0625) + \frac{1}{2}(-0.0625)\cdot 1.0607\right];$$

$$1.0607a_9 = \frac{1}{2}\left[-\frac{1}{2}(1.0607)^3(-0.0625) + \frac{1}{4}(-0.0625)^2\cdot(1.0607)^2\right];$$

$$1.0607a_{13} = \frac{1}{2}\left[-\frac{3}{4}(1.0607)^2(-0.0625)^2\right];$$

$$1.0607a_{17} = 0.$$

This gives the following approximations:

$$a_1 = 1.0672; \quad a_5 = -0.0922; \quad a_9 = 0.0181; \quad a_{13} = -0.0016; \quad a_{17} = 0.$$

These successive approximations can obviously be continued further but to evaluate successive approximations the system (92) must be written in a more specific form, by adding new equations and writing out a greater number of terms

in each equation. To evaluate successive approximations the preceding approximation is substituted in all terms of the system (92) except in the first term on the left, although beginning with the second equation, the values of a_1 are substituted in the first term.

In the case under consideration the values accurate to the first four decimal places, will be as follows:

$$a_1 = 1.0807; \quad a_5 = -0.1081; \quad a_9 = 0.0450; \quad a_{13} = -0.0242; \quad a_{17} = 0.0174;$$
$$a_{21} = -0.0125.$$

Notice that when applying the method of successive approximations with initial values $a_1 = 1$, $a_3 = a_9 = \ldots = 0$, all the coefficients a_{4k+1} in every approximation from a certain coefficient onwards will be equal to zero.

Instead of the above method an expansion in powers of λ can be used, not for the coefficients a_{2j+1} but for the right-hand sides of the equations (77) or (78). This will lead to somewhat different results when the method of successive approximations is used. We are trying to find the normal parametric representation for the curve (78) in the form of series in whole powers of the parameter λ.

$$x = x_0(\varphi) + x_1(\varphi)\lambda + x_2(\varphi)\lambda^2 + \ldots; \quad y = y_0(\varphi) + y_1(\varphi)\lambda + y_2(\varphi)\lambda^2 + \ldots, \tag{93}$$

where $x_0(\varphi)$ and $y_0(\varphi)$ are functions giving the normal parametric representation of the curve (78) when $\lambda = 0$, i.e. they give the circles $x^2 + y^2 - 1 = 0$. In other words, in formula (93) we have:

$$x_0(\varphi) = \cos\varphi; \quad y_0(\varphi) = \sin\varphi.$$

Further coefficients $x_k(\varphi)$ and $y_k(\varphi)$ must be conjugate functions, i.e. they should be represented by conjugate trigonometric series. Substituting the expression (93) in the left-hand side of the equation (78) and equating to zero terms with like powers of λ we obtain equations for the determination of the coefficients of the expansion.

3. Let us apply the above method to the ellipse:

$$z\bar{z} - \frac{\lambda}{2}(z^2 + \bar{z}^2) = 1, \tag{94}$$

which we have also considered in connection with the first method. We shall try to find the normal parametric equation for this curve in the form:

$$z = x + iy = e^{i\varphi} + z_1(\varphi)\lambda + z_2(\varphi)\lambda^2 + \ldots, \tag{95}$$

where every value $z_k(\varphi)$ is an expression of the form:

$$z_k(\varphi) = a_1^{(k)} e^{i\varphi} + a_3^{(k)} e^{i3\varphi} + \ldots \tag{96}$$

Substituting (95) in the left-hand side of (94) we have:

$$(e^{i\varphi} + z_1\lambda + z_2\lambda^2 + \ldots)(e^{-i\varphi} + \bar{z}_1\lambda + \bar{z}_2\lambda^2 + \ldots) - \frac{\lambda}{2}\left[(e^{i\varphi} + z_1\lambda + \ldots)^2 + \right.$$
$$\left. + (e^{-i\varphi} + \bar{z}_1\lambda + \ldots)^2\right] = 1. \tag{97}$$

Equating the coefficients of λ to zero we obtain:

$$e^{i\varphi}\bar{z}_1 + e^{-i\varphi}z_1 = \frac{1}{2}\left(e^{i2\varphi} + e^{-i2\varphi}\right)$$

or

$$\mathcal{R}\left[e^{-i\varphi}z_1\right] = \frac{1}{2}\cos 2\varphi,$$

where $\mathcal{R}$ is the symbol of the real part. In agreement with (96) this gives

$$\mathcal{R}\left[a_1^{(1)} + a_3^{(1)}e^{i2\varphi} + \ldots\right] = \frac{1}{2}\cos 2\varphi,$$

whence

$$a_3^{(1)} = \frac{1}{2}; \quad a_1^{(1)} = a_5^{(1)} = a_7^{(1)} = \ldots = 0.$$

Finally this gives:

$$z_1 = \frac{1}{2}e^{i3\varphi}. \tag{98}$$

Let us return to formula (97) and equate the coefficients of λ^2 to zero:

$$e^{-i\varphi}z_2 + e^{i\varphi}\bar{z}_2 + z_1\bar{z}_1 = e^{i\varphi}z_1 + e^{-i\varphi}z_1$$

or

$$\mathcal{R}\left[e^{-i\varphi}z_2\right] = \mathcal{R}\left[e^{i\varphi}z_1\right] - \frac{1}{2}z_1\bar{z}_1.$$

From (98) we have:

$$\mathcal{R}\left[e^{-i\varphi}z_2\right] = -\frac{1}{8} + \frac{1}{2}\cos 4\varphi,$$

i.e. remembering (96):

$$\mathcal{R}\left[a_1^{(2)} + a_3^{(2)}e^{i2\varphi} + \ldots\right] = -\frac{1}{8} + \frac{1}{2}\cos 4\varphi,$$

whence $a_1^{(2)} = -1/8$, $a_5^{(2)} = 1/2$ and the remaining $a_{2j+1}^{(2)}$ are equal to zero, i.e.

$$z_2 = -\frac{1}{8}e^{i\varphi} + \frac{1}{2}e^{i5\varphi}.$$

Continuing in the same way we obtain

$$z_3 = -\frac{5}{16}e^{i3\varphi} + \frac{5}{8}e^{i7\varphi}; \quad z_4 = \frac{3}{128}e^{i\varphi} - \frac{5}{8}e^{i5\varphi} + \frac{7}{8}e^{i9\varphi}$$

and, finally, substituting in (95) and replacing $e^{ik\varphi}$ by τ^k we obtain the approximate expression for the unknown conformal transformation:

$$z = \tau + \frac{1}{2}\tau^3\lambda + \left(-\frac{1}{8}\tau + \frac{1}{2}\tau^5\right)\lambda^2 + \left(-\frac{5}{16}\tau^3 + \frac{5}{8}\tau^7\right)\lambda^3 +$$

$$+ \left(\frac{3}{128}\tau - \frac{5}{8}\tau^5 + \frac{7}{8}\tau^9\right)\lambda^4. \tag{99}$$

The method described above was developed by Prof. L. V. Kantorovich. The detailed description of this method with the proof of convergence can be

found in the work of the author (*Mathematical Manual*, vol. 40 : 3). Notice that if in formula (91) we consider no terms containing powers of λ higher than the fourth power then formula (99) will be obtained.

41. The two-dimensional established flow of liquids. Having explained the theoretical basis of the theory of conformal transformation we shall now consider the application of the theory of a complex variable to hydrodynamics. Let there be a two-dimensional established flow of liquid, where the potential of velocity is equal to $\varphi(x, y)$ and the function of the current is equal to $\psi(x, y)$ [II, 74]. Let us recall that the components of current at every point are expressed by the formulae:

$$v_x = \frac{\partial \varphi(x, y)}{\partial x}; \quad v_y = \frac{\partial \varphi(x, y)}{\partial y} \tag{100}$$

and the difference

$$\psi(x_1, y_1) - \psi(x_0, y_0) = \psi(M_1) - \psi(M_0) \tag{101}$$

gives the quantity of liquid which flows in unit time across an arbitrary contour connecting the points M_0 and M_1. The flow is independent of time and is the same in all planes parallel to the XY-plane; the liquid is of unit density. More strictly, the expression (101) gives the quantity of liquid which flows in unit time across a cylindrical surface, parallel to the Z-axis, of unit height, the surface having the shape of a contour l in the XY-plane which connects the points $M_0(x_0, y_0)$ and $M_1(x_1, y_1)$. Hence from above the functions $\varphi(x, y)$ and $\psi(x, y)$ are connected by the relationships:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}; \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

which are exactly the same as the Cauchy–Riemann equations. We can therefore say that the function of the complex variable

$$f(z) = \varphi(x, y) + i\psi(x, y) \tag{102}$$

has a derivative in the domain occupied by the flowing liquid. This function (102) is usually known as *the complex potential of flow.*

As we said above, the functions $\varphi(x, y)$ and $\psi(x, y)$ can be many-valued viz. they can acquire constant terms by encircling a certain point, or more generally, by encircling certain holes in the region under consideration. For the function $\psi(x, y)$ this many-valuedness indicates the presence of a source at the corresponding point and for the function $\varphi(x, y)$, the presence of an elementary turbulence at

that point. In such cases the function $f(z)$ will also be many-valued, i.e. it will acquire constant terms by encircling certain points (or holes).

From (100) we can see that the velocity vector corresponds to the complex number

$$\frac{\partial \varphi}{\partial x} + i\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial x} - i\frac{\partial \psi}{\partial x}.$$

The latter expression is the function conjugate with the derivative $\overline{f'(z)}$ [2]. Hence *a function $g(z)$, conjugate with the derivative $f'(z)$, gives the velocity vector of the flow.*

Consider the isothermic net which corresponds to the function (102):

$$\varphi(x, y) = C_1; \quad \psi(x, y) = C_2. \tag{103}$$

The first family of lines represents a family of lines of equal velocity potential or, in other words, a family of equipotential lines. The second family (lines of current) represents, as can easily be seen, a family of trajectories of liquid particles. In fact, as we know, the two families will be orthogonal but the velocity vector, which is equal to grad $\varphi(x, y)$, is directed, as it happens, along the normal to $\varphi(x, y) = c_1$, which corresponds to a line of the second family (103). Thus in the given established flow the velocity vector at every point is directed along the tangent to the line from the second family (103), which passes through that point, i.e. in fact this family is a family of lines of current, and the latter, in an established flow, gives the trajectories of liquid particles.

Until now we have considered the kinematic picture and thought that any kinematically possible picture of movement can be given by a complex potential which is a regular function; conversely, any complex potential gives a kinematically possible picture of movement. We will now show that *we can in this way also satisfy other hydrodynamic equations which give us the value of the pressure.* Let us write down the hydrodynamic equations for the two-dimensional established flow, assuming that exterior capacity forces have a potential $U(x, y)$. Bearing (100) in mind we obtain two hydrodynamic equations and a continuity equation [II, 115]:

$$\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial U}{\partial x} - \frac{1}{\varrho}\frac{\partial p}{\partial x},$$

$$\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial U}{\partial y} - \frac{1}{\varrho}\frac{\partial p}{\partial y},$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

where ϱ is the density of the liquid and $p(x, y)$ is the pressure. The continuity equation is evidently satisfied, for the real part of the regular function is a harmonic function. The first two equations can be rewritten in the form:

$$\frac{\partial}{\partial x}\left\{\frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2+\left(\frac{\partial \varphi}{\partial y}\right)^2\right]-U+\frac{1}{\varrho}p\right\}=0,$$

$$\frac{\partial}{\partial y}\left\{\frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2+\left(\frac{\partial \varphi}{\partial y}\right)^2\right]-U+\frac{1}{\varrho}p\right\}=0.$$

It follows that the expression inside the shaped brackets, must be a constant and we thus obtain the following integral:

$$\frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2+\left(\frac{\partial \varphi}{\partial y}\right)^2\right]-U+\frac{1}{\varrho}p=C, \tag{104}$$

which determines the value of the pressure $p(x, y)$. If the capacity forces are absent and if we assume that $\varrho = 1$, we obtain the formula:

$$p=C-\frac{1}{2}\left|\mathbf{V}\right|^2=C-\frac{1}{2}\left|f'(z)\right|^2, \tag{105}$$

where $|V|$ denotes the velocity.

Note that if instead of $f(z) = \varphi + i\psi$ we take the complex potential $if(z) = -\psi + i\varphi$, then the equipotential lines will be transformed into lines of current and vice versa. Hence *every isothermic net of a regular function essentially gives two different pictures of the flow of a liquid.*

42. Examples. 1. All examples of isothermic nets which we considered earlier can now be interpreted from the point of view of hydrodynamics when, as we have shown above, every example gives two hydrodynamic pictures.

Let us now consider some other examples. To begin with we consider the elementary function

$$f(z) = A \log (z-a) = A \log |z-a| + iA \arg (z-a),$$

where a is a point of the plane and A is a real constant. In this case the equipotential lines are circles, centre at a, and the lines of current are straight lines which originate at that point. The function $f(z)$ acquires the constant $i2\pi A$ by describing this point and therefore the imaginary part of the complex potential $\varphi(x, y)$ (the function of current) acquires the term $2\pi A$. The velocity vector is determined by the complex number

$$\overline{f'(z)}=\frac{A}{\overline{z-a}}\ .$$

If we denote by ϱ and φ the modulus and amplitude of the complex number $z - a$, then the velocity vector corresponds to the complex number $Ae^{i\varphi}/\varrho$. One result of this is that on approaching the source, the velocity tends to infinity and, when A is positive, this velocity is directed from the source to infinity, i.e. we have a source but no flow.

Let us now consider the more general function:

$$f(z) = A \log \frac{z-a}{z-b} = A \log \left| \frac{z-a}{z-b} \right| + iA \arg \frac{z-a}{z-b}, \qquad (106)$$

where a and b are distinct points in the plane and A is a real constant. In this case the isothermic net is defined by the equations:

$$\left| \frac{z-a}{z-b} \right| = C_1; \quad \arg \frac{z-a}{z-b} = C_2.$$

As we know the first of these equations corresponds to a family of circles with respect to which a and b are symmetrical and the second equation to

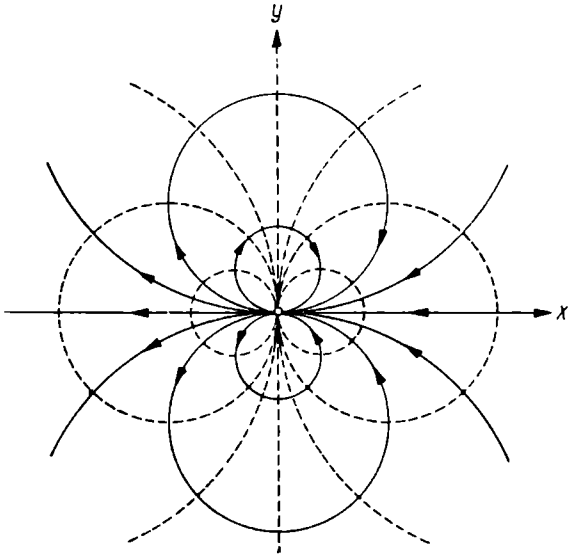

FIG. 45

a family of circles which pass through the points a and b [31]. In the case under consideration we have a source of intensity $2\pi A$ at the point a and a flow of the same intensity at the point b.

2. Let us suppose that the points a and b lie at the points $-h$ and 0 on the real axis and take $A = 1/h$. In this case the function (106) has the form:

$$f(z) = \frac{\log(z+h) - \log z}{h}.$$

Taking the limit as $h \to 0$ we obtain the complex potential which characterizes a so-called *dipole* at the origin:

$$f_1(z) = \frac{1}{z}.$$

It can easily be shown that in this case the isothermic net consists of circles passing through the origin and touching the Y axis (equipotential lines) and of circles passing through the origin and touching the X axis (lines of current) (Fig. 45) [31].

3. Consider the function:

$$f(z) = iA \log (z - a) = - A \arg (z - a) + iA \log |z - a|,$$

where A, as before, is a real constant. In this case circles, centre at a, serve as lines of current and the straight lines which radiate from the point a are equipotential lines. By encircling the point a in the positive sense the real part of $f(z)$ (velocity potential) receives an increment $-2\pi A$ and we have at the point A an elementary turbulence of intensity $-2\pi A$.

4. Take the function

$$f(z) = \frac{k}{2}\left(z + \frac{1}{z}\right), \tag{107}$$

which we investigated in [33]. Separating the real and imaginary parts we obtain an equation for the lines of current in the form:

$$\frac{k}{2}\left(y - \frac{y}{x^2 + y^2}\right) = C$$

or

$$ky(x^2 + y^2 - 1) - 2C(x^2 + y^2) = 0.$$

In the general case these lines are certain curves of the third order. In the particular case when $C = 0$ we have a circle $x^2 + y^2 = 1$ and the axis $y = 0$. We are only considering that part of the plane outside the above circle. We can say that the lines of current consist of the lines $(-\infty, -1)$ and $(1, \infty)$ on the $y = 0$ axis and of the above circle. In this case we have considered the flow of liquid outside the circle with the liquid circulating round the circle. Evaluating the derivative

$$f'(z) = \frac{k}{2}\left(1 - \frac{1}{z^2}\right),$$

we see that the velocity of flow at infinity is equal to $k/2$ (where k is real) and this velocity is equal to zero at the points $z = \pm 1$, i.e. at points where the lines of current enter the circle.

We now add a logarithmic term to our function and thus construct a new function

$$f_1(z) = \frac{k}{2}\left(z + \frac{1}{z}\right) - iA \log z. \tag{108}$$

The real part of the second term also remains constant on the above circle, i.e. this circle, even for the complex potential (108) is one of the lines of current, but in the case under consideration the velocity potential receives an increment $2\pi A$ on encircling this circle, i.e. the potential (108) gives the flow round our circle with an elementary turbulence. Figures 46_1, 46_2 and 46_3 give the appearance of the lines of flow for various values of the constant A/k. The flow represented in Fig. 46_2 shows that the points of entrance and exit of the lines of flow coincide on the circle round which the liquid circulates.

5. As we saw earlier in [33] the isothermic net for the function $f(z) = \arccos z/k$ consists of confocal ellipses and hyperbolae with foci at $\pm k$

on the real axis. This net is shown in Fig. 47. If we take the hyperbolae as the the lines of current we obtain the picture of flow through the aperture $(-k, +k)$ on the real axis. If we take the ellipses as the lines of current we obtain the picture of flow round the ellipse or round the line $(-k, +k)$.

6. Frequently when studying the hydrodynamic picture it is more convenient to give not the complex potential $w = f(z)$ but its inverse function

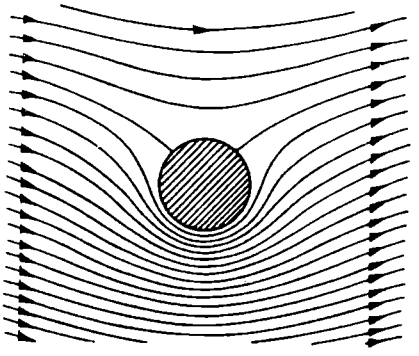

FIG. 46_1

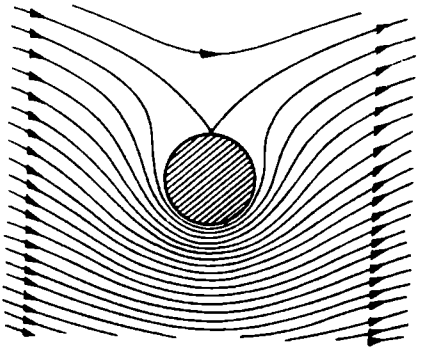

FIG. 46_2

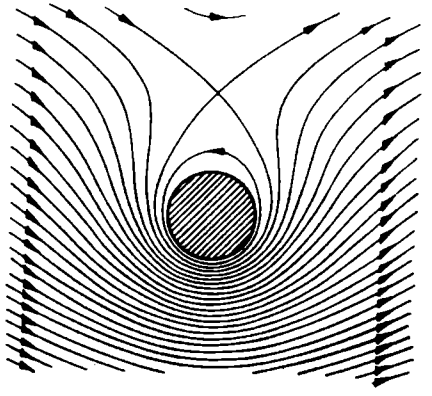

FIG. 46_3

$z = \varphi(w)$. Consider an example of this kind. Suppose that the complex potential is given by its inverse function

$$z = w + e^w.$$

Separating the real and imaginary parts

$$z = x + iy; \quad w = \varphi + i\psi,$$

we have:

$$x = \varphi + e^{\varphi} \cos \psi; \quad y = \psi + e^{\varphi} \sin \psi.$$

Assuming that $\psi = C$ we obtain an equation for the lines of current in the parametric form

$$x = \varphi + e^{\varphi} \cos C; \quad y = C + e^{\varphi} \sin C,$$

where φ is the variable parameter. Consider two lines of current, viz. the lines corresponding to $C = \pi$ and $C = -\pi$. In the first case we have

$$x = \varphi - e^{\varphi}; \quad y = \pi.$$

It can easily be seen that in this case the lines of current consist of a double line $-\infty < x \leqslant -1$ on the line $y = \pi$. In the second case when $C = -\pi$ the lines of current consist of a double line $-\infty < x \leqslant -1$ on the line $y = -\pi$.

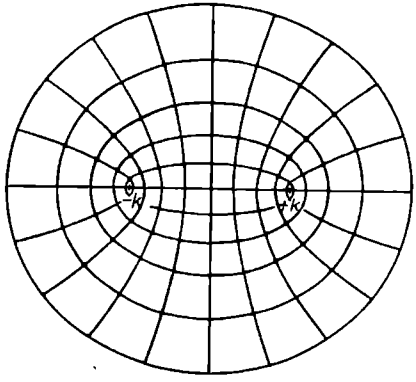

FIG. 47

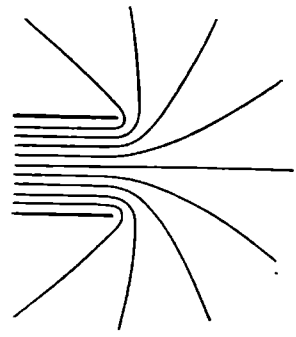

FIG. 48

Also, when $C = 0$, the axis $y = 0$ itself serves as a line of current. Figure 48 represents the appearance of the lines of current in this case.

43. The problem of flow round a contour. Suppose that we are given a simple closed contour l in a plane and that we are investigating the flow of liquid outside this contour, which must satisfy the following two conditions: (1) the contour l must be one of the lines of current and (2) the velocity at infinity must be finite and have a definite direction. It is also necessary that the complex potential $f(z)$ be a single-valued function. We assume that the velocity at infinity is given by a positive real number c (i.e. we choose the positive direction of the real axis as the direction of the velocity at infinity).

Suppose that we know the function which conformally transforms that part of the z-plane outside l, into the outside of the unit circle $|\tau| > 1$. We know that there is an infinite number of such functions and we choose the function which translates the point at infinity into itself and has no direction at that point. $\omega'(\infty)$ is a real positive number

for this function and we have the following expansion for it in the neighbourhood of the point $z = \infty$:

$$\tau = \omega(z) = bz + b_0 + \frac{b_1}{z} + \dots \qquad (b > 0). \tag{109}$$

As we already know, the complex potential in the problem of flow round a circle, can be expressed as follows:

$$f_1(\tau) = \frac{k}{2}\left(\tau + \frac{1}{\tau}\right), \tag{110}$$

where k is a real constant which we shall determine later. If we substitute τ by its equivalent expression from (109) in the equation (110) we obtain a single-valued function which is regular outside the contour l; its imaginary part remains constant on the contour l, in the same way as the imaginary part of (110) remained constant on the circle $|\tau| = 1$:

$$f(z) = f_1[\omega(z)] = \frac{k}{2}\left[\omega(z) + \frac{1}{\omega(z)}\right]. \tag{111}$$

The constant k only remains to be chosen so that the velocity at infinity is equal to c, i.e. so that $f'(\infty) = c$. From the formulae (109) and (110) we must have at infinity

$$f'(z) = \frac{k}{2}\left[1 - \frac{1}{\omega^2(z)}\right]\omega'(z) \quad \text{and} \quad f'(\infty) = \frac{k}{2} \cdot b,$$

from which it follows directly that we should take $k = 2c/b$. We thus see that *the problem of flow round a contour involves the conformal transformation of that part of the plane outside that contour into the exterior of a unit circle.*

It can be shown that if the function $f(z)$ is single-valued the solution of the problem is unique provided that $f(z)$ has no singularities outside l other than the simple pole $z = \infty$.

44. N. E. Zhukovskij's formula. Let $f(z)$ be the complex potential which gives the flow round the contour l and let the velocity at infinity be equal to the positive number c. We assume that $f(z)$ is not a single-valued function but that in describing a circuit round the contour l its real part $\varphi(z, y)$ gains a constant term γ. The component of pressure on the body about which the liquid circulates can be expressed by line integrals:

$$F_x = \int_l p(x, y) \cos(n, x)\, ds; \quad F_y = \int_l p(x, y) \cos(n, y)\, ds, \tag{112}$$

where $p(x, y)$ is the pressure and n is the direction of the normal inside the contour.

The element of the contour ds, like the vector, corresponds to a a complex number $dz = e^{i\theta}\,ds$, where θ is the angle between the tangent to the contour and the OX axis. The multiplication of a complex number by i is equivalent to the addition of $\pi/2$ to the amplitude and therefore the complex number $ie^{i\theta}\,ds$ corresponds to a vector ds, directed along the inside normal to l, and we evidently have:

$$F_x + iF_y \equiv \int_l pi\,dz. \tag{113}$$

According to formula (105)

$$p = C - \frac{1}{2}|f'(z)|^2 = C - \frac{1}{2}\left|\frac{df}{dz}\right|^2,$$

and therefore

$$F_x + iF_y = i\int_l C\,dz - \frac{1}{2}i\int_l \left|\frac{df}{dz}\right|^2 dz.$$

It is obvious that

$$\int_l dz = 0;$$

it is convenient to change to complex conjugate values in the above equation, after which we obtain:

$$F_x - iF_y = \frac{1}{2}i\int_l \left|\frac{df}{dz}\right|^2 \overline{dz} = \frac{1}{2}i\int_l \frac{df}{dz}\cdot\frac{\overline{df}}{\overline{dz}}\,\overline{dz} = \frac{1}{2}i\int_l \frac{df}{dz}\,\overline{df}. \tag{114}$$

The contour l is a line of current, and therefore $\psi(x, y)$ is a constant on this line; $\psi(x, y) = C_1$ and therefore, on l:

$$f(z) = \varphi(x, y) + iC_1, \quad \overline{f(z)} = \varphi(x, y) - iC_1,$$

from which it follows that $df = \overline{df}$. Multiplying both sides of (114) by i we obtain a complex value which fully characterizes the vector of the total pressure upon the body:

$$R = F_y + iF_x = -\frac{1}{2}\int_l \frac{df}{dz}\,df,$$

or finally:

$$R = F_y + iF_x = -\frac{1}{2}\int_l \left(\frac{df}{dz}\right)^2 dz. \tag{115}$$

The function $f'(z)$ is regular and single-valued outside l. In the neighbourhood of infinity it can be expanded as follows:

$$f'(z) = c + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots, \tag{116}$$

where c is the given value of the velocity at infinity. For the function $f(z)$ we have the following expression in the neighbourhood of infinity:

$$f(z) = C + cz + b_1 \log z - \frac{b_2}{z} + \ldots,$$

and by describing a circuit in the positive sense around l, the function $f(z)$ acquires the term $i2\pi b_1$, which we earlier denoted by γ. Thus $b_1 = (1/2\pi i)\,\gamma$ and instead of (116) we can write:

$$f'(z) = c + \frac{\gamma}{2\pi iz} + \frac{b_2}{z^2} + \ldots.$$

whence, by squaring, we obtain an expansion in the form:

$$[f'(z)]^2 = c^2 + \frac{c\gamma}{\pi iz} + \frac{d_2}{z^2} + \ldots. \tag{117}$$

When evaluating the integral (114) we can, as a result of Cauchy's theorem, integrate not round the contour l but round a closed curve which surrounds l and which lies in the neighbourhood of the point at infinity. We can then use the expansion (117) when we obtain the following expression for R:

$$R = F_y + iF_x = -\frac{c\gamma}{2\pi i}\,2\pi i = -c\gamma,$$

i.e.

$$F_y = -c\gamma; \quad F_x = 0. \tag{118}$$

45. The two-dimensional problem of electrostatics. We shall now apply the theory of functions of a complex variable to the problems of electrostatics. We frequently meet here problems analogous with those considered above. First of all we shall explain what is meant by the two-dimensional problem of electrostatics. As we know the point charge e creates lines of force in space which act in accordance with Coulomb's law and the intensity of the field is expressed by the well known formula

$$f = \frac{e}{\varrho^2},$$

where ϱ is the distance from the charge e to the point M, at which the vector of force is determined. This vector of force takes the direction of the line connecting the charge and the point M. Imagine now that we have a charged straight line parallel to the Z-axis which crosses the XY-plane at the point O, and that the density of the charge is the same at every point. Denote this charge, which is proportional to

unit length of the line, by e. The picture of the electrostatic field is obviously the same in all planes parallel to the XY-plane; therefore it is sufficient to consider the XY-plane alone; here again, as a result of the principle of symmetry, the vector of force must lie in this plane and take the direction of the line joining the point O with the point M in the plane at which the force is calculated. The elementary charge on the section dz of the straight line is expressed by the product $e \cdot dz$, and the value of the force at the point M with coordinates $(x, y, 0)$ by the sum of the projections of component forces multiplied by the direction $\overline{OM}$ of the above line.

We have the following expression for the force:

$$\frac{e\,dz}{x^2+y^2+z^2},$$

when O is the origin. The above expression must be multiplied by the cosine of the angle φ, made by the direction $\overline{NM}$, from the variable point N on the Z-axis, and the direction $\overline{OM}$; from the right-angled triangle ONM we have:

$$\cos\varphi = \frac{r}{\sqrt{x^2+y^2+z^2}} \quad \text{and} \quad z = r\tan\varphi,$$

where $r = \sqrt{x^2+y^2}$. Replacing the variable φ in the integral

$$\int_{-\infty}^{+\infty} \frac{e\cos\varphi\,dz}{x^2+y^2+z^2}$$

by z we obtain the following expression for the force:

$$f = \frac{e}{r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\,d\varphi$$

or

$$f = \frac{2e}{r} \qquad (r = \sqrt{x^2+y^2}). \tag{119}$$

The corresponding potential of force is:

$$V(x, y) = 2e\log\frac{r_0}{r}, \tag{120}$$

where r_0 is an arbitrary constant which we assume to be positive. Hence the logarithmic potential (120) is the elementary potential and it originates, as it were, from this point charge, if we disregard the whole space and consider the XY-plane alone. Note that this

elementary potential (120) does not vanish at infinity like the usual Newton's three-dimensional potential $1/r$, but becomes infinity; this is the essential difference of two-dimensional electrostatic problems. If, instead of a charged line, we have a charged cylinder, the base B of which lies in the XY-plane, then instead of the elementary potential (120) we obtain a potential expressed by a double integral

$$V(x, y) = 2 \iint_B \varrho(\xi, \eta) \log \frac{r_0}{r} \, d\xi \, d\eta, \tag{121}$$

where $\varrho(\xi, \eta)$ is the density and r the distance from a variable point (ξ, η) in the domain B to the point $M(x, y)$:

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}.$$

Similarly, if the surface of a cylinder is charged, then the potential is expressed by a line. We also know that the functions $\log r$ and (120) satisfy the Laplace equation [II, 119]:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

The potential (121) outside the charge, i.e. outside the domain B, also satisfies this equation.

We can assume that any harmonic function is the real or imaginary part of a regular function of a complex variable. In this case we shall consider the potential $V(x, y)$ to be the imaginary part of a regular function

$$f(z) = U(x, y) + iV(x, y). \tag{122}$$

Hence *every electrostatic picture outside the charge gives a regular function $f(z)$ (complex potential) and, conversely, any such regular function gives the electrostatic picture of the two-dimensional field.*

In this case both families of the isothermic net of functions

$$U(x, y) = C_1; \quad V(x, y) = C_2 \tag{123}$$

have a simple physical meaning. The second family in (123) gives a family of equipotential lines and the first, which as we know is orthogonal to the second, gives a family of lines of force, i.e. it gives lines the tangents to which define at every point the direction of the acting force. The components of the vectors of force can be expressed as follows

$$F_x = -\frac{\partial V(x, y)}{\partial x}; \quad F_y = -\frac{\partial V(x, y)}{\partial y},$$

or, from the Cauchy–Riemann equations:

$$F_x = -\frac{\partial V}{\partial x}; \quad F_y = -\frac{\partial U}{\partial x}.$$

Thus the vector of force corresponds to the complex number

$$F_x + iF_y = -\frac{\partial V}{\partial x} - i\frac{\partial U}{\partial x} = -\overline{if'(z)}. \tag{124}$$

If we have a closed finite conductor then inside this conductor, as we know, the potential remains constant and the density of charge on its surface, as proved in electrostatics, is calculated according to the formula:

$$\varrho = \frac{1}{4\pi}\sqrt{F_x^2 + F_y^2}$$

or, with the aid of the complex potential, by the formula

$$\varrho = \frac{1}{4\pi}|f'(z)|. \tag{125}$$

The analogy between the above concepts and the corresponding concepts in problems of two-dimensional hydrodynamics can readily be seen.

46. Examples 1. All examples of isothermic nets which we considered above can now be interpreted from the point of view of electrostatics. Consider, for example, the function

$$f(z) = i\,2e\,\log\frac{z-a}{z-b}. \tag{126}$$

Its imaginary part remains constant on the circles with respect to which the points a and b are symmetrical [31]. Take two such circles C_1 and C_2 and suppose that the imaginary part of the function (126) has constant values V_1 and V_2 on these circles. If we imagine two cylinders formed by lines parallel to the z axis for which the above circles serve as bases, then the complex potential (126) gives the picture of the electrostatic field between two such cylinders, where V_1 and V_2 are the values of the potentials on the respective cylinders.

Notice that in the general determination of the electrostatic field in an annulus between two conducting curves l_1 and l_2 we have a complex potential, the imaginary part of which remains constant on the curves l_1 and l_2. Thus the complex potential $f(z)$ transforms the above annulus into a strip, bounded by two straight lines, parallel to the real axis. Such a transformation cannot be single-valued, for an annulus is a multiply connected region and a strip is a connected region. The function (126) in the above example is evidently many-valued in the annulus confined between the circles C_1 and C_2.

Note one other property of the field defined by the function (126). This function can be written as follows:

$$f(z) = i\,2e\,\log(z-a) - i\,2e\,\log(z-b).$$

By using this expression it can be shown that both conductors have equal charges of opposite sign. In agreement with this the function (126):

$$f(z) = i\,2e \log \frac{1 - \frac{a}{z}}{1 - \frac{b}{z}}$$

will be regular at $z = \infty$.

2. If we want to determine the electrostatic field between two conductors, each of which goes off to infinity (Fig. 49) then the domain between these conductors will be connected and the problem essentially involves the transformation of the region into a strip bounded by two straight lines, parallel to the real axis. Thus, for instance, when these lines are the lines $(-\infty < x < -1)$ on the straight lines $y = \pi$ and $y = -\pi$, then the formula $z = w + e^w$ gives the inverse function for the unknown function and Fig. 48 gives the picture of equipotential lines in this case. Notice that at the end of our lines we have $w = \pm \pi i$ and $e^w = -1$. But formula (124) gives the magnitude of the force as:

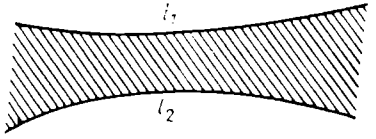

FIG. 49

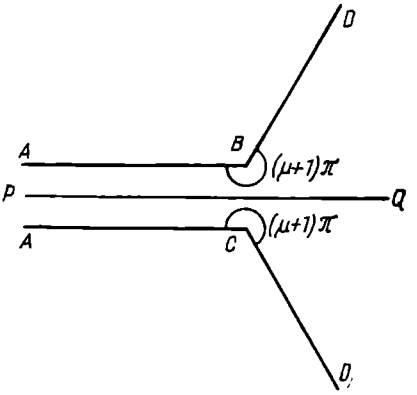

FIG. 50

$$\sqrt{F_x^2 + F_y^2} = |f'(z)| = \left|\frac{dw}{dz}\right| = \left|\frac{dz}{dw}\right|^{-1},$$

in this case

$$\sqrt{F_x^2 + F_y^2} = |1 + e^w|^{-1},$$

i.e. at the ends of the above lines the force becomes infinity.

This is a special case of a more general example which we shall now consider. Assume that our two conductors have the appearance shown in Fig. 50: AB and AC are two parallel halves of straight lines, so that the points B and C lie on their common perpendicular. The directions BD and CD make the same angle α with AB and AC, where $\alpha = \mu\pi$. Draw a straight line PQ, parallel to the above straight lines at an equal distance from either line. Part of the plane bounded by PQ, AB and BD can be regarded as a triangle and the angles at the vertices B and P are respectively equal to $(\mu + 1)\,\pi$ and zero. Let us transform this triangle into the upper half-plane and let the vertices B, P and Q correspond to the points $\tau = -1$, 0 and ∞. Using formula (47) we have

$$z = a \int_0^\tau (\tau + 1)^\mu \tau^{-1}\, d\tau, \tag{127}$$

where the constant a can be regarded as positive; this can be achieved by rotating the z-plane. The z-plane is that shown in Fig. 50 and τ corresponds to the plane in which our triangle is represented by the upper half-plane. If we reflect the above triangle in the line PQ, then the half-plane will be reflected in the line $0 < \tau < +\infty$ on the real axis, and part of the z-plane between our two conductors will be reflected in the τ-plane as a plane with the cut $(-\infty, 0)$. If we now assume that

$$\tau = e^w,$$

and recall the image produced by an exponential function [19], we obtain a strip in the w-plane, bounded by straight lines parallel to the real axis and at a distance π from it, i.e. we obtain the strip

$$-\pi \leqslant I[w] \leqslant \pi,$$

where I is the symbol of the imaginary part. Thus w, as a function of z, has a constant imaginary part $\pm\pi$ on our conductors and represents the complex electrostatic potential of the two-dimensional field between our two conductors. Owing to the fact that $\tau = e^w$ we can rewrite (127) in the form:

$$z = a \int^w (e^w + 1)^\mu \, dw. \tag{128}$$

The value of the constant a evidently depends on the distance between the straight lines AB and AC. Suppose that this distance is equal to $2b$. For points in the z-plane near the point at $-\infty$, the isothermic net giving the families of equipotential lines and the lines of force will evidently be close to the net of Cartesian coordinates and this net corresponds to the net of Cartesian coordinates in a strip of width 2π in the w-plane. The point $z = -\infty$ corresponds to $\tau = 0$ and therefore to $w = -\infty$. Bearing in mind that the width of the strip in the z-plane is $2b$ and that when w tends to $-\infty$ the function e^w tends to zero and, consequently, from (128), dz/dw tends to a, we can see that the value of the constant a is equal to b/π. Consider the particular case when $\mu = 1/2$, i.e. when the straight lines BD and CD are perpendicular to AB and AC.

In this case we have:

$$z = \frac{b}{\pi} \int^w \sqrt{e^w + 1} \, dw, \tag{129}$$

whence the integral can easily be evaluated by substituting $e^w + 1 = t^2$.

The absence of the lower limit of integration means that any constant number can be added to z, i.e. it involves a parallel transition of the z-plane, which is of no significance.

3. Let l be a simple closed contour which is the trace of the cylindrical conductor in the XY-plane. Let e be a given charge on that conductor, per unit length along the z-axis. We have to evaluate the two-dimensional electrostatic field outside l. Reflect that part of the z-plane outside l onto the outside of a unit circle in the τ-plane, i.e. into the domain $|\tau| > 1$; we take it that the point at infinity is translated into itself, so that in the neighbourhood of $z = \infty$ the expansion of the function which performs the conformal transformation

is as follows:

$$\tau = \omega(z) = cz + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \tag{130}$$

We shall show that it is possible to construct the complex potential for our electrostatic field from a knowledge of the above conformal transformation. In fact, let us construct the function:

$$f(z) = i\,2e \log \frac{\tau_0}{\tau} \qquad [\tau = \omega(z)],$$

where τ_0 is a constant which plays no essential part. The imaginary part of the above function will obviously be:

$$\mathrm{I}[f(z)] = 2e\,[\log|\tau_0| - \log|\tau|].$$

Since $|\tau| = 1$ lies on the curve l we can say that the imaginary part remains constant on l. Let us now determine the value of our potential in the neighbourhood of the point at infinity. From the expansion (130) we have the following expression for $f(z)$ near $z = \infty$:

$$f(z) = -i\,2e \log z + d_0 + \frac{d_1}{z} + \dots$$

The first term of the above expansion gives the potential $-2e \log|z|$ which, according to (120), corresponds to the given quantity of electricity on the conductor. Formula (125) gives the following expression for the density of distribution of the charge along l:

$$\varrho = \frac{1}{4\pi}|f'(z)| = \frac{e}{2\pi} \cdot \left|\frac{1}{\tau}\frac{d\tau}{dz}\right|$$

or, since $|\tau| = 1$:

$$\varrho = \frac{e}{2\pi}\left|\frac{d\tau}{dz}\right| = \frac{e}{2\pi}\left|\frac{dz}{d\tau}\right|^{-1}. \tag{131}$$

If l is a square then the relationship between τ and z can be written in the form [38]:

$$z = a\int^{\tau} \frac{\sqrt{\tau^4 + 1}}{\tau^2}\,d\tau, \tag{132}$$

when we take on the circle $|\tau| = 1$ the points

$$\exp\left[\frac{(\pi + 2k\pi)i}{4}\right] \qquad (k = 0, 1, 2, 3),$$

which correspond to the vertices of the square.

Formula (132) gives:

$$\frac{dz}{d\tau} = b\,\frac{\sqrt{\tau^4 + 1}}{\tau^2},$$

and, in this case, we can rewrite (131) as follows:

$$\varrho = \frac{e}{2\pi a}\left|\frac{\tau^2}{\sqrt{\tau^4 + 1}}\right|,$$

where a is a constant which depends on the length of side of the square. On the contour of the square we have $\tau = e^{i\varphi}$, where φ is a polar angle of the unit circle. A side of the square corresponds to the change of this angle through the interval $(\pi/4, 3\pi/4)$ and we therefore have the following expression for the length of side of the square:

$$s = a \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sqrt{e^{i4\varphi} + 1}}{e^{i2\varphi}}\, ie^{i\varphi}\, d\varphi,$$

from which, after performing a simple change of variables we arrive without difficulty at the following formula which connects the lengths of the sides of the square s with the constant a:

$$s = a\sqrt{2} \int_0^{\frac{\pi}{2}} \sqrt{\cos\vartheta}\, d\vartheta. \tag{133}$$

The integral on the right-hand side cannot be expressed in a finite form and belongs to a class of so called elliptic integrals.

47. The two-dimensional magnetic field. Above we explained the connection between the analytic functions of a complex variable and a two-dimensional electrostatic field. In exactly the same way we can consider a two-dimensional magnetic field originating from infinite straight currents, which are perpendicular to the XY-plane. We shall give the fundamental results which apply here without proof. For the vector of magnetic force we have the formulae:

$$H_x = \frac{\partial\varphi}{\partial y}; \quad H_y = -\frac{\partial\varphi}{\partial x}, \tag{134}$$

where the function φ satisfies the Laplace equation outside the field's source and is therefore the real part of a certain analytic function:

$$f(z) = \varphi + i\psi. \tag{135}$$

Using the Cauchy–Riemann equations we can rewrite (134) in the form:

$$H_x = -\frac{\partial\psi}{\partial x}; \quad H_y = -\frac{\partial\psi}{\partial y}$$

or

$$\mathbf{H} = -\operatorname{grad}\psi,$$

so that ψ is the potential of the field.

The lines $\varphi(x, y) = C_1$ which are orthogonal to $\psi(x, y) = C_2$ are thus lines of force of the field. When we have one straight current of force q directed along the Z axis, we have for the function (135):

$$f(z) = -2q \log z,$$

hence $\varphi = -2q \log r$ and $\psi = -2q \arg z$. The lines of force are circles, centre the origin, and by encircling the origin the potential receives an increment $(-4\pi q)$. In this case one of the lines $\varphi(x, y) = C_2$ must lie on the surface of the magnetic conductor (magnetic permeability equal to infinity).

48. Schwarz's formula. The above applications of analytic functions of a complex variable to problems of hydrodynamics and electrostatics were essentially based on the close connection existing between harmonic and analytic functions of a complex variable. We mentioned this earlier in [2].

Let us formulate once again the main points: *the real and imaginary parts of an analytic function are harmonic functions and, conversely, every harmonic function can be regarded as the real part of an analytic function; its imaginary part can be determined accurately as far as the constant term, i.e. the function can be determined as far as the purely imaginary constant term.* From what was said earlier in [II, 194] a harmonic function is defined uniquely in a connected domain by its limiting values on the contour of that domain (Dirichlet's problem). We can therefore say that *the function $f(z)$, which is regular in a domain B with contour l, is defined accurately as far as its purely imaginary constant term by the given values of its real part on the contour l.* There is no simple formula which would give the solution of this problem for every domain, i.e. which would define the regular function from the given contour values of its real part. It is not difficult however to construct such a function for a circle and we shall do this now.

Consider a circle, centre the origin and radius R. Let $u(x, y)$ be the part of the unknown analytic function. This harmonic function is defined from its contour values $u(\varphi)$ by Poisson's integral, which as we know, has the following form [II, 196]:

$$u(x, y) = u(r, \vartheta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\varphi)\frac{R^2 - r^2}{R^2 - 2rR\cos(\varphi - \vartheta) + r^2}\,d\varphi \quad (r < R). \quad (136)$$

It can easily be seen that the core of Poisson's integral i.e. the fraction integrand, is the real part of an analytical function viz.:

$$\frac{R^2 - r^2}{R^2 - 2rR\cos(\varphi - \vartheta) + r^2} = \text{real}\left[\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right] \qquad (z = re^{i\vartheta} = x + iy).$$

If, instead of Poisson's core we substitute an analytic function of the complex variable z in the integral, we obtain a function of the complex variable z, the real part of which coincides with $u(x, y)$. This function is given below:

$$f(z) = u(x, y) + iv(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\varphi)\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\,d\varphi. \quad (137)$$

By assuming that $z = 0$ in this formula we obtain a purely real value for $f(z)$, i.e. formula (137) gives one solution of our problem where the real value is at the origin. If we denote by Ci the imaginary part of the unknown function at the origin, then the general solution of the problem takes the form:

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\varphi) \frac{Re^{i\varphi} + z}{Re^{i\varphi} - z} \, d\varphi + Ci. \tag{138}$$

This formula is usually known as *Schwarz's formula.*

If we separate the imaginary part of the fraction in the integrand

$$\text{im. part} \left[\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z} \right] = \frac{2rR \sin(\vartheta - \varphi)}{R^2 - 2rR \cos(\varphi - \vartheta) + r^2},$$

we obtain *an expression for the imaginary part of the regular function in the circle in terms values of its real part on the contour:*

$$v(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\varphi) \frac{2rR \sin(\vartheta - \varphi)}{R^2 - 2rR \cos(\varphi - \vartheta) + r^2} \, d\varphi + C. \tag{139}$$

All we said above is closely connected with the concept of conjugate trigonometric series.

Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi)$$

be a Fourier series of the function $u(\varphi)$ representing the limiting values of the real part of $f(z)$. As we know from [II, 195] the same real part can be represented in the circle not by Poisson's integral but by a series of the type:

$$u(x, y) = u(r, \vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\vartheta + b_n \sin n\vartheta) r^n. \tag{140}$$

We have for the imaginary part of a conjugate trigonometric series [25]:

$$v(x, y) = v(r, \vartheta) = C + \sum_{n=1}^{\infty} (-b_n \cos n\vartheta + a_n \sin n\vartheta) r^n. \tag{141}$$

If the function $u(\varphi)$ has satisfactory properties, for example its first derivative satisfies Dirichlet's conditions, then the series (141), like the series (140), is uniformly convergent in the whole closed circle and the function $v(r, \theta)$ is harmonic in the circle and continuous in the

closed circle. $v(r, \theta)$ is usually known as *a function conjugate with* $u(r, \theta)$ [2], and the same name also applies to all its limiting values $v(1, \varphi)$ with respect to $u(\varphi)$.

Assume that two Schwarz's integrals give one and the same function which is regular in the circle

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u_1(\varphi)\,\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\,d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_2(\varphi)\,\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\,d\varphi, \qquad (142)$$

where $u_1(\varphi)$ and $u_2(\varphi)$ are continuous real functions. It can be seen that these functions coincide for they are the limiting values of the same harmonic function, viz. of the real part of our regular function. Therefore *the identity* (142) *of z is equivalent to the identity* $u_1(\varphi) = u_2(\varphi)$ *of* φ. This is essentially Harnak's theorem which we mentioned in [8].

49. The core cot $(s-t)/2$. We shall now apply the fundamental theorem of limiting values of Cauchy's integrals [28] to the circle $|z| = 1$, centre the origin and unit radius. Assume that we are given a real function $u(\tau)$ on this circle where $\tau = e^{is}$, which satisfies a Lipschitz condition. By using Schwarz's formula [48] we can construct a function which is regular in this circle and the real part of which has the limiting value $u(\tau)$ on the circle:

$$u\left(re^{i\varphi}\right) + v\left(re^{i\varphi}\right)i = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau)\,\frac{\tau+z}{\tau-z}\,ds \qquad \left(z = re^{i\varphi}\right), \qquad (143)$$

or, since $d\tau = i\tau ds$,

$$u\left(re^{i\varphi}\right) + v\left(re^{i\varphi}\right)i = \frac{1}{2\pi i}\int_{|\tau|=1} u(\tau)\,\frac{\tau+z}{\tau(\tau-z)}\,d\tau.$$

Putting $\tau + z = (\tau - z) + 2z$ and breaking the right-hand side up into two integrals we obtain:

$$u\left(re^{i\varphi}\right) + v\left(re^{i\varphi}\right)i = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau)\,ds + \frac{2z}{2\pi i}\int_{|\tau|=1}\frac{u(\tau)}{\tau}\cdot\frac{1}{\tau-z}\,d\tau.$$

Let us suppose that the point $z = re^{i\varphi}$ tends to a point $\xi = e^{it}$ on the circle $|z| = 1$. Using the theorem of limiting values of Cauchy's integrals [28] we obtain the limiting value for our function:

$$u\left(e^{it}\right) + v\left(e^{it}\right)i = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau)\,ds + \xi\,\frac{u(\xi)}{\xi} + \frac{2\xi}{2\pi i}\int_{|\tau|=1}\frac{u(\tau)}{\tau}\cdot\frac{1}{\tau-\xi}\,d\tau,$$

or

$$u\left(e^{it}\right) + v\left(e^{it}\right)i = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau)\,ds + u(\xi) + \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau)\,\frac{2\xi}{\tau-\xi}\,ds, \qquad (144)$$

but

$$\frac{2\xi}{\tau - \xi} = \frac{2e^{it}}{e^{is} - e^{it}} = -1 + i \cot \frac{t-s}{2},$$

and by separating the imaginary part in (144) we obtain a formula where the limiting values of the imaginary parts are expressed in terms of the real part:

$$v\left(e^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(e^{is}\right) \cot \frac{t-s}{2} \, ds,$$

and where the integral must be taken in its principal value sense. We shall write $u(s)$ and $v(t)$ instead of $u(e^{is})$ and $v(e^{it})$:

$$v(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(s) \cot \frac{t-s}{2} \, ds. \tag{145}$$

We recall that (143) gives a regular function in the circle $|z| < 1$, the imaginary part of which is zero at the centre of the circle. Bearing in mind that the value of a harmonic function at the centre of a circle is equal to the average arithmetic mean on the circle [II, 194], we can write:

$$\int_{-\pi}^{\pi} v(t) \, dt = 0. \tag{146}$$

The function $u(s)$ is a periodic function of period 2π and the function $v(t)$ is also obtained in periodic form; in formula (145) we can therefore take any period, 2π in length, for the interval of integration. The function $\cot z$, has a simple pole with unit residue when $z = 0$, [21] and we can express the core of the linear transformation by Cauchy's core:

$$\frac{1}{2} \cot \frac{t-s}{2} = -\frac{1}{s-t} + P(t-s), \tag{147}$$

where $P(z)$ is an analytic function, regular at all points on the line $2\pi < z < 2\pi$. It can be shown in the same way as in [27] that if the periodic function $u(s)$ satisfies a Lipschitz condition of order α, then $v(t)$ also satisfies a Lipschitz condition of the same order, when $\alpha < 1$, or of any order smaller than unity, when $\alpha = 1$. It appears from (147) that this statement can also be derived from an analogous case for Cauchy's core.

On applying the linear transformation (145) to the function $v(t)$ we obtain a new function $w(t_1)$, which satisfies a Lipschitz condition:

$$w(t_1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(t) \cot \frac{t_1 - t}{2} \, dt.$$

The function $w(t_1)$ gives the limiting values of the imaginary part if we take $v(t)$ for the limiting values of the real part when:

$$\int_{-\pi}^{\pi} w(t_1) \, dt_1 = 0. \tag{148}$$

On the other hand, if the regular function (143) is multiplied by $(-i)$ we obtain the regular function $v(re^{i\varphi}) - u(re^{i\varphi})\,i$. Having thus taken the real part, the imaginary part can be determined accurately as far as the constant term and we can therefore write:

$$w(t_1) = -u(t_1) + C.$$

To determine the constant C we integrate both sides of this equation over the interval $(-\pi, +\pi)$ and, remembering (148) we have:

$$0 = -\int_{-\pi}^{\pi} u(t_1)\,dt_1 + 2\pi C,$$

and finally:

$$w(t_1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t)\cot\frac{t_1-t}{2}\,dt = -u(t_1) + \frac{1}{2\pi}\int_{-\pi}^{\pi} u(s)\,ds, \qquad (149)$$

i.e. the two applications of the transformation (145) give us minus the original function accurately as far as the constant term. The result can be written in the form:

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} u(s)\cot\frac{t-s}{2}\,ds\right]\cot\frac{t_1-t}{2}\,dt = -u(t_1) + \frac{1}{2\pi}\int_{-\pi}^{\pi} u(s)\,ds. \qquad (150)$$

This formula is known as Hilbert's formula and the core of the transformation (145) is usually known as Hilbert's core. Notice that on the left-hand side of formula (149), as in Fourier's integral, we cannot change the order of integration. Denoting the transformation (145) by a single letter h we can write formula (145) in the form:

$$v(s) = h[u(s)],$$

where s denotes the amplitude of both functions. In this case Hilbert's formula (150) can be written:

$$h^2[u(s)] = u(s) - \frac{1}{2\pi}\int_{-\pi}^{\pi} u(s)\,ds.$$

Formula (145) can be regarded as an integral equation of $u(s)$, where $v(t)$ is the given function. It follows from above that this equation is soluble only when the condition (146) is satisfied. One solution of this equation, according to (149), will be given by the function:

$$u(s) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} v(t)\cot\frac{s-t}{2}\,dt. \qquad (151)$$

This is also the solution of the equation (149) which satisfies the condition

$$\int_{-\pi}^{\pi} u(s)\,ds = 0.$$

In other words, this is the imaginary part of the real function $v(re^{i\varphi}) - iu(re^{i\varphi})$, which vanishes at the origin. If the value of the function $u(re^{i\varphi})$ at the origin is equal to C then

$$u(s) = C - \frac{1}{2\pi} \int_{-\pi}^{\pi} v(t) \cot \frac{s-t}{2} \, dt, \tag{152}$$

where $u(s) = \text{const.}$ is the solution of the homogeneous equation

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(s) \cot \frac{t-s}{2} \, ds = 0,$$

for when $u(s) = \text{const.}$, the imaginary part v, which vanishes at the origin, must be equal to zero. Formula (152) gives all the solutions of the equation (145), for the imaginary part is determined accurately as far as the constant term in terms of its real part. We have assumed throughout that both the given and the unknown functions satisfy a Lipschitz condition.

The transformation (145) can be written in the form of an indefinite integral similar to the one used in the core of Cauchy's integral. In fact, taking into consideration that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cot \frac{t-s}{2} \, ds = 0,$$

since the homogeneous equation (145) has a solution equal to a constant $u(s) = c$ we can rewrite the formula (145) in the form:

$$v(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [u(s) - u(t)] \cot \frac{t-s}{2} \, ds. \tag{153}$$

Let us suppose that the function $u(s)$ has a continuous derivative. Taking into consideration that

$$\cot \frac{t-s}{2} = -\frac{d}{ds} \log \left(\sin^2 \frac{t-s}{2} \right),$$

and applying the formula for integration by parts in the intervals $(-\pi, t-\varepsilon)$, and $(t+\varepsilon, \pi)$ to the integral (145) and also taking into account the formula:

$$\lim_{\varepsilon \to +0} [u(t+\varepsilon) - u(t-\varepsilon)] \log \left(\sin^2 \frac{\varepsilon}{2} \right) = u'(\xi) \, 2\varepsilon \log \sin^2 \frac{\varepsilon}{2}$$

$$(t - \varepsilon < \varepsilon < t + \varepsilon),$$

we obtain the following expression for $v(t)$:

$$v(t) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \omega'(s) \log \left(\sin^2 \frac{t-s}{2} \right) ds,$$

where the integral on the right-hand side is undefined.

If $u(\tau)$ satisfies a Lipschitz condition, then the function of the complex variable $z = re^{i\varphi}$, given by formula (143), is continuous as far as the circumference

$|z| = 1$, as we saw above. Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos ks + b_k \sin ks) \tag{154$_1$}$$

be a Fourier series of the function $u(s)$. For the function $v(s)$ we then have the following Fourier series [48]:

$$\sum_{k=1}^{\infty} (-b_k \cos ks + a_k \sin ks). \tag{154$_2$}$$

As a result of the equation [II, 147]:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} u^2(s)\,ds = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \quad \text{and} \quad \frac{1}{\pi}\int_{-\pi}^{\pi} v^2(s)\,ds = \sum_{k=1}^{\infty}(b_k^2 + a_k^2),$$

and consequently:

$$\int_{-\pi}^{\pi} v^2(s)\,ds \leqslant \int_{-\pi}^{\pi} u^2(s)\,ds,$$

where the sign of equality applies only when $a_0 = 0$. Thus, as a result of the transformation (145), the integral of the square of the function in the interval $(-\pi, +\pi)$ can only decrease. Notice that we have supposed that the function $u(s)$ is real. We can thus see that the transformation (145) is equivalent to the transition from the Fourier series (154$_1$) to the series (154$_2$).

50. Limiting problems. Dirichlet's problem is the simplest case of limit problems in which harmonic functions are involved. Let us formulate the general limiting problem for harmonic functions where Dirichlet's problem is a particular case: it is necessary to find a harmonic function in a connected domain B with contour l, which satisfies on this contour a limiting condition of the form:

$$au + b\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = d, \tag{155}$$

where a, b, c and d are real functions given on the contour l, which we assume to be functions of the length of the arc s of that contour. We also assume that u is the real part of a regular function

$$f(z) = u(x, y) + iv(x, y).$$

As we know:

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y},$$

and, consequently, we have

$$b\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = \mathcal{R}[(b + ic) f'(z)],$$

where $\mathcal{R}$ is the symbol of the real part.

Condition (155) can be rewritten in the form:

$$\mathcal{R}\,[af(z) + (b + ic)\,f'(z)] = d, \tag{156}$$

and therefore the problem involves the finding of a function, regular in B, which would satisfy the condition (144) on the contour.

Let $z = \omega(\tau)$ be the known function which conformally transforms our domain B into the unit circle $|\tau| < 1$. We can assume that the unknown function is a function $F(\tau)$ which is regular in the unit circle

$$F(\tau) = f[\omega(\tau)]; \qquad f'(z) = F'(\tau)\frac{1}{\omega'(\tau)}.$$

We have instead of (144)

$$\mathcal{R}\left[aF(\tau) + \frac{b + ic}{\omega'(\tau)}F'(\tau)\right] = d \qquad (|\tau| = 1),$$

where, as a result of the transformation $z = \omega(\tau)$, we can take it that a, b, c and d are defined on the circle $|\tau| = 1$. Hence our problem simply involves a circle.

Consider in greater detail the case when the limiting condition (155), which applies to the circle $|z| = 1$, does not contain the unknown function u. In this case the problem can be formulated as follows: it is necessary to find a harmonic function $u(x, y)$ in a unit circle, which satisfies on this circle a limiting condition of the form:

$$b\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = d.$$

We suppose that u is the real part of a regular function $f(z)$. In this case $\partial u/\partial x$ and $-\partial u/\partial y$ are the real and imaginary parts respectively of a regular function $f'(z)$ and the above problem is thus equivalent to the following problem, usually known as *Hilbert's problem: find a function $f(z)$ which is regular in a unit circle, the real and imaginary parts of which satisfy on the circle a limiting condition of the form:*

$$l(\varphi)\,u(\varphi) + m(\varphi)\,v(\varphi) = d(\varphi) \qquad (0 \leqslant \varphi \leqslant 2\pi), \tag{157}$$

where $l(\varphi)$, $m(\varphi)$ and $d(\varphi)$ are the given functions of the polar angle φ on the unit circle. We assume that the coefficients are continuous functions and that $l(\varphi)$ and $m(\varphi)$ do not vanish simultaneously. When both sides of the equation (157) are divided the coefficients satisfy the condition:

$$l^2(\varphi) + m^2(\varphi) = 1. \tag{158}$$

We can assume:

$$l(\varphi) = \cos\omega(\varphi); \qquad m(\varphi) = -\sin\omega(\varphi), \tag{159}$$

where $\omega(\varphi)$ is a function of φ, viz.

$$\omega(\varphi) = -\operatorname{arc\,tan}\frac{m(\varphi)}{l(\varphi)}. \tag{160}$$

Let us consider in detail the case when formula (159) gives $\omega(\varphi)$ as a single-valued function of φ. This will be so, for example, when $l(\varphi)$ and $m(\varphi)$ do not vanish in the

interval $(-\pi, +\pi)$. Using the function $\omega(\varphi)$ we can write the limiting condition (157) in the form:

$$\mathcal{R}\left[e^{i\omega(\varphi)} f(z)\right] = d(\varphi) \qquad (z = e^{i\varphi}). \tag{161}$$

Let us construct a function $\pi(z)$ from its real part $\omega(\varphi)$ by using Schwarz's formula:

$$\pi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(\varphi) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \tag{162}$$

Denote by $\omega_1(\varphi)$ the limiting values of its imaginary part. The function

$$e^{i\pi(z)} f(z)$$

has a real part on the unit circumference $z = e^{i\varphi}$ which is equal to

$$e^{-\omega_1(\varphi)} \mathcal{R}\left[e^{i\omega(\varphi)} f(z)\right]_{z = e^{i\varphi}}$$

and, consequently, the limiting condition (161) is equivalent to the following limiting condition:

$$\mathcal{R}\left[e^{i\pi(z)} f(z)\right] = d(\varphi) e^{-\omega_1(\varphi)}.$$

By knowing the real part on the contour we can define the function inside the contour by again using Schwarz's formula

$$e^{i\pi(z)} f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d(\varphi) e^{-\omega_1(\varphi)} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi + iC,$$

where $\omega_1(\varphi)$ are the limiting values of the imaginary part of the function (162):

$$\omega_1(\varphi) = \lim_{r \to 1} \mathcal{J}\left[\int_{-\pi}^{\pi} \omega(\psi) \frac{e^{i\psi} + re^{i\varphi}}{e^{i\psi} - re^{i\varphi}} d\psi\right]. \tag{163}$$

where $\mathcal{J}$ is the symbol of the imaginary part.

Finally we obtain the following expression for $f(z)$:

$$f(z) = e^{-i\pi(z)}\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} d(\varphi) e^{-\omega_1(\varphi)} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi + iC\right]. \tag{164}$$

Consider now the case when the function $\omega(\varphi)$ acquires the increment $(-2n\pi)$ on describing a circuit round the unit circle, where n is a positive integer

$$\omega(\pi) - \omega(-\pi) = -2n\pi. \tag{165}$$

Let us construct a single-valued function on the unit circle:

$$\chi(\varphi) = \omega(\varphi) + n\varphi,$$

and construct a similar corresponding function for the complex variable $\delta(z)$, for which $X(\varphi)$ is the limiting value of the real part. The limiting values of

the real part of the function

$$\sigma_1(z) = \sigma(z) + in \log z$$

are equal to $\omega(\varphi)$ and the limiting values of its imaginary part are evidently the same as those of the function $\delta(z)$. We denote them again by $\omega_1(\varphi)$. It can be shown similarly that the limiting values of the real part of the function

$$e^{i\sigma_1(z)} f(z) = z^{-n} e^{i\sigma(z)} f(z).$$

must be equal to

$$\mathcal{R}\left[z^{-n} e^{i\sigma(z)} f(z)\right] = d(\varphi) e^{-\omega_1(\varphi)} \tag{166}$$

Owing to the presence of the factor z^{-n} this function can have a pole of an order not higher than n at the origin. To begin with, let us construct with the aid of Schwarz's formula a function, regular in a unit circle, the limiting values of the real part of which are

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d(\varphi) e^{-\omega_1(\varphi)} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi + iC. \tag{167}$$

We must add to the above function a term, the real part of which is equal to zero on the unit circle but which can have a pole of order n at the origin. It can readily be seen that this term will have the form

$$\sum_{k=1}^{n} \left[A_k \left(\frac{1}{z^k} - z^k\right) + iB_k \left(\frac{1}{z^k} + z^k\right)\right],$$

where A_k and B_k are arbitrary real constants.

Adding the latter expression to the expression (167) we obtain the general solution of the problem

$$f(z) = z^n e^{-i\sigma(z)} \left\{Ci + \sum_{k=1}^{n} \left[A_k \left(\frac{1}{z^k} - z^k\right) + iB_k \left(\frac{1}{z^k} + z^k\right)\right] + \right.$$

$$\left. + \frac{1}{2\pi} \int_{-\pi}^{\pi} d(\varphi) e^{-\omega_1(\varphi)} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi \right\}. \tag{168}$$

When n is a negative integer in formula (165) the solution of the problem will be different. The function under the symbol of the real part in the expression (166) will then not only be regular in the unit circle but it will also have zero of order n at the origin. When constructing a regular function on the right-hand side of formula (166) with the aid of Schwarz's integral, we must also write down the condition that the function obtained must have a zero of order n at the origin. We therefore have several conditions which must be satisfied by the function $d(\varphi)$ if the problem is to be solved.

Consider another particular case, viz.: suppose that *the limiting values of a harmonic function on the unit circle have the form:*

$$\frac{\partial u}{\partial n} + l \frac{\partial u}{\partial s} + mn = d(\varphi), \tag{169}$$

where l and m are constants, $d(\varphi)$ is the given function, n is the direction of the outside normal to the circle and s is the direction of the tangent to the circle. Instead of taking derivatives along the axes of coordinates we take them, in this case, in the directions connected with the boundary curve which are indicated above. As we know from [II, 108] these derivatives are expressed in terms of each other. The limiting conditions, as expressed by formula (169), are frequently used in mathematical physics. Differentiation along the normal n coincides with differentiation along the radius-vector r, and differentiation along s coincides with differentiation with respect to the polar angle φ, when $r = 1$. In general by assuming that $z = re^{i\varphi}$, and $u = \mathcal{R} \mid f(z) \mid$, we have the following, when $\mathcal{I} \mid f(0) \mid = 0$:

$$\frac{\partial u}{\partial n} = \mathcal{R}\,[z' f'(z')]; \quad \frac{\partial u}{\partial s} = \mathcal{R}\,[z' if'(z')] \qquad \left(z' = e^{i\varphi}\right),$$

and the limiting condition (169) can be rewritten in the form:

$$\mathcal{R}\,[(1 + il)\, z' f'(z') + mf(z')] = d(\varphi) \qquad \left(z' = e^{i\varphi}\right).$$

We multiply both sides of the equation by

$$\frac{1}{2\pi} \frac{z' + z}{z' - z}\, d\varphi$$

and integrate with respect to φ. We then obtain a new equation which is equivalent to the one above [48]. Using Schwarz's formula it can readily be seen that this new equation will have the form:

$$(1 + il)\, zf'(z) + mf(z) = F(z), \tag{170}$$

where

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d(\varphi) \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, d\varphi = \frac{1}{2\pi i} \int_{|z'|=1} d(\varphi) \frac{z' + z}{z'(z' - z)}\, dz'. \tag{171}$$

Equation (170) is a linear differential equation of the first order. Solving with the usual formula [II, 4] we obtain the following expression for the unknown function:

$$f(z) = z^{-k}\left[C + \frac{k}{m} \int_{z_0}^{z} z^{k-1} F(z)\, dz\right], \tag{172}$$

where

$$k = \frac{m}{1 + il}.$$

In equation (172) the arbitrary constant C can be determined from the condition that the point $z = 0$ is a regular point for $f(z)$. If

$$d(\varphi) = A_0 + \sum_{s=1}^{n} (A_s \cos s\varphi + B_s \sin s\varphi),$$

then we have for $F(z)$

$$F(z) = A_0 + \sum_{s=1}^{n} (A_s - iB_s) z^s.$$

Substituting in (172) and integrating we finally obtain the following expansion for $f(z)$:

$$f(z) = \frac{A_0}{m} + \sum_{s=1}^{n} \frac{A_s - iB_s}{m + s(1 + il)} z^s.$$

51. The biharmonic equation. We shall now consider the connection between the theory of analytic functions of the complex variable and the theory of the so called *biharmonic functions*, i.e. functions which satisfy the condition

$$\Delta\Delta u(x, y) = 0, \tag{173}$$

where Δ is the Laplace operator, which expresses the sum of the second derivatives of the variables x and y (we are considering the two-dimensional case). Equation (173) can be written as follows:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

or

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0. \tag{174}$$

Let u be a function, which together with its derivatives, is continuous in a connected domain B where it satisfies the equation (174). According to (173) the function

$$\Delta u = p(x, y) \tag{175}$$

is a harmonic function. Suppose that $q(x, y)$ is a conjugate function so that

$$p(x, y) + iq(x, y) = f(z) \tag{176}$$

is an analytic function of the complex variable $z = x + iy$.

Construct the analytic function

$$\varphi(z) = \frac{1}{4}\int f(z)\, dz = r(x, y) + is(x, y). \tag{177}$$

We obviously have:

$$\Delta r = \Delta s = 0; \quad \frac{\partial r}{\partial x} = \frac{\partial s}{\partial y} = \frac{1}{4}\mathcal{R}[f(z)] = \frac{1}{4}p. \tag{178}$$

We now evaluate the Laplace operator for the expression $u - (rx + sy)$. We have from (178):

$$\Delta[u - (rx + sy)] = p - 2\frac{\partial r}{\partial x} - 2\frac{\partial s}{\partial y} = 0,$$

i.e. the above expression is a harmonic function which we shall denote by p_1. Introducing the conjugate function q_1 and the corresponding function of the complex variable $\psi(z) = p_1 + iq_1$ we can write:

$$u - (rx + sy) = p_1; \quad u = (rx + sy) + p_1 = \mathcal{R}\,[(x - iy)\,(r + is)] + p_1$$

or

$$u = \mathcal{R}[\bar{z}\varphi(z) + \psi(z)]. \tag{179}$$

Hence *according to formula* (179) *every biharmonic function can be expressed by two functions of a complex variable.* The converse can also be readily proved by choosing arbitrarily the analytic functions $\varphi(z)$ and $\psi(z)$, where formula (179) gives the biharmonic function, i.e. *formula* (179), *which contains two arbitrary analytic functions, gives the general expression for a biharmonic function. This formula is usually known as Hurse's formula.*

For any given biharmonic function u the functions $\varphi(z)$ and $\psi(z)$ in formula (179) are not fully defined, for they do not contain arbitrary constants. The real function $q(x, y)$ is defined accurately except for the constant term, i.e. the function $f(z)$ is defined except for the purely imaginary constant term. Also, when defining the function $\varphi(z)$ in formula (177) the arbitrary constant complex term must be taken into consideration. In its final form the function $\varphi(z)$ will contain arbitrary elements of the following form:

$$C + iaz,$$

where C is an arbitrary complex constant and a an arbitrary real constant. These arbitrary constants can be determined when certain additional conditions are made, for example conditions of the form

$$\varphi(0) = 0; \quad \mathcal{J}\,[\varphi'(0)] = 0 \qquad (\mathcal{J} \text{ being the imaginary part}) \tag{180}$$

Similarly, when determining $\psi(z)$, we obtain the arbitrary constant as a purely imaginary constant, which can be determined if the function $\psi(z)$ is, for example, subjected to the condition

$$\mathcal{J}\,[\psi(0)] = 0. \tag{181}$$

The conditions (180) and (181) fully define the functions $\varphi(z)$ and $\psi(z)$; we do, of course, assume that the point $z = 0$ belongs to our region.

Let us now consider the fundamental limiting problem for biharmonic functions. It can be formulated as follows: find a biharmonic function inside a closed contour l from the given values of the function and its normal derivative on this contour:

$$u = \omega_1(s); \quad \frac{\partial u}{\partial n} = \omega_2(s) \qquad (\text{on } l). \tag{182}$$

We will show that the limiting conditions (182) also give directly the limiting values of the coordinates of the derivatives of the function u. In fact, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cos(s, x) + \frac{\partial u}{\partial n}\cos(n, x); \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cos(s, y) + \frac{\partial u}{\partial n}\cos(n, y),$$

where s is the direction tangential to the contour l. Hence from the limiting conditions (182) follow the following limiting conditions:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \omega_1' \cos(s, x) + \omega_2 \cos(n, x) = \omega_3(s), \\ \frac{\partial u}{\partial y} &= \omega_1' \cos(s, y) + \omega_2 \cos(n, y) = \omega_4(s). \end{aligned}\right\} \tag{183}$$

The functions $\omega_3(s)$ and $\omega_4(s)$ cannot be taken arbitrarily in the above expressions, viz. the line integral

$$\int \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy, \tag{184}$$

which gives the increments of the function round the closed contour, must be equal to zero, since the function u must be single-valued. We thus arrive at the following condition for the functions $\omega_3(s)$ and $\omega_4(s)$ in the limiting conditions (183):

$$\int_l [\omega_3(s) \cos(s, x) + \omega_4(s) \cos(s, y)]\,ds = 0. \tag{185}$$

Apart from this the choice of these functions can be arbitrary.

We shall try to find the biharmonic function by using Hurse's formula (179). Differentiating with respect to x and y and using z and $\bar{z}$ we have:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \mathcal{R}[\varphi(z) + \bar{z}\,\varphi'(z) + \psi'(z)], \\ \frac{\partial u}{\partial y} &= \mathcal{R}[-i\varphi(z) + i\bar{z}\,\varphi'(z) + i\psi'(z)] = \mathcal{I}[\varphi(z) - z\,\overline{\varphi}'(z) - \psi'(z)]. \end{aligned}\right\} \tag{186}$$

We thus obtain two equations which must be satisfied by the unknown functions $\varphi(z)$ and $\psi(z)$ on the contour l:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} &= \overline{\varphi(z)} + \bar{z}\,\varphi'(z) + \psi'(z) = \omega_3(s) - i\omega_4(s), \\ \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} &= \varphi(z) + \overline{z\,\varphi'(z)} + \overline{\psi'(z)} = \omega_3(s) + i\omega_4(s). \end{aligned}\right\} \tag{187}$$

The second of these equations is obviously obtained from the first as a result of the transition to conjugate values. We thus obtain a limiting problem for two analytic functions.

Here, as in the case of harmonic functions, we are only considering the interior problem, i.e. the bounded part of the plane.

In the two-dimensional problem of the theory of elasticity the tensions X_x, Y_y and X_y are expressed in terms of a biharmonic function according to the formulae:

$$X_x = \frac{\partial^2 u}{\partial y^2}; \quad Y_y = \frac{\partial^2 u}{\partial x^2}; \quad X_y = -\frac{\partial^2 y}{\partial x\,\partial y} \tag{188}$$

and by using Hurse's formula, the tension can be expressed by two analytic functions. Without going into details of the proof we shall just give the final result. Using the symbols from formula (179) we have:

$$\left.\begin{aligned} X_x + Y_y &= 4\mathcal{R}\,[\varphi'(z)]. \\ 2X_y + i\,(X_x - Y_y) &= -\,2i\,[\psi''(z) + \bar{z}\varphi''(z)]. \end{aligned}\right\} \tag{189}$$

With the aid of these formulae the two-dimensional statics problems in the theory of elasticity when tensions are given on the contour, can be solved as a limiting problem in the theory of functions of a complex variable.

An explanation of the connection between the theory of functions of a complex variable and the two-dimensional statics problems in the theory of elasticity was given by Prof. G. V. Kolosoff in his work: "*One application of the theory of functions of a complex variable to the two-dimensional mathematical problem in the theory of elasticity*". A systematic account of the applications of the theory of functions of a complex variable to problems of the theory of elasticity can be found in the book by Prof. N. I. Muskhelishvili *Some Fundamental Problems in The Mathematical Theory of Elasticity*.

52. The wave-equation and analytic functions. We saw in Volume II that for spreading waves, for example, acoustic or electromagnetic waves, the following equation is of the greatest importance

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{190}$$

it is usually known as the wave-equation. We shall now only consider the two-dimensional case, i.e. when the unknown function u does not depend on one of the coordinates, e.g. the z coordinate. In this case the wave-equation has the form:

$$a^2\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \left(c^2 = \frac{1}{a^2}\right). \tag{191}$$

where u is a function of the variables t, x and y. By using the analytic functions of the complex variable we can separate a certain class of solutions of the equation (191), which have important applications in physics; the use of analytic functions considerably simplifies all operations with this class of solutions.

Let us construct an auxiliary equation which plays an important part in all that follows:

$$l(\tau)\,t + m(\tau)\,x + n(\tau)\,y + p(\tau) = 0, \tag{192}$$

where $l(\tau)$, $n(\tau)$ and $p(\tau)$ are analytic functions of the complex variable τ. The equation (192) defines τ as a function of the variables t, x and y. Let us suppose that we have an analytic function $f(\tau)$ which, in

the final analysis, is a function of the variables t, x and y. We shall derive the formulae for the derivatives of this function. Denoting by δ' the derivative of the left-hand side of the equation (192) with respect to the variable τ and applying the usual rules for the differentiation of complicated and undefined functions we obtain without difficulty the following expressions for the derivatives of the function τ:

$$\frac{\partial\tau}{\partial t} = -\frac{l(\tau)}{\delta'}; \quad \frac{\partial\tau}{\partial x} = -\frac{m(\tau)}{\delta'}; \quad \frac{\partial\tau}{\partial y} = -\frac{n(\tau)}{\delta'}. \tag{193}$$

When evaluating derivatives of the second order it must be remembered that

$$\delta' = l'(\tau)\,t + m'(\tau)\,x + n'(\tau)\,y + p'(\tau) \tag{194}$$

depends, for example, on t both directly and through τ:

$$\frac{\partial^2\tau}{\partial t^2} = \frac{\partial}{\partial\tau}\left[\frac{l(\tau)}{\delta'}\right]\frac{l(\tau)}{\delta'} + \frac{l(\tau)\,l'(\tau)}{\delta'^2} = \frac{2l(\tau)\,l'(\tau)}{\delta'^2} - \frac{l^2(\tau)}{\delta'^3}\,\delta'', \tag{195}$$

which can be written as follows:

$$\frac{\partial^2\tau}{\partial t^2} = \frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[\frac{l^2(\tau)}{\delta'}\right]. \tag{196}$$

We obtain similarly:

$$\frac{\partial^2\tau}{\partial x^2} = \frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[\frac{m^2(\tau)}{\delta'}\right]; \quad \frac{\partial^2\tau}{\partial y^2} = \frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[\frac{n^2(\tau)}{\delta'}\right];$$

$$\frac{\partial^2\tau}{\partial x\,\partial y} = \frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[\frac{m(\tau)\,n(\tau)}{\delta'}\right]. \tag{197}$$

The given analytic function $f(\tau)$ depends on t, x and y through τ, and its derivatives are obtained by using the rules for differentiating complicated formulae. Bearing in mind earlier formulae we obtain:

$$\frac{\partial^2 f(\tau)}{\partial t^2} = f''(\tau)\left(\frac{\partial\tau}{\partial t}\right)^2 + f'(\tau)\frac{\partial^2\tau}{\partial t^2} =$$

$$= f''(\tau)\frac{l^2(\tau)}{\delta'^2} + f'(\tau)\frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[\frac{l^2(\tau)}{\delta'}\right], \tag{198}$$

which can be written as follows:

$$\frac{\partial^2 f(\tau)}{\partial t^2} = \frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[f'(\tau)\frac{l^2(\tau)}{\delta'}\right], \tag{199}$$

and in exactly the same way

$$\frac{\partial^2 f(\tau)}{\partial x^2} = \frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[f'(\tau)\frac{m^2(\tau)}{\delta'}\right]; \quad \frac{\partial^2 f(\tau)}{\partial y^2} = \frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[f'(\tau)\frac{n^2(\tau)}{\delta'}\right];$$

$$\frac{\partial^2 f(\tau)}{\partial x\,\partial y} = \frac{1}{\delta'}\frac{\partial}{\partial\tau}\left[f'(t)\frac{m(\tau)\,n(\tau)}{\delta'}\right]. \tag{200}$$

If we suppose that $u = f(\tau)$ and substitute in (191) we obtain an equation of the form:

$$\frac{1}{\delta'}\frac{\partial}{\partial \tau}\left[f'(\tau)\,\frac{m^2(\tau)+n^2(\tau)-a^2 l^2(\tau)}{\delta'}\right]=0,$$

from which it follows that $f(\tau)$ is the solution of the equation (191) if the coefficients of the equation (192) satisfy the relationship

$$m^2(\tau)+n^2(\tau)=a^2 l^2(\tau). \tag{201}$$

If we want to obtain a real solution, we must only take the real part of $f(\tau)$, which should separately satisfy the equation (191) in the same way as the imaginary part satisfies it.

Consider now the three-dimensional space (S) with coordinates (t, x, y). If in a domain B of this space the equation (192) gives real values for τ, then we wrongly supposed that the function $f(\tau)$ was an analytic function, since its argument takes real values only. It is sufficient to assume that $f(\tau)$ is an arbitrary function of a real variable with continuous derivatives of the second order.

This brings us to the following theorem which states the class of the solutions of the equation (191) which we mentioned above.

53. The fundamental theorem. *If in a domain B in the space (S) the equation* (192) *and the condition* (201) *define τ as a complex function of the variables t, x and y, then the real and imaginary parts of any analytic function $f(\tau)$ give a solution of the equation* (191). *If, however, τ is a real function of (t, x, y) in a given domain, then any arbitrary real function of τ with continuous derivatives up to the second order gives a solution of the equation* (191).

If $l(\tau) \neq 0$, then dividing both sides of (192) by $l(\tau)$, we can assume that $l(\tau) = 1$. We can also put $m(\tau)$ as a new complex variable $(-\theta)$. In this case the condition (201) gives $n^2(\tau) = a^2 - \theta^2$, since the equation (192) can be rewritten, for example, in the form:

$$t - \theta x + \sqrt{a^2-\theta^2}\,y + p(\theta) = 0 \tag{202}$$

where $p(\theta)$ is any analytic function of θ. Instead of $f(\tau)$ we should, of course, write $f(\theta)$.

Let us consider in greater detail the particular case when $p(\theta) = 0$. In this case the equation (202) has the form:

$$t - \theta x + \sqrt{a^2-\theta^2}\,y = 0 \quad \text{or} \quad 1 - \theta\frac{x}{t} + \sqrt{a^2-\theta^2}\,\frac{y}{t} = 0, \tag{203}$$

which defines θ as a function of two arguments

$$\xi = \frac{x}{t}; \quad \eta = \frac{y}{t}. \tag{204}$$

In this case the constructed solutions $f(\theta)$ of the equation (191) are also functions of the arguments (204), i.e. they are homogeneous functions of zero order of t, x and y. Such functions [I, 154] are defined from the relationship

$$u(kt, kx, ky) = u(t, x, y),$$

which should be an identity. The converse can also be shown, viz. that any such homogeneous solution of the equation (191) can be obtained in the way described above. In future we shall simply call such solutions *homogeneous solutions.*

Let us consider the equation (203) in greater detail. The radical $\sqrt{a^2 - \theta^2}$, will be a single-valued function in the θ-plane with a cut $(-a, +a)$ along the real axis [19]. We can fix the value of the above radical by the condition that it should be positive above the imaginary axis, i.e. when $\theta = ib$, where $b > 0$. This condition is equivalent to the fact that the above radical must be negative imaginary when $\theta > a$ or positive imaginary when $\theta < -a$ on the real axis. This can easily be proved by considering the continuous change of the argument of the above radical. The equation (203) can be rewritten in the form:

$$1 - \theta\xi + \sqrt{a^2 - \theta^2}\,\eta = 0. \tag{205}$$

Eliminating the radical and solving the quadratic equation so obtained we derive the following expression for θ:

$$\theta = \frac{\xi - i\eta\sqrt{1 - a^2(\xi^2 + \eta^2)}}{\xi^2 + \eta^2} = \frac{xt - iy\sqrt{t^2 - a^2(x^2 + y^2)}}{x^2 + y^2}. \tag{206}$$

We suppose that one of the following inequalities applies:

$$\xi^2 + \eta^2 < \frac{1}{a^2} \tag{207}$$

or

$$x^2 + y^2 < \frac{1}{a^2}t^2. \tag{208}$$

In formula (206) the sign of the radical must be taken as positive. This can easily be proved by considering the equation (205) in which the radical must have a definite value. In fact, if we suppose in the equation (205) that $\xi = 0$, we obtain purely imaginary values for θ

and, from (205), the sign of the radical $\sqrt{a^2 - \theta^2}$ must be opposite to the sign of η, i.e. if, for example, $\eta < 0$, then according to the above condition, θ should lie above the imaginary axis, which coincides with the choice of the sign in formula (206), where we assumed that the radical is positive.

When the values of ξ and η are fixed we have, from (204), a straight line in the space (S), which passes through the origin. We shall only consider that part of the straight line where $t > 0$ and we shall call this line a ray. It appears from the conditions (207) and (208) that these rays form a conical beam with its apex at the origin and an angle equal to arc tan $1/a$ at the apex, and the t-axis as the axis of the beam. The equations (205) or (206) give complex values corresponding to the rays of this beam in the θ-plane with the cut $(-a, +a)$. By using formula (206) this relationship can be followed more accurately. Let us emphasize some essential facts which follow directly from formula (206). Notice, first of all, that the rays which form the surface of the conical beam, i.e. the rays which satisfy the equations

$$\xi^2 + \eta^2 = \frac{1}{a^2} \quad \text{or} \quad x^2 + y^2 = \frac{1}{a^2} t^2,$$

correspond to points of the cut in the θ-plane. The axes of our conical beam, which are defined by the values $x = y = 0$ or $\xi = \eta = 0$, correspond to the point at infinity of the θ-plane. Notice finally, that rays situated in the $y = 0$ plane for which $\eta = 0$, correspond to real values of θ, the modulus of which is greater than a, i.e. they correspond to points on the real axis of the θ-plane which lie outside the cut $(-a, +a)$. If we divide our beam of rays into two parts by the plane, $y = 0$, then one part corresponds to the upper half-plane θ, and the other to the lower half-plane, viz. the half where $y > 0$ corresponds to the lower half-plane and the half where $y < 0$ to the upper half-plane.

If we take the solution of the equation (191) constructed by the above method, i.e. the solution which is the real part of a certain analytic function $f(\theta)$, then this solution will have a constant value on each one of the above rays.

Let us now investigate the values of θ for points of the space (S) which lie outside the above conical beam, i.e. for all points which satisfy the inequalities

$$\xi^2 + \eta^2 > \frac{1}{a^2} \quad \text{or} \quad x^2 + y^2 > \frac{1}{a^2} t^2.$$

The equation (205) gives us two real zeros, which lie on the line $(-a, +a)$:

$$\theta = \frac{\xi \pm \eta \sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2} = \frac{xt \pm yt\sqrt{a^2(x^2+y^2)-t^2}}{x^2+y^2}. \quad (209)$$

This line $(-a, +a)$ is the cut of the plane, and on opposite edges of this cut the radical $\sqrt{a^2-\theta^2}$ has opposite signs, so that in the equation (205) we should take into consideration the double sign of the radical; we must also take both signs of the radical in formula (209). Let $M_0(t_0, x_0, y_0)$ be a point outside our conical beam and θ_1 and θ_2 the corresponding values of θ, obtained from formula (209). If we substitute these values $\theta = \theta_1$ and θ_2 in the left-hand side of the equation (205) we obtain two real equations of the first order with respect to t, x and y and, consequently, we have two planes through the point M_0. This can be expressed in a different way, viz. any value $\theta = \theta_0$ on the cut $(-a, +a)$ corresponds to a plane P in the space (S). Let λ be the generating line which corresponds to the point $\theta = \theta_0$ on the cut. This generating line λ must lie in the plane P. It is not difficult to show that the plane P will be tangential to the surface of our conical beam along the generating line λ. In fact, if the plane P is not tangential to the surface of the cone along λ then it would cut this surface, and part of the plane would then lie within the conical beam. In that case points within the conical beam would correspond to real values of $\theta = \theta_0$ in the interval $(-a, +a)$ which, as we saw above, is not possible. Hence [*from* (205)] *any real* θ *on the cut* $(-a, +a)$ *corresponds to a plane tangential to the surface of the conical beam along the generating line which corresponds to the given value of* θ.

Instead of talking about a conical beam and tangential planes to its surface we can use a two-dimensional diagram, i.e. we can cut our conical beam with a plane perpendicular to the t axis. In this case the conical beam is represented by a circle to which the tangential planes are tangents. In particular, we can use the variables ξ and η in the transition to the two-dimensional diagram. Instead of the conical beam we have in the (ξ, η) plane the circle K:

$$\xi^2 + \eta^2 \leqslant \frac{1}{a^2}, \quad (210)$$

where every point of this circle corresponds to a definite ray of our beam and vice versa. The tangent to the above circle corresponds to the tangential plane to the surface of the beam. The half-plane $\eta > 0$

corresponds to that part of the space where $y > 0$. The axis $\eta = 0$ corresponds to the plane $y = 0$.

Let $f(\theta)$ be an analytic single-valued function in the θ-plane with the cut $(-a, +a)$. Take the corresponding solution of the equation (194):

$$u = \mathcal{R}[f(\theta)] \quad (\mathcal{R} \text{ being the real part}). \tag{211}$$

This solution will be defined within our conical beam or, in the case of a plane (ξ, η), within the circle (210). We shall give one method for continuing this solution outside the conical beam which has very important applications. In order to do this we draw a family of tangential half-planes to the surface of our conical beam, all in the same direction, i.e. their corresponding tangents to the circle

$$\xi^2 + \eta^2 = \frac{1}{a^2} \tag{212}$$

should have the appearance shown in Fig. 51. These tangential half-planes do not cross one another and fill part of the space (S) outside the conical beam. $f(\theta)$ remains constant on every one of these half-planes and we can thus define the solution of u outside the conical beam uniquely by using the same formula (211) which gave the solution within the conical beam. In this case the values of the solution remain constant not on the rays but on half-planes outside the conical beam. Notice that the directions of the tangents to the circle (212) can evidently be selected and we therefore obtain two different methods for continuing the solution, when the above method is used.

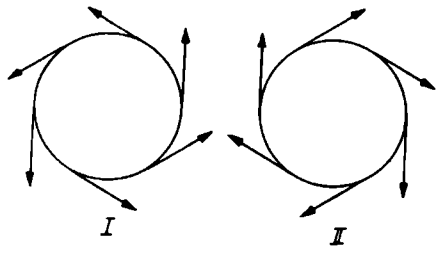

FIG. 51

The corresponding surfaces of the conical beam θ belong to the cut $-a < \theta < +a$. We can, in this case, separate the values of u as given by formula (211) into two real terms $u = u_1(\theta) + u_2(\theta)$ and continue one of the terms along the half-tangents I (Fig. 51) and the other along half-tangents II. This also gives a solution of the equation outside the circle. We therefore have an infinite number of different ways of continuation and in all cases the continuity of the solution u is preserved in transit through the circle. In actual problems the method of continuation is determined from the movement of the frontal wave.

All that was said above referred to the solution in a space (S). Let us now suppose that we are only interested in the half of the space

where $y \geqslant 0$, or in the plane (ξ, η), where $\eta \geqslant 0$. Assume that formula (211) gives the solution in a semicircle and that it is equal to zero on an arc AB of this semicircle, as shown in Fig. 52. This case has many applications in problems of propagation of vibrations and we arrive at a single-valued continuation of the solution (211) by using half-tangents to the circle shown in Fig. 52, i.e. by using the corresponding half-planes which are tangential to the surface of the conical beam. In this case the solution will be equal to zero outside the contour $A_1ABB_1A_1$.

Analogous considerations can also be applied to the general case of the equation (202) but, instead of the conical beam, we shall, of

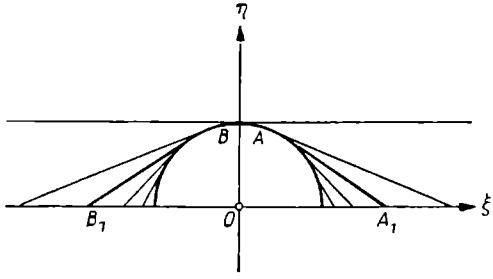

FIG. 52

course, have a much more complicated geometrical picture (a family of straight lines with two parameters), which depends on the choice of $p(\theta)$.

In the equations (202) and (203) we can substitute another variable z instead of θ, which is connected with θ by a functional dependence. We shall give here one particularly convenient choice of this complex variable. Let the connection between z and θ be in accordance with the formula

$$\theta = \frac{a}{2}\left(z + \frac{1}{z}\right). \tag{213}$$

In this case, as we know from [33], instead of the θ-plane with the cut $(-a, +a)$ we have a unit circle $|z| \leqslant 1$ for the variable z. By using formula (213) it is not difficult to see that for our choice of the value of the radical we have the formula

$$\sqrt{a^2 - \theta^2} = i\frac{a}{2}\left(z - \frac{1}{z}\right). \tag{214}$$

Let us consider in greater detail equation (203) in this case. It will have the following form:

$$t-\frac{a}{2}\left(z+\frac{1}{z}\right)x+i\frac{a}{2}\left(z-\frac{1}{z}\right)y=0 \tag{215}$$

or

$$1-\frac{a}{2}\left(z+\frac{1}{z}\right)\xi+i\frac{a}{2}\left(z-\frac{1}{z}\right)\eta=0,$$

which can also be rewritten as follows:

$$1-\frac{a}{2}z\,(\xi-i\eta)-\frac{a}{2}\frac{1}{z}\,(\xi+i\eta)=0. \tag{216}$$

We introduce polar coordinates for the circle (210) according to the formulae

$$\xi=\varrho\cos\varphi;\quad \eta=\varrho\sin\varphi\quad\left(0\leqslant\varrho\leqslant\frac{1}{a}\right).$$

In this case the equation (216) can be rewritten as follows:

$$a\varrho e^{-i\varphi}z^2-2z+a\varrho e^{i\varphi}=0,$$

and we have a solution for z in the form $z=re^{i\varphi}$, where r is determined from the quadratic equation

$$a\varrho r^2-2r+a\varrho=0\qquad(0\leqslant r\leqslant 1),$$

i.e. in this case every point on the circle (210) (i.e. every ray) corresponds to the value of the complex variable $z=re^{i\varphi}$ with the same amplitude, and points on the circle (212) correspond to points on the unit circle with the same amplitude. In other words, every radius of the circle (210) corresponds to the radius of the unit circle $|z|\leqslant 1$ with the same polar angle.

The fundamental ideas about applications of the theory of functions of a complex variable to the solution of the wave-equation (191) with which we have dealt in this section, have wide applications in problems of propagation of vibrations (acoustic, electromagnetic), as well as in more complicated problems of propagation of elastic vibrations. The above method only gives one class of solutions of the equation (191) but it so happens that this class includes the solutions which are of great importance in physics; by using the latter it is possible to bring problems connected with the reflection and diffraction of waves, to a final form suitable for evaluation.

The equation (191) is an equation of a two-dimensional wave (cylindrical wave), but by using the principle of superposition it is possible to construct new solutions from solutions of the above type,

and thus to investigate the wave equation in three dimensions. The theoretical basis of the above method can be found in works by S. L. Sobolev and in papers by this author printed in *The Works of The Seismological Institute at the Academy of Science*. Its applications to definite problems can be found in works by E. A. Naryshkina and S. L. Sobolev. Without going into details which would take us outside the scope of this book, we shall describe briefly the application of this method to two problems: the diffraction of a two-dimensional wave and the reflection of elastic vibrations from a flat object.

54. The diffraction of a two-dimensional wave. Consider the (x, y)-plane, cut in the direction of the straight line $y = x$, where $x > 0$. Assume further that in the remaining part of the plane where $t < 0$ we have a two-dimensional wave, which is propagated parallel to the X axis with a velocity $1/a$, so that at the instant when $t=0$, it reaches the end of the cut (the origin). Assume that this two-dimensional wave has the following elementary form:

$$u = 1 \text{ when } x < \frac{1}{a}t; \quad u = 0 \text{ when } x > \frac{1}{a}t, \tag{217}$$

Behind the frontal of propagation u is constant and equal to 1 and in front of the frontal, which has not yet been disturbed, $u = 0$.

In the case under consideration the function u satisfies the equation (191) when $t < 0$ and it is the homogeneous solution of this equation; it depends only on ξ and η and is defined by the conditions

$$u = 1 \text{ when } \xi < \frac{1}{a}, \quad u = 0 \text{ when } \xi > \frac{1}{a}. \tag{218}$$

The frontal of this wave moves with the velocity $1/a$, which agrees with the wave equation (191).

We shall now investigate the problem of diffraction of the wave (217) at the above cut and we shall suppose that after diffraction, i.e. when $t > 0$, the wave will still be represented by a homogeneous solution of the equation (191), i.e. by the real part of an analytic function $f(z)$ of the complex variable z, as defined by the equation (216). This assumption is quite natural, for the line which causes diffraction is a cut which ends at the origin. We take it that on both sides of the cut the following condition is satisfied:

$$u = 0 \text{ (on the cut)}. \tag{219}$$

At the instant when $t = 0$ our two-dimensional wave reaches the cut after which diffraction takes place. Take any positive time $t > 0$. Bearing in mind that according to the wave-equation (191) the velocity of propagation of the disturbance is equal to $1/a$, we have at the given instant the following picture of the disturbances. To begin with, the straight frontal $ABCD$ is torn in two by the obstacle through which the frontal has passed. The line of this frontal is perpendicular to the X-axis and $OB = (1/a)\,t$. We next have a straight frontal formed by the wave reflected from OG, according to the usual law,

(Fig. 53). This will be the straight line EC, parallel to the X axis. Also tho presence of O creates an additional disturbance in the circle, centre the origin and radius $(1/a)t$. It is the main object of this problem to determine the function u in this circle. Let us list those values of u, which apply outside the circle. In front of the line ABF below the cut OG we evidently have $u = 0$. Also $u = 0$ above this cut and in front of the line CD. Now in the part of the plane bounded by the contour $ECFE$, the falling wave is joined by the reflected wave and from the limiting condition (219) we have again $u = 0$. In the part of the plane outside the above circle and behind the frontal of the wave, $u = 1$ everywhere except in the above domain $ECFE$. The circle, centre the origin and radius $(1/a)\,t$ happens to be the circle (210). In this case, however, it is cut along the radius arc tan $(\eta/\xi) = \pi/4$.

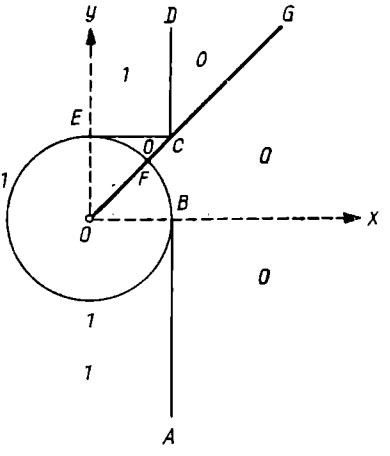

FIG. 53

According to the equation (216) we have on transit to the z-plane, a unit circle $z < 1$ cut along the radius $\arg z = \pi/4$. We know from above that the radii of the circle (210) correspond to the radii of the unit circle $|z| < 1$ with the same central angle.

Bearing in mind the above values of u and the limiting conditions and making the transition to the z-plane we obtain the following problem: find a function $f(z)$, regular in the cut circle $|z| < 1$ and $-7\pi/4 < \arg z < \pi/4$, so that its real part should vanish on both edges of the cut, i.e. on the radii:

$$\arg z = \frac{\pi}{4} \quad \text{and} \quad \arg z = -\frac{7\pi}{4},$$

and also on the arcs

$$-\frac{7}{4}\pi < \arg z < -\frac{3\pi}{2}, \quad \text{and} \quad 0 < \arg z < \frac{\pi}{4},$$

and be equal to unity on the remaining part of the circle $|z| = 1$. It is not difficult to write the solution of this problem in a definite form.

Rotating the z-plane about the origin by an angle $7\pi/4$:

$$w_1 = e^{i\frac{7\pi}{4}} z,$$

we obtain the circle $|w_1| < 1$ and $0 < \arg w_1 < 2\pi$, which is cut along the radius $\arg w_1 = 0$. By extracting the square root this cut is transformed into the interval $(-1, +1)$ of the real axis and the circle is transformed into the upper part of the unit circle. Therefore the transformation:

$$w = \sqrt{w_1} = e^{i\frac{7\pi}{8}} z^{\frac{1}{2}}$$

transforms our cut circle in the z-plane into an upper semicircle in the w-plane. The boundary conditions for the unknown function $f(w)$ will then be that the real part of $f(w)$, should be zero in the interval $(-1, +1)$ of the real axis and

$$\mathcal{R}\left[f\left(e^{i\varphi}\right)\right] = 0 \text{ when } 0<\varphi<\frac{\pi}{8} \text{ and } \frac{7}{8}\pi<\varphi<\pi,$$

$$\mathcal{R}\left[f\left(e^{i\varphi}\right)\right] = 1 \text{ when } \frac{\pi}{8}<\varphi<\frac{7}{8}\pi.$$

$f(w)$ thus transforms the interval $(-1, +1)$ of the real axis into an interval of the imaginary axis and, according to Schwarz's principle of symmetry, $f(w)$ can be analytically continued into the lower part of the unit circle when at points w, symmetrical with respect to the real axis, it acquires values symmetrical with respect to the imaginary axis [24].

We thus obtain the following equation:

$$\mathcal{R}\left[f\left(e^{-i\varphi}\right)\right] = -\mathcal{R}\left[f\left(e^{i\varphi}\right)\right].$$

Bearing this in mind we arrive at the following boundary conditions for $f(z)$ on the unit circle:

$$\left.\begin{aligned} &\mathcal{R}\left[f\left(e^{i\varphi}\right)\right] = 0 && -\frac{\pi}{8}<\varphi<\frac{\pi}{8} \text{ and } \frac{7}{8}\pi<\varphi<\frac{9}{8}\pi \\ &\mathcal{R}\left[f\left(e^{i\varphi}\right)\right] = 1 && \frac{\pi}{8}<\varphi<\frac{7\pi}{8} \\ &\mathcal{R}\left[f\left(e^{i\varphi}\right)\right] = -1 && -\frac{7\pi}{8}<\varphi<-\frac{\pi}{8}. \end{aligned}\right\} \tag{220}$$

To construct the solution of this limiting problem consider the function

$$\frac{1}{i}\log\frac{\alpha-w}{\beta-w} = \frac{1}{i}\log\left|\frac{\alpha-w}{\beta-w}\right| + \arg\frac{\alpha-w}{\beta-w}, \tag{221}$$

where α and β are points on the unit circle, situated at opposite ends of the same diameter AB (Fig. 54). Let M be the variable point w. The real part of (221)

$$\arg\frac{\alpha-w}{\beta-w} = \arg(\alpha-w) - \arg(\beta-w)$$

represents the angle between the vectors MA and MB, measured from MB. The function (221) is single-valued and regular in the circle $|w|<1$. When $w=0$ it is equal to π, and has a period 2π. We suppose that in the circle $|w|<1$, it is equal to π and we thus fix a definite branch of the function (221). For this choice of branch we have:

$$\frac{1}{i}\log\frac{\alpha-w}{\beta-w} = \pi + \frac{1}{i}\log\frac{1-\alpha^{-1}w}{1-\beta^{-1}w} =$$

$$= \pi + \frac{1}{i}\log(1-\alpha^{-1}w) - \frac{1}{i}\log(1-\beta^{-1}w),$$

where we take the principal value of both logarithms, as defined by the usual power series. If w lies on the arc APB, then the above angle BMA is equal to $\pi/2$ and when it lies on the arc AQB it is equal to $3\pi/2$, i.e. for the given choice of the single-valued branch of the function (221) in the circle $|w|<1$ its real part is equal to $\pi/2$ on the arc APB and to $3\pi/2$ on the arc AQB.

Let us apply this result to the function:

$$\psi(w)=\frac{1}{i}\log\frac{e^{i\frac{7\pi}{8}}-w}{e^{-i\frac{\pi}{8}}-w}+\frac{1}{i}\log\frac{e^{-i\frac{7\pi}{8}}-w}{e^{i\frac{\pi}{8}}-w}.$$

Denoting by M_1, M_2, M_3 and M_4 the points

$$e^{-i\frac{\pi}{8}},\ e^{i\frac{\pi}{8}},\ e^{-i\frac{7\pi}{8}},\ e^{i\frac{7\pi}{8}},$$

we can say that the real part of $\psi(w)$ is equal to 2π on the arcs M_1M_2 and M_3M_4, to π on the arc M_1M_3 and to 3π on the arc M_2M_4. Bearing this in mind we directly obtain the solution of the limiting problem (220) in the following form:

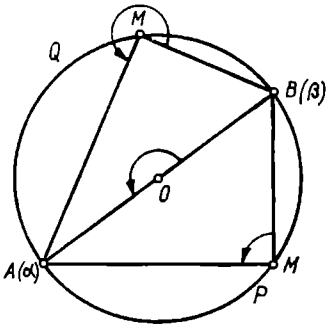

FIG. 54

$$f(w)=\frac{1}{\pi}\psi(w)-2.$$

Returning to the former variable z we obtain the solution of the diffraction problem in the circle

$$x^2+y^2<\frac{1}{a^2}t^2$$

in the form

$$U=\mathcal{R}\left[\frac{1}{\pi i}\log\frac{\left(e^{i\frac{7\pi}{8}}-e^{i\frac{7\pi}{8}}z\frac{1}{2}\right)\left(e^{-i\frac{7\pi}{8}}-e^{i\frac{7\pi}{8}}z\frac{1}{2}\right)}{\left(e^{-i\frac{\pi}{8}}-e^{i\frac{7\pi}{8}}z\frac{1}{2}\right)\left(e^{i\frac{\pi}{8}}-e^{i\frac{7\pi}{8}}z\frac{1}{2}\right)}\right]-2.$$

The above considerations have no strict theoretical basis and the concept of an elementary two-dimensional wave u which is equal to unity behind the frontal and to zero in front of the frontal seems at first to be rather artificial. It can, however, be shown that any two-dimensional wave can be represented by an integral which contains the elementary two-dimensional wave. The result so obtained can therefore be made to include the diffraction of a two-dimensional wave of the most general kind by reducing the problem to the case we considered above.

Let us consider the general appearance of a two-dimensional wave which moves parallel to the X axis. This wave is given by the function $f(t/a-x)$ and we assume that $f(\tau)=0$ when $\tau<0$. The function $f(t/a-x)$ certainly satisfies the equation (191). Above we have considered the elementary case, viz.

$f(\tau) = 1$ when $\tau > 0$ and $f(\tau) = 0$ when $\tau < 0$. Denote $f(\tau)$ by $u(\tau)$ in that particular case as we did in formula (217):

$$u(\tau) = \begin{matrix} 0 \text{ when } \tau < 0 \\ 1 \text{ when } \tau > 0. \end{matrix} \qquad (218_1)$$

Let us now suppose that $f(\sigma)$ is a continuous function with a continuous derivative and that this function is equal to zero when $\tau < 0$. We can write:

$$f(\tau) = \int_0^\infty u(\tau - \lambda) f'(\lambda)\, d\lambda.$$

In fact, taking into consideration the definition of $u(\tau)$ and the condition $f(0) = 0$ we obtain:

$$\int_0^\infty u(\tau - \lambda) f'(\lambda)\, d\lambda = \int_0^\tau f'(\lambda)\, d\lambda = f(\tau) - f(0) = f(\tau).$$

We can therefore write:

$$f\left(\frac{1}{a}t - x\right) = \int_0^\infty u\left(\frac{1}{a}t - x - \lambda\right) f'(\lambda)\, d\lambda = \int_0^\infty u\left(\frac{t - a\lambda}{a} - x\right) f'(\lambda)\, d\lambda.$$

It can be seen from this formula that the general type of falling two-dimensional wave is a sum (an integral, strictly speaking) of elementary falling waves

$$u\left(\frac{t - a\lambda}{a} - x\right) f'(\lambda)\, d\lambda.$$

If we denote by $U(x, y, t)$ the above result obtained for the diffraction of an elementary wave, then in the case of the falling wave $f(t/a - x)$ we have a solution of the form:

$$V = \int_0^\infty U(x, y, t - a\lambda) f'(\lambda)\, d\lambda.$$

We are only considering the result of diffraction with respect to the origin and we denote this by $U(x, y, t)$. When $t > 0$, this takes place in a circle, centre the origin and radius $(1/a)t$, i.e. we suppose that $U(x, y, t) = 0$ when $t < 0$ for any (x, y) and also that $U(x, y, t) = 0$ when $x^2 + y^2 > (1/a^2)t^2$ and $t > 0$. Hence in the expression for V the integral containing λ will, in fact, be propagated in the finite interval in which λ varies.

The above method can be used for solving the problem of diffraction of a two-dimensional wave falling in any arbitrary direction at any given angle.

55. The reflection of elastic waves from rectilinear objects. In two-dimensional problems in the theory of elasticity the component displacements u and v can be expressed by the formulae

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}; \quad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}, \qquad (222)$$

where the function φ is usually known as the potential of longitudinal waves and the function ψ as the potential of transverse waves. These potentials should satisfy wave-equations of the form

$$a^2 \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}, \tag{223}$$

$$b^2 \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}, \tag{224}$$

where

$$a = \sqrt{\frac{\varrho}{\lambda + 2\mu}}; \quad b = \sqrt{\frac{\varrho}{\mu}}, \tag{225}$$

where ϱ is the density of the medium and λ and μ are Lamé's elasticity constants. The numbers $1/a$ and $1/b$, as we know from the theory of elasticity, give the velocity of propagation of the longitudinal and transverse waves and formula (222) gives the subdivision of the general agitation into longitudinal and transverse agitations.

We shall also state formulae which express the tension in the elastic body in terms of the potential. We shall only consider the vector of tension which acts on a surface perpendicular to the Y axis. The components of this vector can be expressed by the following formulae:

$$\left.\begin{aligned} Y_x &= \mu\left[2\frac{\partial^2 \varphi}{\partial x\,\partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right], \\ Y_y &= \mu\left[\left(\frac{b^2}{a^2} - 2\right)\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) + 2\frac{\partial^2 \varphi}{\partial y^2} - 2\frac{\partial^2 \psi}{\partial x\,\partial y}\right]. \end{aligned}\right\} \tag{226}$$

After these preliminaries let us formulate the problem. Let us suppose that at the instant when $t = 0$ an agitation, purely longitudinal in character, is propagated from the point $x = 0$, $y = y_0$; this propagation has a potential φ, which satisfies the equation (223) and which gives the homogeneous solution of the equation with respect to the arguments t, x and $(y - y_0)$, i.e. it is defined as the real part of the analytic function

$$\varphi = \mathcal{R}\left[\Phi(\theta)\right], \tag{227}$$

in which the complex variable θ is determined from the equation:

$$t - \theta x + \sqrt{a^2 - \theta^2}\,(y - y_0) = 0. \tag{228}$$

This latter equation differs from the equation (213) only in so far as y is replaced by $(y - y_0)$. This shows that the potential (227) corresponds to a force which at the time $t = 0$ is concentrated at the point $x = 0$, $y = y_0$. We shall not explain this here from the point of view of the mechanical characteristic of the source.

We assume that the given function $\Phi(\theta)$ in formula (227) is regular in the θ-plane with the cut $(-a, +a)$, except at the point at infinity, and that its real part vanishes on the cut. This later circumstance shows that the given potential φ vanishes on the surface of the conical beam with the vertex of angle arc tan $1/a$ at $t = 0$; $x = 0$, $y = y_0$. This surface corresponds to the frontal of the spreading agitation. We assume, of course, that the potential

is equal to zero everywhere outside the conical beam. Let us suppose that we are given not the whole plane of the spreading agitation, but only the half-plane $y > 0$, with the centre of agitation $x = 0$; $y = y_0 > 0$. The potential φ fully defines the movement only at the instance when $t > ay_0$. When $t = ay_0$ the frontal of agitation reaches the line $y = 0$ which is the edge of our medium, and reflected waves appear; the reflection laws should be obtained from the limiting conditions at this edge. We assume that the medium is free of tension and in future, when writing the corresponding limiting conditions, we shall equate to zero the expression (226) when $y = 0$.

As a result of reflection two other potentials must be added to the given potential φ: one is the reflected longitudinal potential φ_1 and the other is the reflected transverse potential ψ_1. We assume that both potentials are expressed as real parts of analytic functions of a complex variable

$$\varphi_1 = \mathcal{R}\left[\Phi_1(\theta_1)\right]; \quad \psi_1 = \mathcal{R}\left[\Psi_1(\theta_2)\right]. \tag{229}$$

We have to find equations for both complex variables θ_1 and θ_2, as well as the form of the analytic functions $\Phi_1(\theta_1)$ and $\Psi_1(\theta_2)$; the latter are obtained from the given falling potential φ and from the limiting conditions. From [53] and also owing to the fact that the wave-equation for the transverse potential ψ contains the constant b instead of a, the above complex variables will be found from equations of the type:

$$\left.\begin{aligned} t - \theta_1 x \pm \sqrt{a^2 - \theta_1^2}\, y + p_1(\theta_1) = 0, \\ t - \theta_2 x \pm \sqrt{b^2 - \theta_2^2}\, y + p_2(\theta_2) = 0, \end{aligned}\right\} \tag{230}$$

and we must, first of all, select the form of the functions $p_1(\theta_1)$ and $p_2(\theta_2)$ and the signs of the radicals, bearing in mind the fact that the values of radicals in cut planes are always determined in the way explained in [53].

Consider the conical beam of rays which corresponds to the equation (228), with vertex at the point $t = x = 0$, $y = y_0$. In this case the difference $(y - y_0)$ replaces the letter y, if we make comparisons with [53]. The plane $y = y_0$ divides our beam into two parts and the part of the beam where $y > y_0$ will never meet the edge $y = 0$ in the space (S) with the coordinates (t, x, y). The second part of the beam where $y \leqslant y_0$ will meet this plane, and the points of intersection of the straight lines of the beam and the plane will fill a whole domain of this plane defined by the inequality (Fig. 55)

$$x^2 + y_0^2 < \frac{1}{a^2} t^2. \tag{231}$$

This follows directly from the fact that the equation of the beam will, in this case, have the form

$$x^2 + (y - y_0)^2 < \frac{1}{a^2} t^2.$$

The domain (231) therefore represents the interior of a hyperbola in the $y = 0$ plane in the space (S). It follows from [53], that that part of the conical beam which intersects the plane $y = 0$, where $y - y_0 < 0$, corresponds to the upper half-plane of the complex variable θ. At the same time y decreases while t

simultaneously increases along every ray. We select in the equations (230) the signs of the radicals so that they should be opposite to those in the equation (228); the functions $p_1(\theta_1)$ and $p_2(\theta_2)$ are determined in such a way that the equations (232) and (228) should coincide when $y = 0$. We thus obtain for the new complex variables the following equations:

$$t - \theta_1 x - \sqrt{a^2 - \theta_1^2}\,(y + y_0) = 0, \tag{232}$$

$$t - \theta_2 x - \sqrt{b^2 - \theta_2^2}\,y - \sqrt{a^2 - \theta_2^2}\,y_0 = 0. \tag{233}$$

Consider a point $M_1(t_1, x_1)$ in the domain (231) in the plane $y = 0$, in the space (S). This point can be reached by a ray belonging to one half of the beam, which corresponds to a definite value $\theta = \theta'$. If we substitute the coordinates of the point $t = t_1$; $x = x_1$ and $y = 0$ in the equations (232) and (233), we obtain the identical value for the complex variables θ_1 and θ_2. If we now substitute these values $\theta_1 = \theta'$ and $\theta_2 = \theta'$ into the complete equations (232) and (233), then the equations so obtained will determine two rays which we shall in future call the reflected longitudinal and the reflected transverse ray [all this takes place in the space (S)]. Notice one important circumstance, viz. as a result of the definite choice of the signs of radicals in the equations (232) and (233) it is evident that t and y increase simultaneously along the reflected rays, i.e. the reflected rays travel into the depth of our half-plane, as time goes on or, in other words, the reflected waves alter nothing in the picture of disturbance which existed before reflection occurred. Let us test this circumstance for the equation (232). By comparing it with the equation (228) it can easily be shown that it corresponds to a conical beam with vertex at the point $t = x = 0, y = -y_0$, symmetrical with the centre of agitation with respect to the $y = 0$ plane. Bearing in mind the fact that the sign of the radical in the equation (232) is different from the sign of the radical in the equation (228) we can say that the values of θ in the upper half-plane, which we obtained as a result of reflection, correspond to rays, where $t > 0$ and $y + y_0 > 0$; also when t increases y increases along the ray. An analogous circumstance also applies to rays defined by the equation (233), but in this case the beam of rays will no longer be conical. Thus from every point M_1 of the region (231) two different rays will radiate. We are trying to find the potentials of the reflected waves in accordance with the formulae (229), so that they remain constant along the reflected rays. The form of the functions in the formulae (229) remains to be found. As we have already said

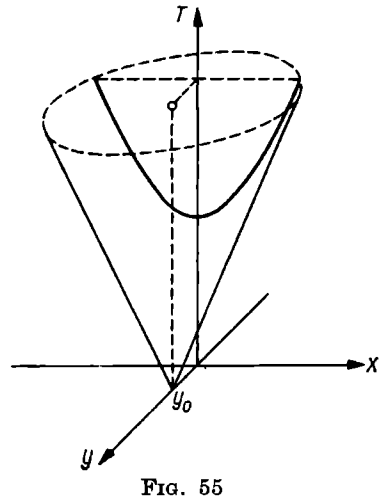

FIG. 55

above, we are considering in this case limiting conditions of the form:

$$2\frac{\partial^2(\varphi+\varphi_1)}{\partial x\,\partial y}+\frac{\partial^2\psi_1}{\partial y^2}-\frac{\partial^2\psi_1}{\partial x^2}\Big|_{y=0}=0.$$

$$\left(\frac{b^2}{a^2}-2\right)\left[\frac{\partial^2(\varphi+\varphi_1)}{\partial x^2}+\frac{\partial^2(\varphi+\varphi_1)}{\partial y^2}\right]+2\frac{\partial^2(\varphi+\varphi_1)}{\partial y^2}-2\frac{\partial^2\psi_1}{\partial x\,\partial y}\Big|_{y=0}=0.$$

To evaluate the derivatives of the functions φ, φ_1, and ψ_1, as determined by the formulae (227) and (229) we can use the formulae (200) by substituting $l(\tau)$, $m(\tau)$ and $n(\tau)$ by the corresponding coefficients from the equations (228), (232) and (233). Notice also that in the reflected transverse potential ψ_1 we must replace a by b. When $y=0$ our complex variables θ, θ_1, θ_2 coincide and we can denote them by the same letter θ. We thus arrive at conditions of the following type:

$$\left.\begin{aligned}&\mathcal{R}\left[\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{-2\theta\sqrt{a^2-\theta^2}\left[\Phi'(\theta)-\Phi_1'(\theta)\right]+(b^2-2\theta^2)\,\Psi_1'(\theta)}{\delta'}\right]=0,\\&\mathcal{R}\left[\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{(b^2-2\theta^2)\left[\Phi'(\theta)+\Phi_1(\theta)\right]-2\theta\sqrt{b^2-\theta^2}\,\Psi_1'(\theta)}{\delta'}\right]=0,\end{aligned}\right\}\tag{234}$$

where

$$\delta'=-x+\frac{\theta}{\sqrt{a^2-\theta^2}}\,y_0.$$

The conditions (234) should be satisfied in the whole domain (231), i.e. in the whole upper half-plane θ.

We obviously obtain the solution of the equations (234) if we determine the unknown functions $\Phi_1(\theta)$ and $\Psi_1(0)$ from the equations:

$$\begin{aligned}&-2\theta\sqrt{a^2-\theta^2}\left[\Phi'(\theta)-\Phi_1'(\theta)\right]+(b^2-2\theta^2)\,\Psi_1'(\theta)=0,\\&(b^2-2\theta^2)\left[\Phi'(\theta)+\Phi_1'(\theta)\right]-2\theta\sqrt{b^2-\theta^2}\,\Psi_1'(\theta)=0.\end{aligned}$$

It can be shown that these equations are not only sufficient but are also necessary if the conditions (234) are to be satisfied. By solving them we obtain expressions for the derivatives of the unknown functions

$$\left.\begin{aligned}\Phi_1'(\theta)&=-\frac{-(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\,\Phi'(\theta),\\\Psi_1'(\theta)&=-\frac{4\theta(2\theta^2-b^2)\sqrt{a^2-\theta^2}}{F(\theta)}\,\Phi'(\theta),\end{aligned}\right\}\tag{235}$$

where

$$F(\theta)=(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}.\tag{236}$$

To obtain the solution of the problem we are only interested in the derivatives of the potentials. From formula (222) we obtain the following formulae for the displacements:

$$\left.\begin{aligned}u&=\mathcal{R}\left[\Phi'(\theta)\frac{\partial\theta}{\partial x}+\Phi_1'(\theta_1)\frac{\partial\theta_1}{\partial x}+\Psi_1'(\theta_2)\frac{\partial\theta_2}{\partial y}\right],\\v&=\mathcal{R}\left[\Phi'(\theta)\frac{\partial\theta}{\partial y}+\Phi_1'(\theta_1)\frac{\partial\theta_1}{\partial y}-\Psi_1'(\theta_2)\frac{\partial\theta_2}{\partial x}\right].\end{aligned}\right\}\tag{237}$$

If neither the falling, nor the reflected ray passes through the point $M(t, x, y)$ then we must cross out the corresponding term in the expressions (237). Notice one important circumstance, viz. from the given condition the real part of $\Phi_1'(\theta)$ is equal to zero when $-a < \theta < +a$. From the formulae (235) and (225) it is clear that the relationship $b > a$ follows directly; it also applies to $\Phi_1'(\theta)$ and $\Psi_1'(\theta)$, so that the reflected potentials φ_1 and ψ_1 are constant on the surfaces of the reflected beams of rays and we can assume that they are equal to zero on these surfaces and outside the beams.

If we were to consider the source of transverse vibration instead of the source of longitudinal vibration then the picture would be somewhat different. In this case we would be given the potential of transverse vibrations in the form of the real part of an analytic function

$$\psi = \mathcal{R}[\Psi(\theta)], \tag{238}$$

regular in the θ-plane with the cut $(-b, +b)$ and the complex variable θ is determined from the equation

$$t - \theta x + \sqrt{b^2 - \theta^2}\,(y - y_0) = 0, \tag{239}$$

where the real part $\psi(\theta)$ is equal to zero when $-b < \theta < +b$. We are looking for reflected longitudinal and transverse potentials of the form:

$$\varphi_1 = \mathcal{R}[\Phi_1(\theta_1)]; \quad \psi_1 = \mathcal{R}[\Psi_1(\theta)], \tag{240}$$

where θ_1 and θ_2 are determined from the equations

$$t - \theta_1 x - \sqrt{a^2 - \theta_1^2}\,y - \sqrt{b^2 - \theta_1^2}\,y_0 = 0, \tag{241}$$

$$t - \theta_2 x - \sqrt{b^2 - \theta_2^2}\,(y + y_0) = 0. \tag{242}$$

Similarly for functions in the expressions (240) we obtain the following expression instead of formula (235):

$$\left.\begin{aligned} \Phi_1'(\theta) &= \frac{4\theta(2\theta^2 - b^2)\sqrt{b^2 - \theta^2}}{F(\theta)}\,\Psi'(\theta), \\ \Psi_1'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2 - \theta^2}\sqrt{b^2 - \theta^2}}{F(\theta)}\,\Psi'(\theta). \end{aligned}\right\} \tag{243}$$

In this case the cut in the θ-plane, points of which correspond to rays on the surface of the conical beam, will be $-b < \theta < +b$. The coefficients of $\psi'(\theta)$ in both expressions (243) contain the radical $\sqrt{a^2 - \theta^2}$ and therefore these coefficients, which remain real when $-a < \theta < +a$, cease to be constant when $-b < \theta < -a$ and $a < \theta < b$. At the same time the product of the imaginary part of the coefficient and the imaginary part of $\Psi'(\theta)$ gives the real part of $\Phi_1'(\theta)$ and $\Psi_1'(\theta)$ which is other than zero when

$$-b < \theta < -a \quad \text{and} \quad a < \theta < b. \tag{244}$$

If we substitute these values of θ in the left-hand side of the equation (241) then, after the separation of the real and imaginary parts, we have:

$$t - \theta x - \sqrt{b^2 - \theta^2}\,y_0 = 0; \quad y = 0.$$

i.e. for the reflected longitudinal potential these critical rays, on which the potential is other than zero, do not penetrate into the medium but travel in the $y = 0$ plane (Fig. 56). For the reflected transverse potential the reflected beam of rays, given by equation (242), will simply be a conical beam with the vertex at $t = x = 0$, $y = -y_0$; along the generating lines of the surface of this beam, which correspond to values of θ satisfying the conditions (244), the values of the reflected potential will be other than zero. In this case we shall have to continue the reflected transverse potential outside the above

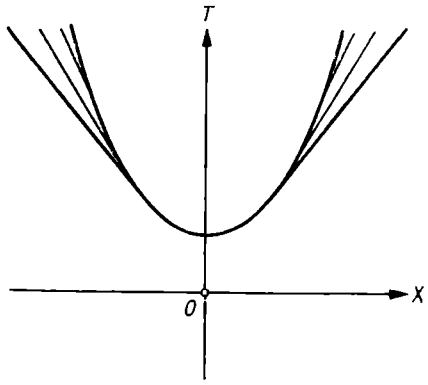

FIG. 56

conical beam by the method described in [53]. This circumstance has a simple mechanical meaning, viz. the transverse waves radiating from the source of vibration originate longitudinal reflected waves when falling on the edge $y = 0$; these are propagated along the edge faster than the transverse waves and they, in their turn, produce a transverse wave, which travels in front of the reflected wave and follows the usual laws for transverse waves.

We have only given here brief indications and not a detailed mechanical investigation of the formulae (235) and (243). Note that the denominator of $F(\theta)$, as given by the equation (236), has real zeros $\theta = \pm c$, which satisfy the inequality $c > b$; the existence of these zeros produce the phenomenon known as the phenomenon of surface waves.

CHAPTER III

THE APPLICATION OF THE THEORY OF RESIDUES, INTEGRAL AND FRACTIONAL FUNCTIONS

56. Fresnel's integral. In [21] we proved the fundamental theorem of residues which is the basis of the application of the theory of analytic functions to various calculations and analytic representations of functions. We will now deal with the problems of evaluating definite integrals, integrating linear differential equations, expanding functions into infinite series and representing them by contour integrals.

Let us start with the evaluation of the definite integral [II, 83]

$$\int_0^\infty \sin(x^2)\,dx \tag{1}$$

usually known as Fresnel's integral and met with in problems of light diffraction. Consider the integral

$$\int_l e^{-z^2}\,dz, \tag{2}$$

where l is a closed contour, which consists of the section OA of the real axis, the arc AB of a circle centre at O, and radius $R = \overline{OA}$ and the section $\overline{BO}$, and we take the angle $\overline{AOB}$ as equal to $\pi/4$. Inside this contour the integrand e^{-z^2} has no singularities and therefore integral (2) is equal to zero. Divide this integral into three parts which correspond to the above three parts of the contour. The variable z will be real along $\overline{OA}$ and we suppose here that $z = x$, where $0 \leqslant x \leqslant R$. Along $\overline{BO}$ we have $z = xe^{i\pi/4}$; $z^2 = ix^2$ and $dz = e^{i\pi/4}\,dx$. Finally, along AB

$$z = Re^{i\varphi} \qquad \left(0 \leqslant \varphi \leqslant \frac{\pi}{4}\right),$$

hence $z^2 = R^2 e^{i2\varphi}$ and $dz = iRe^{i\varphi}\,d\varphi$. We thus obtain the following equation

$$\int_0^R e^{-x^2}dx + e^{i\frac{\pi}{4}} \int_R^0 e^{-ix^2}dx + \int_0^{\frac{\pi}{4}} iRe^{-R^2(\cos 2\varphi + i\sin 2\varphi) + i\varphi}\,d\varphi = 0. \tag{3}$$

We will show that the third integral in the above equation tends to zero as R increases indefinitely. Bearing in mind that the modulus of e^{τ}, when τ is purely imaginary, is unity and substituting the integrand by its modulus, we obtain the inequality:

$$\left| \int_0^{\frac{\pi}{4}} iRe^{-R^2(\cos 2\varphi + i \sin 2\varphi) + i\varphi} d\varphi \right| < R \int_0^{\frac{\pi}{4}} e^{-R^2 \cos 2\varphi} d\varphi.$$

We will prove that the expression on the right-hand side tends to zero as $R \to \infty$. Substituting a new variable $\psi = 2\varphi$ for φ and rejecting the constant factor, which is of no importance, we obtain:

$$R \int_0^{\frac{\pi}{2}} e^{-R^2 \cos \psi} d\psi.$$

We now divide the interval of integration into two parts $(0, a)$ and $(a, \pi/2)$, where a is a certain number between 0 and $\pi/2$:

$$R \int_0^{\frac{\pi}{2}} e^{-R^2 \cos \psi} d\psi = \int_0^{a} Re^{-R^2 \cos \psi} d\psi + \int_a^{\frac{\pi}{2}} Re^{-R^2 \cos \psi} d\psi. \tag{4}$$

In the first of the above integrals we substitute the negative index by its greatest value, i.e. by the smallest absolute value, viz. by $(-R^2 \cos a)$. We multiply the integrand of the second integral by the fraction $\sin \psi / \sin a$, which is always greater than unity in the interval $a < \psi < \pi/2$. Having thus increased the sum (4) we obtain:

$$\int_0^{a} Re^{-R^2 \cos a} d\psi + \int_a^{\frac{\pi}{2}} R \frac{\sin \psi}{\sin a} e^{-R^2 \cos \psi} d\psi,$$

and it is sufficient to show that this latter sum tends to zero. Both integrals can be fully evaluated and their sum will be

$$aRe^{-R^2 \cos a} + \frac{1}{R \sin a} \left[e^{-R^2 \cos \psi} \right]_{\psi = a}^{\psi = \frac{\pi}{2}} = aRe^{-R^2 \cos a} + \frac{1 - e^{-R^2 \cos a}}{R \sin a},$$

from which it follows directly that this sum tends to zero as R increases indefinitely. We have thus shown that the third term on the left-hand side of (3) tends to zero as $R \to \infty$. The first term on the left-hand side has the limit

$$\int_0^{\infty} e^{-x^2} dx,$$

which, as we know from [11, 78], is equal to $(1/2)\sqrt{\pi}$. We can therefore say that the second term has a definite limit; this gives us

$$\frac{1}{2}\sqrt{\pi} + \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \int_{\infty}^{0} e^{-ix^2} dx = 0.$$

or, separating the real and imaginary parts under the integral:

$$\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\int_0^\infty [\cos(x^2)-i\sin(x^2)]\,dx=\frac{1}{2}\sqrt{\pi}.$$

Equating the real and imaginary parts we obtain Fresnel's integral:

$$\int_0^\infty \cos(x^2)\,dx=\int^\infty \sin(x^2)\,dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}. \tag{5}$$

57. Integration of expressions containing trigonometric functions. Consider now the integral:

$$\int_0^{2\pi} R(\cos x,\sin x)\,dx \tag{6}$$

where $R(\cos x, \sin x)$ is a rational function of $\cos x$ and $\sin x$. Let us substitute the complex variable $z = e^{ix}$ for the real variable x. As x varies in the interval $(0,2\pi)$ the complex variable z varies on the circumference of the unit circle $|z| = 1$. Also, from Euler's formula, we know that

$$\cos x=\frac{z+z^{-1}}{2};\qquad \sin x=\frac{z-z^{-1}}{2i},$$

and therefore $dx = (1/iz)\,dz$. Substituting all this in (6) we obtain the integral of a rational fraction on the unit circle $|z| = 1$, which we denote by C.

This integral is equal to the product of $2\pi i$ and the sum of residues of the integrand at poles inside the unit circle.

Example **1.** Consider the integral

$$\int_0^{2\pi}\frac{dx}{1+\varepsilon\cos x}\qquad (0<\varepsilon<1).$$

Performing the transformations shown above we can obtain it in the form:

$$\int_C \frac{dz}{iz\left(1+\varepsilon\dfrac{z+z^{-1}}{2}\right)}$$

or

$$\frac{2}{i}\int_C\frac{dz}{\varepsilon z^2+2z+\varepsilon}.$$

The poles of the integrand will be the same as the zeros of the quadratic equation

$$\varepsilon z^2 + 2z + \varepsilon = 0, \tag{7}$$

in which one of the zeros has a modulus smaller than unity. This zero is determined by the formula

$$z_0 = \frac{-1 + \sqrt{1-\varepsilon^2}}{\varepsilon},$$

where the positive sign of the radical must be taken. The residue of the integrand can be determined by the rule stated in [21], viz. this residue is equal to the quotient obtained by dividing the numerater of the integrand by the derivative of the denominator when $z = z_0$, i.e. in this case the residue is

$$r = \frac{1}{2\varepsilon z_0 + 2} = \frac{1}{2\sqrt{1-\varepsilon^2}},$$

and we finally obtain the following result:

$$\int_0^{2\pi} \frac{dx}{1 + \varepsilon \cos x} = \frac{2\pi}{\sqrt{1-\varepsilon^2}}. \tag{8}$$

2. Let us also consider the integral

$$\int_0^{2\pi} \frac{dx}{(1 + \varepsilon \cos x)^2} \qquad (0 < \varepsilon < 1).$$

On performing the same transformations as above we obtain the integral:

$$\frac{4}{i} \int_C \frac{z}{(\varepsilon z^2 + 2z + \varepsilon)^2} dz.$$

In this case $z = z_0$ will be the only pole inside the unit circle and it will be a pole of order two. From [21], to determine the residue r_2 at this pole we must multiply the integrand by $(z - z_0)^2$, then take the first derivative of this product and put $z = z_0$. Let $z = z_1$ be the other zero of equation (7):

$$z_1 = \frac{-1 - \sqrt{1-\varepsilon^2}}{\varepsilon};$$

its modulus being greater than unity. On performing the above operations we obtain in this case:

$$r = \left[\frac{z}{\varepsilon^2 (z - z_1)^2}\right]'_{z=z_0} = -\left.\frac{z + z_1}{\varepsilon^2 (z - z_1)^3}\right|_{z=z_0},$$

and, subsequently, putting $z = z_0$ and bearing in mind the expressions for z_0 and z_1, we obtain the residue:

$$r = \frac{1}{4(1-\varepsilon^2)^{3/2}}.$$

The theorem of residues gives us finally:

$$\int_0^{2\pi} \frac{dx}{(1+\varepsilon\cos x)^2} = \frac{2\pi}{(1-\varepsilon^2)^{3/2}} \tag{9}$$

58. Integration of a rational fraction. Consider the integral of a rational fraction

$$\int_{-\infty}^{+\infty} \frac{\varphi(x)}{\psi(x)}\,dx. \tag{10}$$

If this integral is to have a meaning it is necessary and sufficient [II, 82] that the polynomial $\psi(x)$ in the denominator should have no real zeros and that its order should be at least two units higher than that of the polynomial $\varphi(x)$. If we also consider the function of the complex variable

$$f(z) = \frac{\varphi(z)}{\psi(z)},$$

then it will have the same property as the product $zf(z)$ in that it will tend uniformly to zero as $z \to \infty$, i.e. it will not depend on the manner in which z tends to infinity. In other words: for any small positive ε there exists a positive R_ε, such that $|zf(z)| < \varepsilon$ when $|z| > R_\varepsilon$. We will show that providing the function $f(z)$ satisfies this condition, its integral along any arc of the circle $|z| = R$ tends to zero as R increases indefinitely.

LEMMA. *If $f(z)$ is continuous in the neighbourhood of the point at infinity u and $zf(z) \to 0$ uniformly as $z \to \infty$, then the integral of $f(z)$ along any arc of the circle $|z| = R$ tends to zero as R increases indefinitely.*

Writing down the upper bound of the integral in the usual way as we did in [4], we have:

$$\left|\int_l f(z)\,dz\right| = \left|\int_l zf(z)\,\frac{1}{z}\,dz\right| \leqslant \max_{\text{on } l} |zf(z)| \cdot \frac{1}{R}\,s,$$

where s is length of the above arc l which evidently, does not exceed $2\pi R$, so that finally

$$\left|\int_l f(z)\,dz\right| \leqslant 2\pi \max_{\text{on } l} |zf(z)|.$$

Bearing in mind that $zf(z)$ tends to zero on the arc as R increases indefinitely, we have the direct proof of our lemma.

Let us now return to our example and integrate the rational fraction $\varphi(z) : \psi(z)$ over the contour consisting of the section $(-R, +R)$ of the real axis and the semicircle in the upper half-plane, for which the above section is the diameter. We can take R so large that all poles of the function, $f(z)$, in the upper half-plane will be included in its constructed semicircle. Denoting it by C_R we have:

$$\int_{-R}^{R} \frac{\varphi(x)}{\psi(x)}\,dx + \int_{C_R} \frac{\varphi(z)}{\psi(z)}\,dz = 2\pi i \sum r, \tag{11}$$

where Σr denotes the sum of residues of the function $f(z)$ at its respective poles in the upper half-plane. As R increases indefinitely, the right-hand side of the equation will not alter and the second term on the left side will, according to the lemma, tend to zero, so that we obtain the limit

$$\int_{-\infty}^{+\infty} \frac{\varphi(x)}{\psi(x)}\,dx = 2\pi i \sum r,$$

i.e. *integral* (10), *which is an integral of a rational fraction, is equal to the product of* $2\pi i$ *and the sum of the residues of the integrand at its respective poles in the upper half-plane.*

Example. Let us consider the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^n}.$$

In this case the integrand has a single pole $z = i$ of order n in the upper half-plane. To determine the residue at this pole we must, according to [21], multiply the integrand $(z^2+1)^{-n}$ by $(z-1)^n$; the product thus obtained must be differentiated $(n-1)!$ times with respect to z and it we can then put $z = i$, i.e. the required residue is determined by the formula:

$$r = \frac{1}{(n-1)!} \frac{d^{n-1}(z-i)^n}{dz^{n-1}(z^2+1)^n}\Big|_{z=i} = \frac{1}{(n-1)!} \frac{d^{n-1}(z+i)^{-n}}{dz^{n-1}}\Big|_{z=i}$$

or

$$r = \frac{(-n)(-n-1)\ldots(-n-n+2)(2i)^{-2n+1}}{(n-1)!} = -\frac{n(n+1)\ldots(2n-2)}{(n-1)!\,2^{2n-1}}\,i,$$

and we finally obtain:

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^n} = \frac{(2n-2)!}{[(n-1)!]^2} \frac{\pi}{2^{2n-2}}. \tag{12}$$

59. Certain new types of integrals containing trigonometric functions. Note that in the deduction of the above rule for evaluating infinite integrals we did not use anywhere the fact that the integrand $f(z)$ was a rational fraction. It is sufficient for our purpose if the function $f(z)$ satisfies the following two conditions: it is regular in the upper half-plane and on the real axis, except at a finite number of poles, and secondly, as $z \to \infty$ in the above domain, $zf(z) \to 0$ uniformly. In this case, as before, we obtain equation (11), in which the second term on the left-hand side tends to zero, so that taking the limit we have:

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum r, \tag{13}$$

where Σr is the sum of the residues of $f(z)$ at its poles in the upper half-plane. Dividing the interval of integration $(-R, +R)$ into two parts $(-R, 0)$ and $(0, +R)$ and substituting $(-x)$ for x in the first integral we can, instead of equation (13), write:

$$\lim_{R\to\infty} \int_{0}^{R} [f(x) + f(-x)]\,dx = 2\pi i \sum r$$

or

$$\int_{0}^{\infty} [f(x) + f(-x)]\,dx = 2\pi i \sum r. \tag{14}$$

Let us apply this result to the particular case when the integrand is

$$f(z) = F(z)\,e^{imz}, \qquad (m > 0) \tag{15}$$

where the function $F(z)$ satisfies the two conditions above. At the same time, as can readily be seen, the function $f(z)$ will also satisfy these two conditions. To prove this it is sufficient to show that the factor e^{imz} is regular in the whole plane and that it remains bounded in the upper half-plane and on the real axis. We must have

$$e^{imz} = e^{im(x+iy)} \text{ and } |e^{imz}| = e^{-my} \qquad (m > 0;\ y \geqslant 0),$$

from which it follows directly that $|e^{imz}| \leqslant 1$ when $z \geqslant 0$. If $F(z)$ satisfies the two conditions above we can write:

$$\int_{0}^{\infty} [F(x)\,e^{imx} + F(-x)\,e^{-imx}]\,dx = 2\pi i \sum r, \tag{16}$$

where Σr is the sum of the residues of the function (15) in the upper half-plane. Let us consider two particular cases. To start with we suppose that $F(z)$ is an even function, i.e. $F(-z) = F(z)$, whence

$$\int_0^\infty F(x) \cos mx \, dx = \pi i \sum r. \tag{17}$$

If, however, $F(z)$ is an odd function, i.e. $F(-z) = -F(z)$ then:

$$\int_0^\infty F(x) \sin mx \, dx = \pi \sum r. \tag{18}$$

Example **1.** Consider the integral

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2} \, dx \qquad (a > 0;\ m > 0).$$

In this case the function

$$F(z) = \frac{1}{a^2 + z^2}$$

obviously satisfies the above two conditions and is an even function, so that we can apply formula (17). The only pole of the function

$$f(z) = \frac{e^{imz}}{a^2 + z^2} \tag{19}$$

in the upper half-plane is a simple pole $z = ia$. We can determine the residue at this pole by the rule which we have used already and which can briefly be formulated as follows: the numerator, divided by the derivative of the denominator. In this ease, the above rule gives us the following expression for the residue of the function (19):

$$r = \frac{e^{-ma}}{i2a},$$

and we finally obtain:

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2} \, dx = \frac{\pi}{2a} e^{-ma}. \tag{20}$$

2. Let us consider the integral

$$\int_0^\infty \frac{x \sin mx}{(x^2 + a^2)^2} \, dx.$$

In this case formula (18) will be valid and the function

$$f(z) = \frac{ze^{imz}}{(z^2 + a^2)^2}$$

will have a single pole $z = ia$ of order two in the upper half-plane. The residue at this pole is determined by the formula

$$r = \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{ze^{imz}}{(z^2+a^2)^2}(z-ia)^2\right]\Big|_{z=ia}$$

or

$$r = \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{ze^{imz}}{(z+ia)^2}\right]\Big|_{z=ia} = \frac{m}{4a}e^{-ma},$$

which gives us directly the final result:

$$\int_0^\infty \frac{x\sin mx}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{\pi m}{4a}e^{-ma}. \tag{21}$$

Note. Generally speaking we have no right to write formula (13) in the form

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 2\pi i\sum r. \tag{22}$$

In fact, the infinite integral

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$$

is determined as the sum of the limits of the integrals

$$\int_0^R f(x)\,\mathrm{d}x \text{ and } \int_{-R}^0 f(x)\,\mathrm{d}x$$

as R tends to $(+\infty)$. If these limits do not exist separately, but the sum of the above integrals tends to a finite limit, i.e. a finite limit exists:

$$\lim_{R\to+\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x,$$

then this limit is known as the principal value of the integral in the infinite interval and is denoted as follows:

$$\text{v. p.}\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = \lim_{R\to+\infty}\int_{-R}^{+R} f(x)\,\mathrm{d}x. \tag{23}$$

The integral in formula (13) should be taken in the sense of its principal value. But if for any particular reason we know that this integral exists as an ordinary indefinite integral, then this should not be assumed, for then the principal value of the integral will be the same as

the usual indefinite integral. In [26] we defined the principal value of an integral in the case when the continuity of $f(x)$ is disrupted at any particular point at a finite distance.

60. Jordan's lemma. The conditions placed on the function $F(z)$ in the above paragraph can be made less restrictive so that formulae (17) and (18) hold, by using a lemma which will be of great importance in what follows:

JORDAN'S LEMMA. *If in the upper half-plane and on the real axis $F(z)$ satisfies the condition: $F(z) \to 0$ uniformly as $z \to \infty$, and m is a certain positive number, then as $R \to +\infty$*

$$\int_{C_R} F(z)\, e^{imz}\, dz \to 0, \tag{24}$$

where C_r is a semicircle in the upper half-plane, centre the origin and radius R.

Introducing the polar coordinates $z = Re^{i\varphi}$, we can rewrite integral (24) as follows:

$$\int_0^{\pi} F(Re^{i\varphi})\, e^{imR(\cos\varphi + i\sin\varphi)}\, iRe^{i\varphi}\, d\varphi,$$

and therefore, taking into consideration the fact that $|ie^{imR\cos\varphi + i\varphi}| = 1$ we have

$$\left|\int_{C_R} F(z)\, e^{ims}\, dz\right| < \int_0^{\pi} |F(Re^{i\varphi})|\, e^{-mR\sin\varphi}\, R\, d\varphi,$$

or

$$\left|\int_{C_R} F(z)\, e^{im^2}\, dz\right| \leqslant \max_{\text{on } C_R} |F(z)| \int_0^{\pi} e^{-mR\sin\varphi}\, R\, d\varphi. \tag{25}$$

It is given that $|F(Re^{i\varphi})|$ tends to zero as $R \to \infty$ uniformly with respect to φ where $0 \leqslant \varphi \leqslant \pi$ and, consequently, it is sufficient to prove that, as $R \to \infty$, the integral

$$\int_0^{\pi} e^{-mR\sin\varphi}\, R\, d\varphi \tag{26}$$

will be bounded. Dividing the interval of integration into two parts: $(0, \pi/2)$ and $(\pi/2, \pi)$ and substituting in the second integral the variable φ by $(\pi - \varphi)$, we obtain integral (26) in the form:

$$2\int_0^{\frac{\pi}{2}} e^{-mR\sin\varphi}\, R\, d\varphi.$$

We will now proceed as we did in [56]. Dividing the interval of integration into two parts and increasing the positive integrand, we obtain the inequality:

$$2\int_0^{\frac{\pi}{2}} e^{-mR\sin\varphi} R\,d\varphi < 2\int_0^{a} e^{-mR\sin\varphi} R\frac{\cos\varphi}{\cos a}\,d\varphi + 2\int_a^{\frac{\pi}{2}} e^{-mR\sin a} R\,d\varphi.$$

The last two integrals are in their final form and we obtain the following inequality:

$$2\int_0^{\frac{\pi}{2}} e^{-mR\sin\varphi} R\,d\varphi < \frac{2}{m\cos a}\left[-e^{-mR\sin\varphi}\right]_{\varphi=0}^{\varphi=a} + 2e^{-mR\sin a} R\left(\frac{\pi}{2}-a\right).$$

The second of the above terms tends to zero as $R\to\infty$, and the first term tends to the finite limit $2/m\cos a$ so that the whole sum remains bounded as $R\to\infty$. The same may be said for integral (26) from which the result of the lemma is derived.

By using the lemma we can, for example, prove formula (18), when less restrictive requirements are made with respect to the function $F(z)$. In fact, we required earlier that in the upper half-plane and on the real axis $zF(z)\to 0$ as $|z|\to\infty$ and this condition was necessary for the integral

$$\int_{C_R} F(z)\,e^{imz}\,dz$$

to tend to zero in the upper semicircle as $R\to\infty$. The lemma shows that it is sufficient for $F(z)\to 0$ as $z\to\infty$ and therefore it is sufficient to use formula (18) in the assumption.

Example. Consider the integral

$$\int_0^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx \qquad (a>0;\ m>0).$$

In this case the function

$$F(z) = \frac{z}{z^2+a^2}$$

satisfies all conditions of formula (18) and therefore we need, as before, determine only the residue of the function

$$F(z)\,e^{imz} = \frac{ze^{imz}}{z^2+a^2}$$

at the pole $z = ia$ in the upper half-plane. This will be a pole of order one and the corresponding residue is determined by the usual rule: the numerator divided by the derivative of the denominator, i.e.

$$r = \frac{ze^{imz}}{2z}\Big|_{z=ia} = \frac{1}{2}e^{-ma},$$

and finally

$$\int_0^\infty \frac{x \sin mx}{x^2 + a^2}\,dx = \frac{\pi}{2}e^{-ma}. \tag{27}$$

61. Contour integrals of certain functions. From a knowledge of the theory of residues it is easy to construct contour integrals for non-continuous functions.

Consider, for example, the function $\varphi(t)$, which is zero when $t < 0$, and unity when $t > 0$, i.e.

$$\varphi(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t > 0). \end{cases} \tag{28}$$

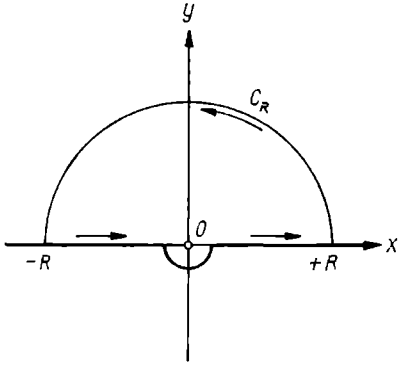

FIG. 57

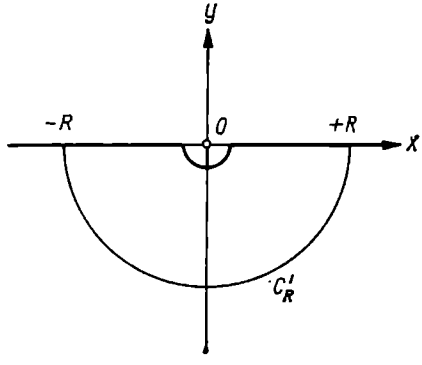

FIG. 58

We will show that such a function can be represented by the contour integral

$$\varphi(t) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{e^{itz}}{z}\,dz, \tag{29}$$

in which t appears as a parameter in the integral. The contour of integration includes the whole real axis but the origin $z = 0$ which is a pole of the integrand is avoided by the circumference of a small circle, centre the origin in the lower half-plane (Fig. 57). Consider the auxiliary contour l_R, which consists not of the whole real axis but only of its section $(-R, +R)$, surrounds the origin and includes the semicircle C_R in the upper half-plane, centre the origin and radius R. If $t > 0$, then Jordan's lemma can be applied to integral (29), so that the integral over the semicircle will tend to zero as R increases. The

integrand has a single pole inside the contour, viz. at the origin $z = 0$, where the residue is unity. Therefore

$$\frac{1}{2\pi i}\int_{l_R}\frac{e^{itz}}{z}\,dz = 1.$$

Passing to the limit we obtain

$$\frac{1}{2\pi i}\int_{\smile}\frac{e^{itz}}{z}\,dz = 1 \qquad (t>0).$$

Let us now suppose that $t<0$. Consider the closed contour which consists of the above section $(-R, +R)$ of the real axis, surrounds the origin and includes the semicircle C'_R of radius R, not in the upper but in the lower half-plane (Fig. 58). Inside this contour our function has no singularities and therefore the integral along this contour is zero.

We will now show that as R increases indefinitely, the integral over the lower semicircle will tend to zero. In fact, if we change the variable of integration z so that $z' = -z$, then the lower semicircle C'_R will be transformed into the upper semicircle C_R and we have:

$$\int_{C'_R}\frac{e^{itz}}{z}\,dz = \int_{C_R}\frac{e^{-itz'}}{z'}\,dz'.$$

It is given that $t < 0$, i.e. $-t > 0$, and Jordan's lemma shows that the latter integral does, in fact, tend to zero. Passing to the limit, as before, we get:

$$\frac{1}{2\pi i}\int_{\smile}\frac{e^{itz}}{z}\,dz = 0 \qquad (t<0).$$

Consider also the integral when $t = 0$.

$$\frac{1}{2\pi i}\int_{\smile}\frac{1}{z}\,dz. \tag{30}$$

If we consider, as before, the section $(-R, +R)$ of the real axis we have to evaluate the increment of $\log z$ as it moves along this section and round the origin. At the end of the path the real part of $\log z$ will be $\log R$ and, consequently, it has received no increment. The imaginary part, which is equal to $i \arg z$, must receive an increment πi when surrounding the origin along the semicircle, but on other sections of the path it remains unaltered. Thus integral (30) is equal to 1/2 along the section $(-R, +R)$. Consequently we should obtain the same result in the limit as $R \to \infty$, i.e.

$$\frac{1}{2\pi i}\int_{\smile}\frac{1}{z}\,dz = \frac{1}{2}. \tag{31}$$

In this case it was essential that both the upper and lower limits should tend to ∞ and the same absolute values, i.e. integral (31) should be taken in its principal value sense in the interval $(-\infty, +\infty)$ when surrounding $z = 0$.

Integral (29), when $t \neq 0$, will be convergent in the usual sense of the word with respect to infinite limits. In fact, separating the real and imaginary parts we obtain

$$\int_a^\infty \frac{\cos tz}{z}\,dz \text{ and } \int_a^\infty \frac{\sin tz}{z}\,dz \qquad (a > 0).$$

We proved the convergence of the second integral in [11, 83]. The convergence of the first integral can be proved in exactly the same way.

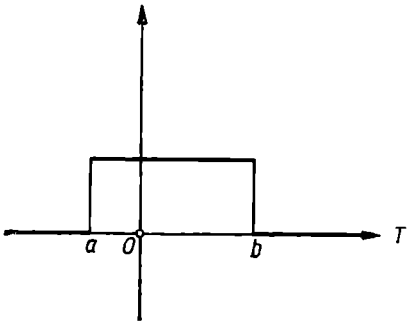

FIG. 59

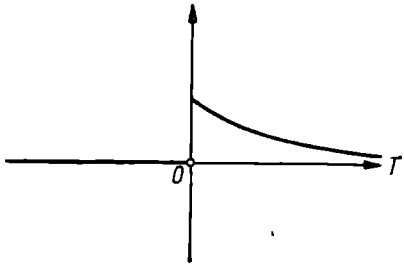

FIG. 60

Thus, when $t \neq 0$ integral (29) gives the function (28). When $t = 0$ the integral only has a meaning in the principal value sense and is equal to 1/2.

Consider now another example where the function is zero everywhere except on a certain finite section where it is unity, i.e.

$$\varphi(t) = 0 \quad \text{when} \quad t < a \quad \text{and} \quad t > b; \quad \varphi(t) = 1 \quad \text{when} \quad a < t < b. \tag{32}$$

It is not difficult to represent the above function as the difference of two functions of this kind; hence

$$\varphi(t) = \frac{1}{2\pi i}\int_{\smile} \frac{e^{i(b-t)z}}{z}\,dz - \frac{1}{2\pi i}\int_{\smile} \frac{e^{i(a-t)z}}{z}\,dz. \tag{33}$$

Both terms vanish when $t > b$. In the interval $a < t < b$ the first term on the right-hand side is unity and the second term is zero so that the difference is unity. Lastly, when $t < a$ and both these terms are unity, the difference is zero and we have, in fact, obtained function (32). The graph of this function is shown in Fig. 59.

Consider now the function which is zero when $t < 0$ and which, by starting with $t = 0$, begins to decrease from unity exponentially:

$$\varphi_1(t) = 0 \quad (t < 0); \quad \varphi_1(t) = e^{-at} \quad (t > 0,\ a > 0). \tag{34}$$

The graph of this function is illustrated in Fig. 60. It can readily be shown that this function can be represented by the contour integral:

$$\varphi_1(t) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{itz}}{z - ia}\,dz, \tag{35}$$

where the real axis is the contour of integration. The proof is exactly the same as for formula (29). In this case the residue of the function

$$\frac{e^{itz}}{z - ia}$$

at the pole $z = ia$ is equal to e^{-at}.

Finally let us consider the function which is zero when $t < 0$ and which gives the sine curve when $t > 0$ (Fig. 61).

$$\left.\begin{aligned} &\psi_1(t) = 0 \quad \text{when} \quad t < 0; \\ &\psi_1(t) = \sin at \quad \text{when} \quad t > 0 \ (a \text{ being real}). \end{aligned}\right\} \tag{36}$$

In the same way as before it can readily be shown that this latter function can be represented by the contour integral:

$$\psi_1(t) = \mathcal{R}\left[-\frac{1}{2\pi}\int_{\smile} \frac{e^{itz}}{z - a}\,dz\right], \tag{37}$$

where the contour of integration is the real axis, which surrounds the pole $z = a$ of the integrand. In this case the residue is

$$e^{ita} = \cos at + i \sin at,$$

so that, separating the real and imaginary parts, we obtain formula (37), where $\mathcal{R}$ is the symbol of the real part.

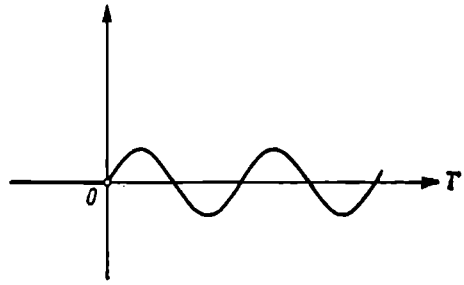

FIG. 61

In some cases the above formulae are written in a different way, viz. we integrate not along the real but along the imaginary axis and the pole is described from the right side, i.e. from that side of the imaginary axis near which the real part of the complex number is positive. To obtain this new contour of integration it is sufficient to rotate the plane about the origin through an angle $\pi/2$ in the counter-clockwise direction, i.e. to replace z by a new variable z', where $z' = iz$ or $z = z'/i$. By introducing this new variable we have in place of formula (29)

$$\varphi(t) = \frac{1}{2\pi i}\int_{\smile} \frac{e^{tz'}}{z'}\,dz'. \tag{29$_1$}$$

In formula (35) we now obtain the pole ia not on the imaginary axis but on the negative part of the real axis and we thus obtain the following expression for the function $\varphi_1(t)$:

$$\varphi_1(t) = \frac{1}{2\pi i} \int_{-\infty i}^{+\infty i} \frac{e^{tz'}}{z' + a} dz'. \qquad (35_1)$$

Similarly, we have the following expression for the function $\psi_1(t)$:

$$\psi_1(t) = \mathcal{R}\left[-\frac{1}{2\pi} \int_{\gamma} \frac{e^{tz'}}{z' - ia} dz'\right].$$

The contents of this paragraph are directly connected with Laplace's transformation with which we shall deal in Vol IV.

62. Examples of integrals of many-valued functions. We will now consider a few examples where the integrand functions are many-valued functions of the complex variable. For the first example consider the integral

$$\int_l (-z)^{a-1} Q(z)\, dz, \qquad (38)$$

where a is a certain real number and $Q(z)$ is a rational function, which is such that $z^a Q(z) \to 0$ if either $z \to 0$ or $z \to \infty$. The integrand is many-valued so that by describing a circuit round $z = 0$ in the counter-clockwise direction, $(-z)$ describes the same circuit and consequently the amplitude of this expression acquires the term 2π; the expression itself acquires the factor $e^{2\pi i}$ and $(-z)^{a-1}$ becomes $(-z)^{a-1} e^{2(a-1)\pi i}$, i.e. in this case the function acquires the factor $e^{2(a-1)\pi i}$, which is other than unity, providing a is not an integer. The origin is therefore the branch-point of the integrand. To make the function single-valued cut along the real axis starting from $z = 0$. In the cut T-plane our function will be single-valued and to determine it fully we have to fix the amplitude of $(-z)$ for a point in the T-plane. Let us agree that on the upper edge of the cut, where z is positive, the amplitude of the negative number $(-z)$ is equal to $(-\pi)$. Describing a circuit about the origin round a closed contour we come from the upper edge of the cut to the lower edge and in the course of this the amplitude of $(-z)$ acquires an additional term 2π, so that the amplitude of $(-z)$ on the lower edge of the cut will be equal to π. Denoting the modulus of z by x we have:

$$(-z) = xe^{-i\pi} \text{ on the upper edge,}$$

$$(-z) = xe^{i\pi} \text{ on the lower edge}$$

and consequently

$$\begin{aligned} (-z)^{a-1} &= x^{a-1} e^{-i(a-1)\pi} \text{ on the upper edge,} \\ (-z)^{a-1} &= x^{a-1} e^{i(a-1)\pi} \text{ on the lower edge.} \end{aligned} \qquad (39)$$

Let us now select the contour of integration for integral (38). We take for the contour of integration the following curve which consists of four parts: the section (ε, R) of the upper edge of the cut, the circumference C_R, centre the

origin and radius R, in the counter-clockwise direction, the section (R, ε) of the lower edge of the cut and, finally, the circumference C_ε, centre the origin and radius ε, in the clockwise direction (Fig. 62). To integrate along the positive part of the real axis we assume that the rational fraction $Q(z)$ has no poles there. In accordance with the fundamental theorem of residues the integral (38) will be equal to the product of $2\pi i$ and the sum of the residues of the integrand at all poles of the rational fraction $Q(z)$; the latter are also poles of the integrand. We are assuming all the time that ε is taken so small and R so large that all the above poles will be included in the region bounded by our contour of integration. We will now show that the integrals along the circles C_R and C_ε tend to zero as $R \to \infty$ and $\varepsilon \to 0$. In fact, applying the usual inequality we have:

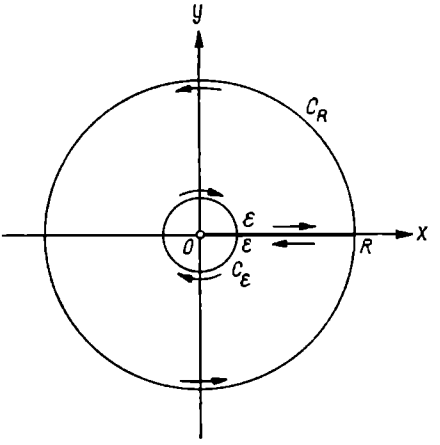

FIG. 62

$$\left| \int_{C_R} (-z)^{a-1} Q(z)\, dz \right| \leqslant$$

$$< 2\pi R \cdot R^{a-1} \max_{\text{on } C_R} |Q(z)| =$$

$$= 2\pi R^a \max_{\text{on } C_R} |Q(z)|.$$

It is given that $z^a Q(z) \to 0$ as $|z| \to \infty$; therefore the above expression also tends to zero as $R \to \infty$. Similarly, on the circumference C_ε we have the inequality:

$$\left| \int_{C_\varepsilon} (-z)^{a-1} Q(z)\, dz \right| < 2\pi \varepsilon^a \max_{\text{on } C_\varepsilon} |Q(z)|,$$

since $z^a Q(z) \to 0$ as $z \to 0$; therefore the above expression also tends to zero as $\varepsilon \to 0$. Hence in the limit only integrals along the upper and lower edges of the cut remain and the value of the integrand on these cuts is determined by formula (39); this gives us:

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon}^{R} \left[x^{a-1} e^{-i\pi(a-1)} Q(x) - x^{a-1} e^{i\pi(a-1)} Q(x) \right] dx = 2\pi i \sum r,$$

where Σr denotes the sum of residues of the function $(-z)^{a-1} Q(z)$ at all poles within a finite distance.

Bearing in mind that $e^{-i\pi} = e^{i\pi} = -1$, we can rewrite the above formula as follows:

$$(e^{i\pi a} - e^{-i\pi a}) \int_0^{\infty} x^{a-1} Q(x)\, dx = 2\pi i \sum r$$

or, (from Euler's formula):

$$\int_0^\infty x^{a-1} Q(x)\,dx = \frac{\pi}{\sin a\pi} \sum r. \tag{40}$$

Formula (40) makes it possible to evaluate many definite integrals in which the primitive is not expressed in final form. Let us remind you once again of the conditions to which the function $Q(z)$ is subject, for the above formula to be valid. *The function $Q(z)$ must be a rational fraction with no poles on the positive part of the real axis and it must also satisfy the conditions:*

$$z^a Q(z) \to 0 \text{ as } z \to 0 \text{ and } z \to \infty.$$

As a particular example let us consider the integral

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx \qquad (0 < a < 1). \tag{41}$$

It can readily be seen that in this case the function

$$Q(z) = \frac{1}{1+z}$$

satisfies all the above conditions and has a single pole $z = -1$. At this pole the function

$$\frac{(-z)^{a-1}}{1+z}$$

has a residue which is evaluated by the rule: numerator divided by the derivative of the denominator, i.e. this residue is

$$r = (-z)^{a-1}\big|_{z=-1}.$$

Note that when evaluating the function $(-z)^{a-1}$ at the point $z = -1$ we must bear in mind the definition of many-valued functions which was given above, viz. on the upper edge of the cut the amplitude of $(-z)$ is equal to $(-\pi)$ and, consequently, when describing half the circuit round the origin on the negative part of the real axis the amplitude of $(-z)$ will be zero. In other words

$$r = 1.$$

We finally obtain the following expression for integral (41) from formula (40):

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi}. \tag{42}$$

As the second example of an integral of a many-valued function consider the integral:

$$\int_{z_1}^{z_2} \sqrt{A + 2\frac{B}{z} + \frac{C}{z^2}}\,dz, \tag{43}$$

and suppose that the trinomial $A + 2B/z + C/z^2$ has real coefficients and distinct real zeros $z = z_1$ and $z = z_2$, where $0 < z_1 < z_2$.

We also suppose that $A < 0$ from which follows directly that the above trinomial will be positive when $z_1 < z < z_2$. We integrate (43) along the section $z_1 \leqslant z \leqslant z_2$ of the real axis on which the radical is taken to be positive. The integrand

$$\sqrt{A + 2\frac{B}{z} + \frac{C}{z^2}} = \frac{\sqrt{A\,(z - z_1)\,(z - z_2)}}{z} \tag{44}$$

will have branch-points of order one at the points z_1 and z_2. If we make a cut along the section (z_1, z_2) of the real axis then the function (44) will be regular and single-valued in the cut T-plane [19].

Let us suppose that the radical is positive on the lower edge of the cut. To reach the upper edge where the radical is negative we have to pass one of the branch-points [19]. Let us take our integral along the whole contour of the cut in the positive direction, i.e. we take the integral of the function (44) along the lower edge from z_1 to z_2 and along the upper edge from z_2 to z_1. The first part of this integral will give integral (43). When integrating along the upper edge the integrand will change its sign but the direction of integration will also be reversed so that the value of the integral along the upper and lower edges will be the same, i.e. the value of the integral along the whole contour of the cut will be twice the value of the integral (43).

According to Cauchy's theorem we can, without changing the value of the integral, continuously deform our closed contour, providing it does not leave the domain in which the function (44) is regular. If l is any closed contour, which includes the above cut and the point $z = 0$, which is a pole of the function (44), remains outside l, then it follows from above:

$$J = \frac{1}{2}\int_l \sqrt{A + 2\,\frac{B}{z} + \frac{C}{z^2}}\,\mathrm{d}z. \tag{45}$$

We will now expand function (44) in the neighbourhood of $z = \infty$ and of $z = 0$. In the first case we can write:

$$\sqrt{A + 2\frac{B}{z} + \frac{C}{z^2}} = \sqrt{A}\left[1 + \left(2\,\frac{B}{Az} + \frac{C}{Az^2}\right)\right]^{\frac{1}{2}},$$

and using Newton's binomial formula we obtain:

$$\sqrt{A + 2\frac{B}{z} + \frac{C}{z^2}} = \sqrt{A}\left(1 + \frac{B}{A}\,\frac{1}{z} + \ldots\right). \tag{46}$$

Let us determine the value of the radical $\sqrt{A}$ in this formula. We use for this purpose the left-hand side of formula (44). We know that the radical is positive along the lower edge of the section (z_1, z_2). To reach the section $(z_2, +\infty)$ of the real axis the point $z = z_2$ must be circumscribed in a counter-clockwise direction. As a result the amplitude of $(z - z_2)$ is increased by π and the amplitude of the expression (44) by $\pi/2$, i.e. instead of zero this amplitude becomes $\pi/2$. In other words the function (44) must be considered to be positively imaginary on

the section $(z_2, +\infty)$ of the real axis. (By a positively imaginary number we mean ai when $a > 0$.) It follows from (46) that the radical $\sqrt{A}$ is positively imaginary.

Similarly, to reach the section $(0, z_1)$ from the lower edge of the section (z_1, z_2) the point $z = z_1$ must be described in the clockwise direction, as a result of which the amplitude of the expression (44) will be $(-\pi/2)$, i.e. this expression will be negatively imaginary on the section $(0, z_1)$.

Expanding the function (44) near $z = 0$:

$$\sqrt{A + 2\frac{B}{z} + \frac{C}{z^2}} = \frac{\sqrt{C}}{z}\left[1 + \left(2\frac{Bz}{C} + \frac{Az^2}{C}\right)\right]^{\frac{1}{2}},$$

or, using the binomial expansion:

$$\sqrt{A + 2\frac{B}{z} + \frac{C}{z^2}} = \frac{\sqrt{C}}{z}\left(1 + \frac{B}{C}z + \ldots\right), \tag{47}$$

and, in agreement with what was said above, $\sqrt{C}$ must be negatively imaginary. It is given that $A < 0$ and it follows from $z_2 > z_1 > 0$ that $C < 0$.

According to Cauchy's theorem the integral of the function (44) along a large closed contour L near $z = \infty$ is equal to the sum of integrals along the above contour l and along the contour λ which surrounds $z = 0$; every integral being taken in the counter-clockwise direction. The integrals along L and λ, are equal to the product of $2\pi i$ and the coefficient of z^{-1} in the expansions (46) and (47) and consequently:

$$\int_l \sqrt{A + 2\frac{B}{z} + \frac{C}{z^2}}\,dz = 2\pi i\left(\frac{B}{\sqrt{A}} - \sqrt{C}\right),$$

hence the formula (45) gives us the final value of our integral (43):

$$J = \int_{z_1}^{z_2} \sqrt{A + 2\frac{B}{z} + \frac{C}{z^2}}\,dz = \pi i\left(\frac{B}{\sqrt{A}} - \sqrt{C}\right). \tag{48}$$

63. Integration of a system of linear equations with constant coefficients. We will now apply the theory of residues to the integration of a system of linear homogeneous equations with constant coefficients. Consider such a system:

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n, \end{aligned}\right\} \tag{49}$$

where the a_{ik} are constant coefficients and the x_s' are derivatives of the required function x_s with respect to the independent variable t. We require the solution of this system in the form:

$$x_s = \sum_R \varphi_s(z)\,e^{tz} \qquad (s = 1, 2, \ldots, n), \tag{50}$$

where $\varphi_s(z)$ are the required rational functions of z; the symbol

$$\sum_R f(z)$$

will denote, here and in future, the sum of residues of the function $f(z)$ with respect to all singularities within a finite distance. In formulae (50) the functions after the symbol of the sum of residues depend not only on the complex variable z, with respect to which we are calculating the residues, but also on the real parameter t, so that the sum of the residues will, generally speaking, also be a function of this parameter t. Owing to the fact that z and t are not connected in any way with each other, we can, when differentiating the function (50) with respect to t, differentiate under the symbol of the sum of residues, i.e. we obtain one and the same result if we differentiate the function

$$\varphi_s(z)\, e^{tz} \tag{51}$$

with respect to t first and then take the sum of its residues, or if we take the sum of residues of the function (51) first and then differentiate with respect to t. Therefore, apart from the formulae (50) we also have the following formulae:

$$x_s' = \sum_R z\varphi_s(z)\, e^{tz} \qquad (s = 1, 2, \ldots, n). \tag{52}$$

We substitute all this into our system (49) and collect all terms together:

$$\begin{aligned}
&\sum_R \left[(a_{11} - z)\varphi_1(z) + a_{12}\varphi_2(z) + \ldots + a_{1n}\varphi_n(z)\right] e^{tz} = 0,\\
&\sum_R \left[a_{21}\varphi_1(z) + (a_{22} - z)\varphi_2(z) + \ldots + a_{2n}\varphi_n(z)\right] e^{tz} = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\sum_R \left[a_{n1}\varphi_1(z) + a_{n2}\varphi_2(z) + \ldots + (a_{nn} - z)\varphi_n(z)\right] e^{tz} = 0.
\end{aligned}$$

These equations will obviously be satisfied if we equate the expressions in the square brackets to any arbitrary constant, so that we get under the symbol of the sum of residues a function in the form Ce^{tz}, which has no singularities at all within a finite distance. Denoting these arbitrary constants by $-C_1, -C_2, \ldots, -C_n$ we obtain a system of ordinary algebraic equations of the first order for the determination

of $\varphi_s(z)$:

$$\begin{aligned}
&(a_{11}-z)\,\varphi_1(z)+ \quad a_{12}\,\varphi_2(z)+\ldots+a_{1n}\,\varphi_n(z)=-C_1,\\
&a_{21}\,\varphi_1(z)+(a_{22}-z)\,\varphi_2(z)+\ldots+a_{2n}\,\varphi_n(z)=-C_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{n1}\,\varphi_1(z)+ \quad a_{n2}\,\varphi_2(z)+\ldots+(a_{nn}-z)\,\varphi_n(z)=-C_n.
\end{aligned}$$

We solve this system by Kramer's formula:

$$\varphi_s(z)=\frac{\Delta_s(z)}{\Delta(z)} \qquad (s=1,2,\ldots,n), \tag{53}$$

where

$$\Delta(z)=\begin{vmatrix} a_{11}-z, & a_{12}, \ldots, a_{1n} \\ a_{21}, & a_{22}-z, \ldots, a_{2n} \\ \cdots\cdots & \cdots\cdots\cdots \\ a_{n1}, & a_{n2}, \ldots, a_{nn}-z \end{vmatrix}, \tag{54}$$

and $\Delta_s(z)$ is obtained by changing the elements of the sth column by terms $(-C_k)$ in the determinant $\Delta(z)$. Note that the determinant $\Delta(z)$ represents the left-hand side of a familiar equation [III$_1$, 17]. The expressions obtained in (53) must now be substituted in formula (50).

Making these substitutions we obtain the solution of our system in the form:

$$x_s=\sum_R \frac{\Delta_s(z)}{\Delta(z)}\,e^{tz} \qquad (s=1,2,\ldots,n), \tag{55}$$

where the meanings of $\Delta(z)$ and $\Delta_s(z)$ are explained above.

We will now show that the result obtained satisfies the original conditions

$$x_1\big|_{t=0}=C_1; \quad x_2\big|_{t=0}=C_2; \quad \ldots; \quad x_n\big|_{t=0}=C_n. \tag{56}$$

We will only test this for x_1. We have:

$$x_1\big|_{t=0}=\sum_R \frac{\Delta_1(z)}{\Delta(z)}. \tag{57}$$

The denominator of the above fraction is given by formula (54) and it is evidently a polynomial of the nth order, the last term of which is $(-1)^n z^n$. The numerator of the fraction in formula (57) is:

$$\Delta_1(z)=\begin{vmatrix} -C_1, & a_{12}, & \ldots, & a_{1n} \\ -C_2, & a_{22}-z, & \ldots, & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -C_n, & a_{n2}, & \ldots, & a_{nn}-z \end{vmatrix}.$$

Multiplying the elements of the first column we obtain a polynomial of the $(n-1)$th order, the last term of which is $(-1)^n C_1 z^{n-1}$; we can therefore rewrite formula (57) as follows:

$$x_1\Big|_{t=0} = \sum_R \frac{(-1)^n C_1 z^{n-1} + \dots}{(-1)^n z^n + \dots}, \tag{58}$$

where the dots indicate terms of the polynomial with lower powers, which are of no importance in subsequent calculations.

We will now establish a certain general rule for the sum of residues of a rational fraction.

LEMMA. *The sum of the residues of a rational fraction with reference to its poles at a finite distance, is equal to the coefficient of* z^{-1} *in the expansion of the rational fraction in the neighbourhood of the point at infinity.*

In fact, suppose that our rational fraction has the following expansion in the neighbourhood of the point at infinity:

$$f(z) = \sum_k b_k z^k. \tag{59}$$

Consider the integral

$$\frac{1}{2\pi i} \int_{C_R} f(z)\, dz,$$

where C_R is the circumference of a circle, centre the origin and radius R. If R is sufficiently large all the poles of $f(z)$ will lie inside C_R and the integral will give the sum of the residues at these poles. Also, if R is sufficiently large C_R will be near the point at infinity and we can therefore apply expansion (59) to solve the integral; it follows directly from this that the integral will be equal to b_{-1}, which proves the lemma.

Note. Earlier in [17] we called the coefficient b_{-1} in the expansion (59) with its sign reversed the residue of the function $f(z)$ at the point at infinity, i.e. this residue is equal to $(-b_{-1})$. For this reason our lemma can be formulated as follows: *the sum of the residues of a rational fraction at all its poles, including the point at infinity, is equal to zero.*

Let us now apply the above lemma to the expression (58). Note that near the point at infinity the fraction can be expanded as follows:

$$\frac{(-1)^n C_1 z^{n-1} + \dots}{(-1)^n z^n + \dots} = \frac{C_1}{z} + \frac{\beta_2}{z^2} + \dots,$$

and the above lemma gives us directly $x_1 \mid_{t=0} = C_1$; it can similarly be shown that $x_s \mid_{t=0} = C_s$. Hence the solution given by formula

(55) satisfies the original conditions (56), i.e. the arbitrary constants C_s in the polynomial $\Delta_s(z)$ take the place of the original conditions. Therefore our formula (55) gives the general solution of the system.

Example. Consider the system

$$\begin{aligned} x_1' &= \quad x_2 + x_3 \\ x_2' &= x_1 \quad + x_3 \\ x_3' &= x_1 + x_2. \end{aligned}$$

In this case

$$\Delta(z) = \begin{vmatrix} -z, & 1, & 1 \\ 1, & -z, & 1 \\ 1, & 1, & -z, \end{vmatrix},$$

or

$$\Delta(z) = -z(z^2-1) + 2(z+1) = (z+1)(-z^2+z+2),$$

and for the first of the required functions we obtain the formula:

$$x_1 = \sum_R \frac{\begin{vmatrix} -C_1, & 1, & 1 \\ -C_2, & -z, & 1 \\ -C_3, & 1, & -z, \end{vmatrix}}{(z+1)(-z^2+z+2)} e^{tz},$$

or, by solving the determinant and cancelling by $(1+z)$:

$$x_1 = \sum_R \frac{C_1(1-z) - C_2 - C_3}{-z^2+z+2} e^{tz}.$$

The denominator has zeros $z=-1$ and $z=2$. We will calculate the residues at these points by the usual rule: the numerator divided by the derivative of the denominator, and we obtain:

$$x_1 = \left(\frac{2}{3}C_1 - \frac{1}{3}C_2 - \frac{1}{3}C_3\right)e^{-t} + \left(\frac{1}{3}C_1 + \frac{1}{3}C_2 + \frac{1}{3}C_3\right)e^{2t}.$$

Notice that in this case the polynomial $\Delta(z)$ has a double zero at $z=-1$, but in spite of this the coefficient of e^{-t} in the expression for x_1 is not a polynomial in the tth power but simply a constant.

In those cases when the equations are not homogeneous (compulsory vibrations):

$$x_s' = a_{s1}x_1 + \ldots + a_{sn} + f_s(t) \quad (s = 1, 2, \ldots, n), \tag{60}$$

where $f_s(t)$ is a given function of t, the required solution should have the following form:

$$x_s = -\sum_R \frac{C_1(t)A_{1s}(z) + \ldots + C_n(t)A_{ns}(z)}{\Delta(z)} e^{tz} \tag{61}$$

where $A_{ik}(z)$ is the algebraic complement of the elements of the determinant $\Delta(z)$, and $C_k(t)$ is a required function of t (the variation method of arbitrary constants) [II, 25]. Substituting (61) in (60) and bearing in mind that when the C_k are arbitrary constants the formulae (61) give the solution of the homogeneous system, we obtain the following equations for the derivatives $C_k'(t)$:

$$-\sum_R \frac{C_1'(t)\,A_{1s}(z)+\ldots+C_n'(t)\,A_{ns}(z)}{\Delta(z)}\,e^{tz}=f_s(t)$$
$$(s=1,2,\ldots,n). \qquad (62)$$

We will show that this system can be satisfied, providing:

$$C_1'(t)=e^{-tz}f_1(t);\ \ldots\ ;C_n'(t)=e^{-tz}f_n(t). \qquad (63)$$

In fact, making these substitutions, we get on the left side of (62):

$$-\sum_R \frac{f_1(t)\,A_{1s}(z)+\ldots+f_n(t)\,A_{ns}(z)}{\Delta(z)}. \qquad (64)$$

If $i\neq k$, then in the algebraic complement $A_{ik}(z)$ two elements situated on the main diagonal of the determinant $\Delta(z)$ will be cancelled, viz. $(a_{ii}-z)$ and $(a_{kk}-z)$; therefore $A_{ik}(z)$ will be a polynomial of the order $(n-2)$ in z. As a result of the above lemma:

$$\sum_R \frac{A_{ik}(z)}{\Delta(z)}=0 \qquad (i\neq k),$$

since the expansion of $A_{ik}(z):\Delta(z)$ near the point at infinity begins with the term a/z^2 and there is therefore no term in z^{-1}.

The algebraic complement $A_{ii}(z)$ will be a polynomial of order $(n-1)$ with the last term $(-1)^{n-1}z^{n-1}$ (cf. above) and consequently:

$$-\sum_R \frac{A_{ii}(z)}{\Delta(z)}=1.$$

It follows directly from this that the preceding expressions (64) are equal to $f_s(t)$. The formulae for $C_k(t)$ give:

$$C_k(t)=\int_0^t e^{-\tau z}f_k(\tau)\,d\tau \qquad (k=1,2,\ldots,n),$$

and we chose the constant of integration so that $C_k(0)=0$ (purely compulsory vibrations).

Substituting in (61) we finally have:

$$x_s=-\sum_R \int_0^t \frac{f_1(\tau)\,A_{1s}(z)+\ldots+f_n(\tau)\,A_{ns}(z)}{\Delta(z)}\,e^{(t-\tau)z}\,d\tau. \qquad (65)$$

64. The expansion of a fractional function into partial fractions. We shall now apply the basic theorem of residues to the expansion of a function into an infinite series. Let the function $f(z)$ be uniform and homogeneous in the whole plane except at a few isolated points which are its poles. Such a function is usually known as a *fractional or meromorphic function.* A rational fraction is one example of a fractional function. As the second example take $\cot z = \cos z/\sin z$ which has poles at the points where $\sin z$ vanishes.

This latter meromorphic function has an infinite number of poles. Notice that when a meromorphic function has an infinite number of poles then in any confined part B of the plane the number of poles should, at any rate, be finite. Otherwise we would have in B at least one limit-point for these poles, i.e. such a point $z = c$, that any small circle with centre at $z = c$ would contain an infinite number of poles of the function $f(z)$. This point $z = c$ would be a singularity of $f(z)$ other than a pole, for it follows from the definition of a pole [17], that it should be an isolated singularity. But it is given that $f(z)$ has no singularities other than poles. Once we have in a confined part of the plane a finite number of poles we can number them in the order of their non-decreasing moduli, so that denoting the poles by a_k we have:

$$|a_1| \leqslant |a_2| \leqslant |a_3| \leqslant \ldots,$$

where $|a_n| \to +\infty$ as n increases indefinitely. At every pole $z = a_k$ our function will have a definite infinite part, which will be a polynomial with respect to the argument $1/(z-a_k)$, without the constant term [17]. Denote this polynomial by

$$G_k\left(\frac{1}{z-a_k}\right) \qquad (k = 1, 2, \ldots). \tag{66}$$

We shall now show that by making certain additional assumptions the fractional function $f(z)$ can be represented by a simple infinite series, the terms of which are expressed by infinite parts (66). Let us formulate the condition to which we must subject the function $f(z)$. Suppose that a sequence of closed contours C_n which surround the origin exists and which are such that every contour C_n lies inside the contour C_{n+1}. Let l_n be length of the contour C_n and δ_n be its shortest distance from the origin. We assume that $\delta_n \to \infty$, i.e. that the contours C_n widen indefinitely in all directions as n increases. We also suppose

that the relationship $l_n : \delta_n$ remains bounded as n increases indefinitely, i.e. a positive number m exists such that

$$\frac{l_n}{\delta_n} \leqslant m. \tag{67}$$

If, for example, C_n are circles, centres the origin and radii r_n, then $l_n = 2\pi r_n$ and $\delta_n = r_n$, so that $l_n : \delta_n = 2\pi$. We now suppose that the modulus of our fractional function $f(z)$ remains bounded on all contours C_n, in other words, a positive number M exists, such that on any contour C_n the inequality given below is satisfied:

$$|f(z)| \leqslant M \quad (\text{on } C_n). \tag{68}$$

Consider the integral:

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(z')}{z' - z} \, dz', \tag{69}$$

where we integrate in the positive direction and the point z lies inside C_n and is other than a_k. Consider also the sum of infinite parts which refer to poles a_k, inside C_n:

$$\omega_n(z) = \sum_{(C_n)} G_k\left(\frac{1}{z - a_k}\right), \tag{70}$$

where (C_n) below the symbol of the sum shows that only poles situated inside C_n must be added.

The integrand of (69), which is a function of z', has in C_n a simple pole $z' = z$, which is due to the vanishing of the denominator, and poles $z' = a_k$, which are due to infinite parts of $f(z')$. The residue at the pole $z' = z$ is determined by the rule: the numerator divided by the derivative of the denominator:

$$\frac{f(z')}{(z' - z)'}\bigg|_{z'=z} = \frac{f(z')}{1}\bigg|_{z'=z} = f(z).$$

The residues at the poles $z' = a_k$ will be the same as for the function

$$\frac{\omega_n(z')}{z' - z}. \tag{71}$$

In this latter function $\omega_n(z')$ is a rational fraction in which the order of the numerator is lower than the order of the denominator and all poles are situated inside C_n. We will show that in this case the sum of residues of the function (71) at the poles a_k will be

$$-\omega_n(z) = -\sum_{(C_n)} G_k\left(\frac{1}{z - a_k}\right). \tag{72}$$

In fact, the function (71) is a rational fraction of z' in which the order of the denominator is at least two units higher than that of the numerator, for $\omega_n(z')$ is already a rational fraction in which the order of the denominator is higher than the order of the numerator. In the neighbourhood of $z' = \infty$ we therefore have the following expansion:

$$\frac{\omega_n(z')}{z'-z} = \frac{a_2}{z'^2} + \frac{a_3}{z'^3} + \dots,$$

and the integral of the function (71) around a circle of a sufficiently large radius will be equal to zero, i.e. the sum of residues of the function (71) at all its poles within a finite distance, is zero. Its residue at the point $z' = z$ is, obviously, equal to $\omega_n(z)$ and, consequently, the sum of residues at the remaining poles a_k is equal to the expression (72). Applying the basic theorem of residues to integral [69] we obtain:

$$\frac{1}{2\pi i}\int_{C_n}\frac{f(z')}{z'-z}\,dz' = f(z) - \sum_{(C_n)} G_k\left(\frac{1}{z-a_k}\right).$$

Suppose in the above formula that $z = 0$, where the point $z = 0$ is not a pole of $f(z)$:

$$\frac{1}{2\pi i}\int_{C_n}\frac{f(z')}{z'}\,dz' = f(0) - \sum_{(C_n)} G_k\left(-\frac{1}{a_k}\right).$$

Subtracting this equation from the one above we have:

$$\frac{z}{2\pi i}\int_{(C_n)}\frac{f(z')}{z'(z'-z)}\,dz' = f(z) - f(0) - \sum_{(C_n)}\left[G_k\left(\frac{1}{z-a_k}\right) - G_k\left(-\frac{1}{a_k}\right)\right]. \quad (73)$$

We will now show that the integral on the left side of the above equation tends to zero as n increases indefinitely. In fact, bearing in mind that

$$|z'| \geqslant \delta_n \text{ and } |z'-z| \geqslant |z'| - |z| \geqslant \delta_n - |z|,$$

we have from (68)

$$\left|\int_{C_n}\frac{f(z')}{z'(z'-z)}\,dz'\right| \leqslant \frac{Ml_n}{\delta_n(\delta_n - |z|)},$$

or from (67)

$$\left|\int_{C_n}\frac{f(z')}{z'(z'-z)}\,dz'\right| < \frac{Mm}{\delta_n - |z|},$$

from which it follows directly that the integral tends to zero, as $\delta_n \to \infty$. Therefore in the limit formula (73) gives

$$f(z) - f(0) - \lim_{n\to\infty} \sum_{(C_n)} \left[G_k\left(\frac{1}{z-a_k}\right) - G_k\left(-\frac{1}{a_k}\right)\right] = 0,$$

or

$$f(z) = f(0) + \lim_{n\to\infty} \sum_{(C_n)} \left[G_k\left(\frac{1}{z-a_k}\right) - G_k\left(-\frac{1}{a_k}\right)\right]. \tag{74}$$

As n increases indefinitely, the contour C_n will widen indefinitely, and more and more new poles a_k will find their way within C_n, so that in the limit we have on the right-hand side of (74) an infinite series; hence formula (74) gives $f(z)$ in the form of an infinite series:

$$f(z) = f(0) + \sum_{k=1}^{\infty} \left[G_k\left(\frac{1}{z-a_k}\right) - G_k\left(-\frac{1}{a_k}\right)\right]. \tag{75}$$

Strictly speaking we should, according to (74), group together all terms in the infinite series (75) which refer to poles situated between C_n and C_{n+1}. However, if we are convinced that the series (75) is convergent without performing this grouping of its terms, then we can deal with the infinite series (75) in the usual way.

If instead of condition (68), which tells us that the modulus of the function $f(z)$ is bounded on the contours C_n, we use a wider condition, viz. that $f(z)$ does not grow on the contours C_n faster than a certain positive power z^n, i.e. on all contours C_n the following inequality holds:

$$\left|\frac{f(z)}{z^p}\right| \leqslant M \quad (\text{on } C_n),$$

then in place of formula (75) we obtain the following expansion formula:

$$f(z) = f(0) + \frac{f'(0)}{1} z + \ldots + \frac{f^{(p)}(0)}{p!} z^p + \\ + \sum_{k=1}^{\infty} \left[G_k\left(\frac{1}{z-a_k}\right) - \chi_k^{(p)}(z)\right], \tag{76}$$

where the symbol $\chi_k^{(p)}(z)$ denotes the first $(p+1)$ terms in the expansion of the function $G_k[1/(z-a_k)]$ into a McLaurin's series.

65. The function cot z. Consider the fractional function

$$\cot z = \frac{\cos z}{\sin z}. \tag{77}$$

From Euler's formula we have:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and from this it follows directly that the equation $\sin z = 0$ is equivalent to $e^{i2z} = 1$; this has zeros $z = k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$), i.e. $\sin z$ has real zeros only, the values of which, which are well known from trigonometry. The function (77) will therefore have poles at the points:

$$z = 0, \pm \pi, \pm 2\pi, \ldots \qquad (78)$$

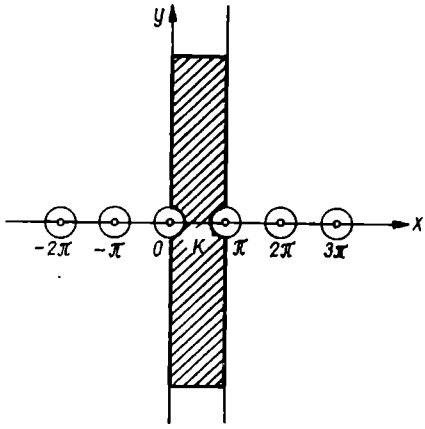

FIG. 63

We can show that the modulus of the function (77) is bounded in the whole plane if we isolate the points (78) by small circles λ_ϱ of same radius ϱ, where ϱ is any given positive number. Owing to the fact that the function (77) has a period π, it is sufficient to investigate it in the strip K, bounded by the straight lines $x = 0$ and $x = \pi$ (Fig. 63), in which the poles $z = 0$ and $z = \pi$ are isolated by the above circles of radius ϱ with centres at the respective points. In any confined part of the strip K our function (77) is continuous and therefore also bounded. It remains to be shown that by moving up or down the strip *ad infinitum*, the modulus of the function (77) remains bounded. Suppose, for example, that we move towards infinity up the strip K, i.e. if we assume that $z = x + iy$, then $y \to \infty$ and x varies in the interval $0 \leqslant x \leqslant \pi$. We have:

$$\cot z = i\,\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = i\,\frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{e^{ix}e^{-y} - e^{-ix}e^{y}},$$

whence, substituting the modulus of the numerator by the sum of the moduli, and the modulus of the denominator by the difference of the moduli, we have:

$$|\cot z| < \frac{e^{y} + e^{-y}}{e^{y} - e^{-y}} = \frac{1 + e^{-2y}}{1 - e^{-2y}}.$$

As y increases indefinitely the right side tends to the limit l and, consequently, for all sufficiently large y's we have, for example, the inequality:

$$|\cot z| < 1.5.$$

In exactly the same way we can act in the lower part of the strip K, and our proposition is thus proved.

Notice that the same proof also applies to the fractional function

$$\frac{1}{\sin z}, \qquad (79)$$

with poles at the same points and a period of 2π. The modulus of the function (79) will be bounded if its poles are isolated by small circles of the same radius, which can be as small as we please.

Let us return to the function (77) and adopt as the contour the circles C_n, centres the origin and radii $(n + 1/2)\pi$. These circles satisfy the condition (67). Also by taking ϱ sufficiently small (for example, smaller than $\pi/2$), we can say that the circles C_n will not pass through the circles λ_ϱ, which are isolated in the plane; hence as a result of the above proof, the modulus of the function (77) will be bounded. The same can evidently be said about the function

$$f(z) = \cot z - \frac{1}{z}, \tag{80}$$

for z^{-1} tends to zero as $z \to \infty$. It can readily be seen that the function (80) no longer has a pole at the origin $z = 0$ and we can therefore apply the expansion (75) to this function. We then calculate the infinite parts of the function (77) at its poles $z = k\pi$. Each of its poles will be a simple zero of $\sin z$ and the residue at this pole will be calculated by the usual formula

$$r_k = \frac{\cos z}{(\sin z)'}\bigg|_{z=k\pi} = 1.$$

Hence the infinite part of the function (77) at the pole $z = k\pi$ will be

$$\frac{1}{z - k\pi} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

In particular, at the pole $z = 0$ the infinite part will be z^{-1} and therefore the function (80) will no longer have a pole at $z = 0$. With regard to the other poles $z = k\pi$, the infinite part of the function (80) will be the same as for the function (77). If we are to apply formula (75) $f(0)$ remains to be calculated. Function (80), which is an odd function, can be expanded as follows near $z = 0$:

$$f(z) = \gamma_1 z + \gamma_3 z^3 + \ldots,$$

from which it follows directly that $f(0) = 0$. Formula (75) finally gives

$$\cot z = \frac{1}{z} + \sum_{k=-\infty}^{+\infty}{}' \left(\frac{1}{z - k\pi} + \frac{1}{k\pi}\right), \tag{81}$$

where the accent above the symbol of the sum shows that the term corresponding to $k = 0$ is excluded.

It can readily be proved that the series on the right-hand side will converge absolutely and uniformly in any bounded part of the plane providing the first few terms with poles in this part of the plane are rejected. In fact, the general term of the series will be

$$\frac{z}{(z - k\pi)\, k\pi}.$$

In any bounded part of the plane we have $|z| < M$; if we assume that the absolute value of k is sufficiently large we can write:

$$\left|\frac{z}{(z - k\pi)\, k\pi}\right| < \frac{1}{k^2} \cdot \frac{M}{\pi\,(\pi - Mk^{-1})}.$$

The coefficient of $1/k^2$ tends to a finite limit M/π^2 as k increases indefinitely, and the series

$$\sum_{k=-\infty}^{+\infty}{}' \frac{1}{k^2},$$

as we know, is convergent. Consequently [I, 147] the series (81) converges absolutely and regularly in any bounded part of the plane.

If in formula (81) we substitute πz for z, then

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k=-\infty}^{+\infty}{}' \left(\frac{1}{z-k} + \frac{1}{k}\right). \tag{81$_1$}$$

Grouping together in pairs terms which refer to values of k, which have opposite signs but the moduli of which are equal, we can rewrite this formula as follows:

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

In exactly the same way the following formula can be proved

$$-\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{k=-\infty}^{+\infty}{}' (-1)^k \left(\frac{1}{z-k} + \frac{1}{k}\right).$$

Differentiating the uniformly convergent series (81) we also have the formula

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{1}{z^2} + \sum_{k=-\infty}^{+\infty}{}' \frac{1}{(z-k)^2} = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k)^2}.$$

We recall that the above formulae were deduced by a different method in the theory of trigonometric series [II, 145].

66. The construction of meromorphic functions. We shall now deal with the construction of a meromorphic function when its poles a_k and infinite parts at these poles are given by

$$g_k\left(\frac{1}{z-a_k}\right) \qquad (k = 1, 2, \ldots). \tag{82}$$

If only a finite number of poles a_k ($k = 1, 2, \ldots, n$) are given, then the function

$$\varphi(z) = \sum_{k=1}^{n} g_k\left(\frac{1}{z-a_k}\right)$$

will give the obvious solution of the problem, where the above function is a rational fraction. Let us now suppose that we are given an infinite number of poles a_k and a corresponding number of infinite parts. We saw in [64] that in every bounded part of the plane there must only be a limited number of poles which can be numbered in the order of non-decreasing moduli, i.e.

$$|a_1| \leqslant |a_2| \leqslant \ldots \qquad (|a_n| \to +\infty).$$

There are no further restrictions as to the position of the poles or the given infinite parts. We only suppose that there is no pole at $z = 0$ among them.

Every infinite part (82) represents a function which is regular in the circle $|z| < |a_k|$, in which it can be expanded into a McLaurin's series:

$$g_k\left(\frac{1}{z-a_k}\right) = a_0^{(k)} + a_1^{(k)} z + a_2^{(k)} z^2 + \ldots \quad (|z| < |a_k|). \tag{83}$$

Take any sequence of positive numbers ε_k which form a convergent series:

$$\sum_{k=1}^{\infty} \varepsilon_k. \tag{84}$$

As a result of the uniform convergence of the series (83) in the circle [13]

$$|z| \leqslant \frac{1}{2}|a_k|$$

we can take a segment of this series

$$q_k(z) = a_0^{(k)} + a_1^{(k)} z + a_2^{(k)} z^2 + \ldots + a_{m_k}^{(k)} z^{m_k},$$

such that

$$\left| g_k\left(\frac{1}{z-a_k}\right) - q_k(z) \right| < \varepsilon_k \text{ in the circle } |z| \leqslant \frac{1}{2}|a_k|. \tag{85}$$

Construct the series

$$\varphi(z) = \sum_{k=1}^{\infty} \left[g_k\left(\frac{1}{z-a_k}\right) - q_k(z) \right] \tag{86}$$

and consider any circle C_R, centre the origin and radius R. Since $|a_k| \to +\infty$ there exists an N such that $R < (1/2)|a_k|$ when $k > N$, and for these values of k the inequality (85) holds in the circle C_R; consequently, as a result of the convergence of the series (84), series (86) is absolutely and uniformly convergent in C_R, providing its first N terms are rejected. These first N terms give poles a_k in the circle C_R with infinite parts (82). An absolutely and uniformly convergent series will give a regular function in C_R. Owing to the fact that the radius R is arbitrary we see that the sum (86) solves the problem of constructing a meromorphic function from its given poles and infinite parts. Notice that the polynomials $q_k(z)$ do not add new characteristics.

If the pole $z = 0$ is also given and its infinite part is

$$g_0\left(\frac{1}{z}\right),$$

then it is sufficient to add this infinite part to the series (86). This solution of the problem is due to the Swedish mathematician Mittag-Löffler.

In [64] we gave a formula for the expansion of a meromorphic function into partial fractions, when certain additional assumptions were made. We shall now give an analogous formula for the general case.

Let $f(z)$ be a certain meromorphic function. By using the method given above we shall construct the meromorphic function $\varphi(z)$ which has the same poles and infinite parts as $f(z)$. This meromorphic function $\varphi(z)$ will be given by a formula similar to (86). The difference $f(z) - \varphi(z)$ must be a regular function

in the whole plane (except at $z = \infty$). Such a function is called an *integral function*. It is expressed in the whole plane by a McLaurin series. Putting

$$f(z) - \varphi(z) = F(z),$$

we obtain the following formula for the meromorphic function:

$$f(z) = F(z) + g_0\left(\frac{1}{z}\right) + \sum_{k=1}^{\infty}\left[g_k\left(\frac{1}{z-a_k}\right) - g_k(z)\right], \tag{87}$$

where $F(z)$ is a certain integral function. This last formula is of great interest in theory though it is more convenient to use the formulae (75) and (76) in particular cases. If $F(z)$ is any integral function then (87) gives the general formula for all meromorphic functions for which the poles and infinite parts are given.

67. Integral functions. As we said above *an integral function is a function which is regular in the whole plane.* It is expressed in the plane by a McLaurin's series. If this series is disrupted then the function is simply a polynomial. Otherwise the point at infinity will be an essential singularity of our function and in this case the function is sometimes called an integral transcendent function. The functions e^z and $\sin z$ are examples of integral transcendent functions. In future we shall simply use the term "integral function".

We know that every polynomial has zeros. An integral function must not necessarily have this property. For example e^z has no zeros at all. We will now construct a general expression for integral functions without zeros. Let $g(z)$ be a certain integral function. In this case the function

$$f(z) = e^{g(z)} \tag{88}$$

is an integral function without zeros. Let us show, conversely, that every integral function $f(z)$ without zeros is of the form (88), where $g(z)$ is a certain integral function, i.e. *formula* (88), *where* $g(z)$ *is any arbitrary integral function, gives the general form of integral functions* $f(z)$ *without zeros.*

When the integral function $f(z)$ has no zeros then the function

$$\frac{f'(z)}{f(z)}$$

will also be an integral function. When integrating this integral function we also obtain an integral function

$$g(z) = \int \frac{f'(z)}{f(z)}\,dz = \log f(z),$$

from which (88) follows directly.

Suppose now that the integral function $f(z)$ has a finite number of zeros, other than $z = 0$:

$$z = a_1, a_2, \ldots, a_m,$$

where multiple zeros are counted as many times as there are units in their number. The relationship

$$f(z) : \prod_{k=1}^{m} \left(1 - \frac{z}{a_k}\right),$$

where the symbol $\prod_{k=1}^{m}$ denotes a product, which embraces all integral values of k from 1 to m, is an integral function without zeros, i.e. it has the form (88). We therefore have the following expression for our function $f(z)$

$$f(z) = e^{g(z)} \prod_{k=1}^{m} \left(1 - \frac{z}{a_k}\right), \tag{89}$$

where $g(z)$ is a certain integral function.

Let us suppose that the point $z = 0$ is not a zero of $f(z)$. If this point is a zero of order p, then in place of formula (89) we must have

$$f(z) = e^{g(z)} z^p \prod_{k=1}^{m} \left(1 - \frac{z}{a_k}\right). \tag{90}$$

In the most interesting case when $f(z)$ has an infinite number of zeros, we can no longer apply formula (90) directly, for on the right-hand side we have an infinite product, which may not have any meaning. To make this product convergent we have to add to the factors $(1 - z/a_k)$ additional exponential factors which do not introduce new zeros but make the infinite product convergent.

Let us investigate this for $\sin z$. Rewrite the formula (81):

$$\cot z - \frac{1}{z} = \sum_{k=-\infty}^{+\infty}{}' \left(\frac{1}{z - k\pi} + \frac{1}{k\pi}\right).$$

In this case both sides will be regular at the point $z = 0$ and we can integrate the infinite series term by term, from $z = 0$ to the variable point z. As a result of integration we have

$$\log \frac{\sin z}{z}\Big|_{z=0}^{z=z} = \sum_{k=-\infty}^{+\infty}{}' \left[\log (z - k\pi) + \frac{z}{k\pi}\right]_{z=0}^{z=z},$$

or, taking the principal value of the logarithm in the neighbourhood of the origin

$$\log \frac{\sin z}{z} = \sum_{k=-\infty}^{+\infty}{}' \left[\log \left(1 - \frac{z}{k\pi}\right) + \frac{z}{k\pi}\right].$$

Hence, discarding the logarithms, we obtain an expression for $\sin z$ in the form of an infinite product:

$$\sin z = z \prod_{k=-\infty}^{+\infty}{}' \left(1 - \frac{z}{k\pi}\right) e^{\frac{z}{k\pi}}, \tag{91}$$

where the accent above the symbol of the product shows that the factor corresponding to $k = 0$, has to be excluded. In this case factors of the exponential type $e^{z/k\pi}$ guarantee the convergence of the infinite product.

Grouping together in pairs factors corresponding to values of k the moduli of which are equal, we have:

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2 \pi^2}\right). \tag{92}$$

Substituting πz for z we can rewrite the formula in the following form

$$\frac{\sin \pi z}{\pi} = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right). \tag{93}$$

To give a more detailed explanation of the expansion of an integral function into an infinite product we have to explain certain basic facts about infinite products.

68. Infinite products. Consider the infinite product

$$\prod_{k=1}^{\infty} c_k = c_1 c_2 \ldots, \tag{94}$$

where c_k are certain complex numbers, other than zero. The convergence of the product (94) is analogous with the convergence of a series. Let us construct the finite products:

$$P_n = \prod_{k=1}^{n} c_k = c_1 c_2 \ldots c_n. \tag{95}$$

If, as n increases indefinitely, the product P_n tends to a finite limit P, which is not zero, then the infinite product (94) is said to be convergent and its limit is P.

If there are zeros among the numbers c_k, then the infinite product (94) is said to be convergent if, after the exclusion of the zeros, the product remains infinite and convergent in the above sense. In this case the value of the infinite product with zeros is taken to be equal to zero. The above remark, that the limit P of the product P_n should not be zero was made so that infinite convergent products should have the usual properties of finite products, viz. they should be equal to zero only when at least one of the factors is zero.

Let us suppose that all of the terms of the product (94) are other than zero and construct the infinite series:

$$\sum_{k=1}^{\infty} \log c_k, \tag{96}$$

where the value of the logarithm in every term is determined in a certain way. The sum of the first n terms of the series (96) will be

$$S_n = \sum_{k=1}^{n} \log c_k. \tag{97}$$

Suppose that for certain values of the logarithms the series (96) is convergent, i.e. a limit $S_n \to S$ exists. Formula (95) gives $P_n = e^{S_n}$, and consequently, there is a limit $P_n \to e^S$, other than zero, i.e. it follows from the convergence of the series (96) that the product (94) will also be convergent. Conversely, let us now suppose that the infinite product (94) is convergent, i.e. there is a limit $P_n \to P$, other than zero. We determine the values of the logarithms in the series (96) in such a way that the right side of formula (97) always contains the principal value of the logarithm of the product $c_1 c_2 \ldots c_n$:

$$S_n = \log |c_1 c_2 \ldots c_n| + i \arg (c_1 c_2 \ldots c_n),$$

where

$$-\pi < \arg (c_1 c_2 \ldots c_n) \leqslant \pi.$$

In thus case S_n will also tend to a definite limit, viz.:

$$\lim S_n = \log |P| + i \arg P = \log P,$$

and, consequently, the series (96) will be convergent.

We are assuming all the time that P is not a real negative number, for $\arg P$ lies in the interval $(-\pi, +\pi)$. When P is a real negative number we should select the amplitudes in such a way that $\arg (c_1 c_2 \ldots c_n)$ would be confined to the interval $(0, 2\pi)$. The proof would be the same as above.

We thus arrive at the following general proposition: *if all the numbers c_k are other than zero, then for the infinite product* (94) *to be convergent it is necessary and sufficient that the series* (96) *should be convergent when the values of the logarithms are determined in a certain way. This infinite product will be equal to*

$$P = e^S. \tag{98}$$

The general term of the series (96) will be:

$$\log c_k = \log |c_k| + i \arg c_k$$

Hence, discarding the logarithms, we obtain an expression for $\sin z$ in the form of an infinite product:

$$\sin z = z \prod_{k=-\infty}^{+\infty}{}' \left(1 - \frac{z}{k\pi}\right) e^{\frac{z}{k\pi}}, \tag{91}$$

where the accent above the symbol of the product shows that the factor corresponding to $k = 0$, has to be excluded. In this case factors of the exponential type $e^{z/k\pi}$ guarantee the convergence of the infinite product.

Grouping together in pairs factors corresponding to values of k the moduli of which are equal, we have:

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2 \pi^2}\right). \tag{92}$$

Substituting πz for z we can rewrite the formula in the following form

$$\frac{\sin \pi z}{\pi} = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right). \tag{93}$$

To give a more detailed explanation of the expansion of an integral function into an infinite product we have to explain certain basic facts about infinite products.

68. Infinite products. Consider the infinite product

$$\prod_{k=1}^{\infty} c_k = c_1 c_2 \ldots, \tag{94}$$

where c_k are certain complex numbers, other than zero. The convergence of the product (94) is analogous with the convergence of a series. Let us construct the finite products:

$$P_n = \prod_{k=1}^{n} c_k = c_1 c_2 \ldots c_n. \tag{95}$$

If, as n increases indefinitely, the product P_n tends to a finite limit P, which is not zero, then the infinite product (94) is said to be convergent and its limit is P.

If there are zeros among the numbers c_k, then the infinite product (94) is said to be convergent if, after the exclusion of the zeros, the product remains infinite and convergent in the above sense. In this case the value of the infinite product with zeros is taken to be equal to zero. The above remark, that the limit P of the product P_n should not be zero was made so that infinite convergent products should have the usual properties of finite products, viz. they should be equal to zero only when at least one of the factors is zero.

Let us suppose that all of the terms of the product (94) are other than zero and construct the infinite series:

$$\sum_{k=1}^{\infty} \log c_k, \tag{96}$$

where the value of the logarithm in every term is determined in a certain way. The sum of the first n terms of the series (96) will be

$$S_n = \sum_{k=1}^{n} \log c_k. \tag{97}$$

Suppose that for certain values of the logarithms the series (96) is convergent, i.e. a limit $S_n \to S$ exists. Formula (95) gives $P_n = e^{S_n}$, and consequently, there is a limit $P_n \to e^S$, other than zero, i.e. it follows from the convergence of the series (96) that the product (94) will also be convergent. Conversely, let us now suppose that the infinite product (94) is convergent, i.e. there is a limit $P_n \to P$, other than zero. We determine the values of the logarithms in the series (96) in such a way that the right side of formula (97) always contains the principal value of the logarithm of the product $c_1 c_2 \ldots c_n$:

$$S_n = \log |c_1 c_2 \ldots c_n| + i \arg (c_1 c_2 \ldots c_n),$$

where

$$-\pi < \arg (c_1 c_2 \ldots c_n) \leqslant \pi.$$

In thus case S_n will also tend to a definite limit, viz.:

$$\lim S_n = \log |P| + i \arg P = \log P,$$

and, consequently, the series (96) will be convergent.

We are assuming all the time that P is not a real negative number, for arg P lies in the interval $(-\pi, +\pi)$. When P is a real negative number we should select the amplitudes in such a way that arg $(c_1 c_2 \ldots c_n)$ would be confined to the interval $(0, 2\pi)$. The proof would be the same as above.

We thus arrive at the following general proposition: *if all the numbers c_k are other than zero, then for the infinite product* (94) *to be convergent it is necessary and sufficient that the series* (96) *should be convergent when the values of the logarithms are determined in a certain way. This infinite product will be equal to*

$$P = e^S. \tag{98}$$

The general term of the series (96) will be:

$$\log c_k = \log |c_k| + i \arg c_k$$

and, bearing in mind that the general term of a convergent series tends to zero, we should, in any case, have $\arg c_k \to 0$, i.e. the series (96) can only be convergent providing that, by starting from a certain place, we take the principal values of logarithms. The determination of the values of logarithms for a finite number of the first few terms can, naturally, have no effect on the convergence of the series and will only add $2m\pi i$ to the sum of the series, where m is a certain integer. This additional term will not alter the value S and, according to the formula (98), it will not affect P. Therefore the determination of the values of logarithms in the series (96) is only significant in so far that *by starting from a certain place which can be chosen arbitrarily, only the principal values of logarithms should be taken.*

Let us now consider the infinite product, the terms of which are integral functions of z:

$$F(z) = \prod_{k=1}^{\infty} u_k(z) = u_1(z) \cdot u_2(z) \dots \tag{99}$$

We take a circle C_R in the z-plane, centre the origin and radius R. Suppose that for any R the terms $u_k(z)$, starting with a certain value of k, no longer have zeros inside the circle C_R. Let, for example, this start with $k = k_0$ for the given radius R (this number will, generally speaking, depend on R). Consider the infinite series

$$S(z) = \sum_{k=1}^{\infty} \log u_k(z), \tag{100}$$

which can be rewritten as follows:

$$\sum_{k=1}^{k_0-1} \log u_k(z) + \sum_{k=k_0}^{\infty} \log u_k(z). \tag{101}$$

The terms of this latter sum are regular and single-valued functions in the circle C_R, for $u_k(z)$ does not vanish in this circle. Assume that for a certain set of values of these regular functions $\log u_k(z)$ the latter series will uniformly converge in the circle C_R. Denoting its sum by $f_{k_0}(z)$, where $f_{k_0}(z)$ is a certain regular function [12] we have:

$$\prod_{k=1}^{\infty} u_k(z) = e^{f_{k_0}(z)} \prod_{k=1}^{k_0-1} u_k(z),$$

i.e. in this case (99) will be a regular function in the circle C_R and its zeros inside the circle will be determined by the zeros of the terms

$u_k(z)$, when $k < k_0$. Since R is chosen arbitrarily, we can, in general say that *if the series* (100) *converges uniformly in any confined part of the plane (when the first few terms are excluded which is unimportant) the infinite product* (99) *will be convergent in the whole plane; it is an integral function and its zeros are fully determined by the factors* $u_k(z)$.

Differentiating the uniformly convergent series (100) we obtain:

$$S'(z) = \sum_{k=1}^{\infty} \frac{u_k'(z)}{u_k(z)},$$

but

$$F(z) = e^{S(z)} \quad \text{and} \quad F'(z) = S'(z)\, F(z),$$

i.e.

$$F'(z) = F(z) \sum_{k=1}^{\infty} \frac{u_k'(z)}{u_k(z)}. \tag{102}$$

This formula shows that when the series (100) is uniformly convergent, the infinite product (99) can be differentiated as shown by (102), which is analogous to the differentiation of a finite product.

69. The construction of an integral function from its given zeros. Using the above considerations we can construct an integral function from its given zeros. Notice, first of all, that the zeros of an integral function cannot have limit-points at a finite distance. If such a point $z = c$ were to exist, i.e. if any small circle with the centre at $z = c$ contained an infinite number of zeros of an integral function, then it would be identically zero [18]. Repeating the same arguments as in [64] we can see that, in any case, the zeros a_k of the integral function can be grouped in the order of their non-decreasing moduli:

$$|a_1| \leqslant |a_2| \leqslant \ldots,$$

where $|a_n| \to +\infty$ as $n \to \infty$. Notice that if a certain number a appears q times among the numbers a_k, this shows that the corresponding zero a is a zero of order q. We are also supposing, for the moment, that $z = 0$ does not appear among the given numbers a_k.

We will only consider one particular case which is of great practical importance, viz. when a_k moves towards infinity so quickly that a positive number m exists, such that the series

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^m} \tag{103}$$

is convergent. We assume that $m \geqslant 2$.

Let us construct the infinite product:

$$F(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \ldots + \frac{1}{m-1}\left(\frac{z}{a_k}\right)^{m-1}} \tag{104}$$

and show that it will satisfy all the conditions mentioned in the previous section. Consider a certain circle C_R such that starting with a certain symbol $k = k_0$, the numbers a_k will lie outside the circle C_R, Then when $k \geqslant k_0$, the terms of the product (104) will have no zeros in the circle C_R and for any z in C_R we will have:

$$\left|\frac{z}{a_k}\right| < \vartheta < 1 \qquad (k \geqslant k_0), \tag{105}$$

where ϑ is a definite positive number smaller than unity. Consider the series (100) in this case:

$$\sum_{k=k_0}^{\infty} \log\left[\left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \ldots + \frac{1}{m-1}\left(\frac{z}{a_k}\right)^{m-1}}\right]. \tag{106}$$

From (105) we can use the expansion of a logarithm into a power series and for this determination of the logarithm we obtain the following formula for the series (106):

$$\sum_{k=k_0}^{\infty} \left[\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \ldots + \frac{1}{m-1}\left(\frac{z}{a_k}\right)^{m-1} - \sum_{s=1}^{\infty} \frac{1}{s}\left(\frac{z}{a_k}\right)^s\right] =$$

$$= \sum_{k=k_0}^{\infty} \left[-\frac{1}{m}\left(\frac{z}{a_k}\right)^m - \frac{1}{m+1}\left(\frac{z}{a_k}\right)^{m+1} - \ldots\right].$$

Consider the general term of this series

$$v_k(z) = -\frac{1}{m}\left(\frac{z}{a_k}\right)^m - \frac{1}{m+1}\left(\frac{z}{a_k}\right)^{m+1} - \ldots$$

We obviously have:

$$|v_k(z)| \leqslant \frac{1}{m}\left|\frac{z}{a_k}\right|^m + \frac{1}{m+1}\left|\frac{z}{a_k}\right|^{m+1} + \ldots$$

or, from (105) taking $(1/m)|z/a_k|^m$ outside the brackets and bearing in mind that in the circle C_R $|z| \leqslant R$:

$$|v_k(z)| \leqslant \frac{R^m}{m|a_k|^m}(1 + \vartheta + \vartheta^2 + \ldots),$$

i.e.

$$|v_k(z)| \leqslant \frac{R^m}{m(1-\vartheta)} \frac{1}{|a^k|^m}.$$

As a result of the convergence of the series (103), the positive numbers on the right-hand side of the above inequality form a convergent

series and, consequently, series (106) will be absolutely and uniformly convergent in the circle C_R. We can therefore say that the infinite product (104) is an integral function and that its zeros are determined from the zeros of the factor $(1 - z/a_k)$, i.e. the numbers a_k are the zeros of this integral function.

If we have any arbitrary function $f(z)$ the zeros of which are a_k, then the quotient $f(z) : F(z)$ will be an integral function without zeros, i.e. this quotient will have the form $e^{g(z)}$, and we will have the following formula for the integral function $f(z)$:

$$f(z) = e^{g(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \ldots + \frac{1}{m-1}\left(\frac{z}{a_k}\right)^{m-1}}, \qquad (107)$$

where $g(z)$ is a certain integral function. We have so far that the point $z = 0$ is not a zero of the function. If this point is in fact a zero of order p, then we should add the factor z^p to the right-hand sides of the formulae (104) and (107).

Let us consider, for example, the function $\sin z$. It has a simple zero $z = 0$ and zeros simple $z = k\pi$ $(k = \pm 1, \pm 2, \ldots)$.

In this case we have $m = 2$, so that the series

$$\sum_{k=-\infty}^{+\infty}{}' \frac{1}{|k\pi|^2}$$

as we have already said above, is convergent. Using formula (107) and adding the factor z we have

$$\sin z = e^{g(z)} z \prod_{k=-\infty}^{+\infty}{}' \left(1 - \frac{z}{k\pi}\right) e^{\frac{z}{k\pi}}.$$

The integral function $g(z)$ cannot be determined from the above considerations. The results [67] show that in this case the function is identically zero.

Note that when $m = 1$, i.e. the series below is convergent

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|};$$

we can by using the same arguments as before, write instead of formula (107):

$$f(z) = e^{g(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right).$$

We shall give more examples of the application of formula (104) later; it is usually known as *the infinite product of Weierstrass*.

It may happen that the numbers a_k are such that the series (103) will be divergent for any positive m. This will happen if we suppose, for example,

that $a_k = \log(k+1)$ $(k = 1, 2, \ldots)$. In fact a series with the general term $[\log(k+1)]^{-m}$ will be divergent for any positive m, so that the sum of its first few terms is greater than

$$\frac{k}{[\log(k+1)]^m},$$

and this latter expression can easily be shown to grow indefinitely with k by applying, for instance, l'Hôpital's rule [I, 66]. If the series (103) is divergent for any positive m we can construct the infinite product:

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{a_k}\right)e^{Q_k(z)}, \tag{108}$$

where

$$Q_k(z) = \frac{z}{a_k} + \frac{z^2}{2a_k^2} + \ldots + \frac{z_k^m}{m_k a_k^m k}$$

and m_k depends on k. Repeating the above inequalities we can see that the infinite produot (108) will be convergent providing that the series given below is convergent for any $R > 0$

$$\sum_{k=1}^{\infty}\left(\frac{R}{|a_k|}\right)^{m_k+1}.$$

For this purpose it is sufficient to take $m_k = k - 1$. In fact, applying Cauchy's analysis [I, 121] to the series

$$\sum_{k=1}^{\infty}\left(\frac{R}{|a_k|}\right)^k$$

we obtain

$$\sqrt[k]{\left(\frac{R}{|a_k|}\right)^k} = \frac{R}{|a_k|} \to 0,$$

i.e. the series does, in fact, converge. It can be shown that the series will be convergent providing m_k is so chosen that the inequality $m_k + 1 > \log k$ holds.

70. Integrals which depend on parameters. In future we shall have to determine functions in the form of integrals which depend on parameters. We have already met these functions in [61]. We have also considered this problem for real variables and determined conditions for such a function to have a derivative and when differentiation under the integral is possible [II, 84].

We shall now consider an analogous process for complex variables.

THEOREM. *Let $f(t, z)$ be a continuous function of two variables t and z, where z belongs to the closed domain B with contour l and t to a finite interval $a \leqslant t \leqslant b$ of the real axis. Also $f(t, z)$ is a regular function of*

z in the closed domain B for any t in the above interval. In this case the function $\omega(z)$ given by the inequality:

$$\omega(z) = \int_a^b f(t, z)\, dt, \tag{109}$$

is a regular function in B; when evaluating its derivative we can differentiate under the integral, i.e.

$$\omega'(z) = \int_a^b \frac{\partial f(t, z)}{\partial z}\, dt. \tag{109_1}$$

According to Cauchy's formula we can write:

$$f(t, z) = \frac{1}{2\pi i} \int_l \frac{f(t, z')}{z' - z}\, dz',$$

where z lies inside B and t is any point in the interval $a \leqslant t \leqslant b$. Consequently:

$$\omega(z) = \int_a^b \left[\frac{1}{2\pi i} \int_l \frac{f(t, z')}{z' - z}\, dz\right] dt.$$

When integrating a continuous function we can change the order of integration [II, 78 and 97]:

$$\omega(z) = \frac{1}{2\pi i} \int_l \frac{\int_a^b f(t, z')\, dt}{z' - z}\, dz'.$$

This formula gives $\omega(z)$ in the form of Cauchy's integral and, consequently, $\omega(z)$ is a regular function in B; its derivative is determined by the formula [8]:

$$\omega'(z) = \frac{1}{2\pi i} \int_l \frac{\int_a^b f(t, z')\, dt}{(z' - z)^2} dz'.$$

Changing the order of integration we can write:

$$\omega'(z) = \int_a^b \left[\frac{1}{2\pi i} \int_l \frac{f(t, z')}{(z' - z)^2}\, dz'\right].$$

According to Cauchy's formula, the expression in the square brackets gives the derivative $\partial f(t, z)/\partial z$, the above formula is the same as formula (109_1) and the theorem is thus proved. Notice that we could have assumed that t varies not within a finite interval (a, b)

of the real axis but along any finite curve. This would not have altered the proof of the theorem. Notice also that in the above proof the integral

$$\int_a^b f(t, z')\,\mathrm{d}t,$$

which appears in the numerator of Cauchy's integral for $\omega(z)$, is a regular function of z on l. This is directly due to the fact that $f(t, z)$ is given as a continuous function of its two arguments [II, 80].

Let us now consider indefinite integrals. To prove the theorem for this case it is sufficient to add the condition that the integral (109) must be convergent. To be more specific, we will consider the integral in the infinite interval $(a, +\infty)$, but the proof is equally valid for other kinds of indefinite integrals.

THEOREM. *Let $f(t, z)$ be a continuous function of two variables, where z belongs to the closed domain B and $t \geqslant a$. Also $f(t, z)$ is a regular function in the closed domain B for any $t \geqslant a$ and the integral*

$$\int_a^\infty f(t, z)\,\mathrm{d}t$$

is uniformly convergent with domain to z in the closed domain B. Then

$$\omega(z) = \int_a^\infty f(t, z)\,\mathrm{d}t \tag{110}$$

is a regular function of z in B and

$$\omega'(z) = \int_a^\infty \frac{\partial f(t, z)}{\partial z}\,\mathrm{d}t.$$

Construct the sequence of functions

$$\omega_n(z) = \int_a^{a_n} f(t, z)\,\mathrm{d}t,$$

where a_n is any sequence of numbers greater than a, which tends to $(+\infty)$. It follows from the theorem we have just proved that $\omega_n(z)$ is a regular function in B and

$$\omega_n'(z) = \int_a^{a_n} \frac{\partial f(t, z)}{\partial z}\,\mathrm{d}t.$$

It follows from the condition of uniform convergence of the integral (110) that $\omega_n(z)$ tends uniformly to the function $\omega(z)$ as given by

the formula (110) and, according to Weierstrass's theorem, this function $\omega(z)$ is a regular function in B; also $\omega_n'(z) \to \omega'(z)$, i.e.

$$\lim_{n\to\infty} \int_a^{a_n} \frac{\partial f(t,z)}{\partial z}\,dt = \omega'(z)$$

when a_n tends to $(+\infty)$ in any manner. It follows from this that

$$\omega'(z) = \int_a^{\infty} \frac{\partial f(t,z)}{\partial z}\,dt,$$

where the integral on the right-hand side must have a meaning. The theorem is thus fully proved.

In this proof of the theorem we could have assumed that integration with respect to t takes place along a certain infinite contour C. Such an indefinite integral must be taken as the limit of integrals along finite contours which form part of the contour C. This theorem applies, word for word, to indefinite integrals in which the integrand $f(t, z)$ becomes infinite, for example, when t approaches a.

Notice, finally, that the following rule which gives the sufficient condition for the integral to be absolutely and uniformly convergent applies [II, 84]: if we integrate with respect to t along the real axis and if, when $t \geqslant a$ and z belongs to the closed domain B, the inequality $|f(t, z)| \leqslant \varphi(t)$ is valid, when the integral

$$\int_a^{\infty} \varphi(t)\,dt$$

is convergent, then the integral (110) will converge absolutely and uniformly. Absolute convergence is determined in exactly the same way as for the real $f(t, z)$.

71. Euler's integral of the second class. Consider the function given by Euler's integral of the second class:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\,dt, \tag{111}$$

where $t^{z-1} = e^{(z-1)\log t}$, and the real value of the logarithm of the positive number t is taken. We write the above integral as the sum of two integrals:

$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1}\,dt + \int_1^{\infty} e^{-t} t^{z-1}\,dt. \tag{112}$$

Consider, first of all, the second term on the right-hand side:

$$\omega(z) = \int_1^\infty e^{-t} t^{z-1}\, dt. \tag{113}$$

When $t \geqslant 1$ the integrand:

$$e^{-t} t^{z-1} = e^{-t+(z-1)\log t} \tag{114}$$

is a continuous function of t and z for any z and $t \geqslant 1$, and it is an integral function of z for any $t \geqslant 1$. Suppose that z belongs to a certain confined domain B of the z plane and that $z = x + iy$. In the closed domain B the abscissa has its greatest value which we denote by x_0. Bearing in mind that $\log t \geqslant 0$ when $t \geqslant 1$ and that the modulus of the exponential function $2^{\varphi i}$ with a purely imaginary index is unity, we obtain for z in B:

$$|e^{-t} t^{z-1}| = |e^{-t+(x-1)\log t + iy \log t}| \leqslant e^{-t+(x_0-1)\log t} = e^{-t} t^{x_0-1}.$$

The integral

$$\int_1^\infty e^{-t} t^{x_0-1}\, dt$$

is convergent [II, 82], and consequently the integral (113) is uniformly convergent with respect to z in B. Bearing in mind the second theorem of the previous paragraph and our complete freedom in choosing B we can say that $\omega(z)$, as given by formula (113), is an integral function which can be differentiated under the integral.

Consider now the first term of the formula (112):

$$\varphi(z) = \int_0^1 e^{-t} t^{z-1}\, dt. \tag{115}$$

In this case the continuity of the integrand (114) may be disrupted when $t = 0$, for $\log t$, when $t = 0$, becomes $(-\infty)$. As before, the modulus of the function (114) will be:

$$e^{-t} t^{x-1}.$$

If $x > 1$ when $t = 0$, then the continuity of the integrand will not be disrupted and, applying the first theorem of the preceding section, we can see that the function (115) will be regular when $x > 1$, i.e. when it lies to the right of the line $x = 1$. We shall now prove that it will also be regular when it lies to the right of the imaginary axis. In fact, take any finite domain B to the right of the imaginary axis. Let x_1 be the smallest abscissa of points of the closed domain B.

Since the closed domain lies to the right of the imaginary axis $x_1 > 0$. bearing in mind that $\log t \leqslant 0$, when $t \leqslant 1$, we obtain:

$$|e^{-t} t^{z-1}| \leqslant e^{-t} t^{x_1-1},$$

if z lies in B. But when $x_1 > 0$ the integral

$$\int_0^1 e^{-t} t^{x_1-1} dt$$

is convergent and, consequently, as before, the function (115) is regular to the right of the imaginary axis and can be differentiated under the integral. It follows from all that was said above that *formula* (111) *determines a regular function* $\Gamma(z)$ *to the right of the imaginary axis.*

We shall now analytically continue the function to the left of the imaginary axis and show that $\Gamma(z)$ is a meromorphic function with simple poles at the points

$$z = 0, -1, -2, \ldots \tag{116}$$

Owing to the fact that the second term on the right-hand side of the formula (112) is an integral function, we have to deal with the function (115).

Along the finite interval $0 \leqslant t \leqslant 1$ the function e^{-t} can be expanded into a uniformly convergent series:

$$e^{-t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!},$$

where, as always, we take $0! = 1$. Multiplying by t^{z-1} and integrating by parts along the interval (0, 1) we obtain:

$$\int_0^1 e^{-t} t^{z-1} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{t^{n+z}}{n+z}\right]_{t=0}^{t=1}.$$

We suppose that z lies to the right of the imaginary axis and, consequently, the real axis $(n + z)$ is positive and $t^{n+z} = 0$ when $t = 0$, i.e.

$$\int_0^1 e^{-t} t^{z-1} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{z+n}.$$

We thus obtain the following expression for $\Gamma(z)$ to the right of the imaginary axis:

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{z+n} + \int_1^{\infty} e^{-t} t^{z-1} dt. \tag{117}$$

Owing to the presence of $n!$ in the denominator, the infinite sum on the right-hand side will be absolutely and uniformly convergent in any bounded part of the plane, providing the first few terms, with poles at the points (116) are rejected. This sum will, consequently, give a meromorphic function with simple poles (116) where the residue at the pole $z = -n$ is $(-1)^n/n!$. The second term on the right-hand side is, as we have said already, an integral function. The right-hand side of formula (117) thus gives the analytic continuation of the function $\Gamma(z)$ in the whole plane of the complex variable z, which formula (111) only defines to the right of the imaginary axis; *$\Gamma(z)$ appears to be a meromorphic function with simple poles* (116) *and with a residue* $(-1)^n/n!$ *at the pole* $z = -n$. The values of $\Gamma(z)$, when the argument is a positive integer, can readily be obtained. Assume that $z = n + 1$, where n is a positive integer. We obtain in this case [II, 81]:

$$\Gamma(n+1) = \int_0^\infty e^{-t} t^n \, dt = n!$$

and

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1.$$

Hence values of $\Gamma(z)$, when z is a positive integer, give the factorials of integers:

$$\Gamma(1) = 1; \ \Gamma(n+1) = n! \qquad (n = 1, 2, 3, \ldots). \tag{118}$$

We shall now derive the fundamental properties of the function $\Gamma(z)$. Assuming that $z > 0$ and integrating by parts we have:

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z \, dt = \left[-e^{-t} t^z \right]_{t=0}^{t=\infty} + z \int_0^\infty e^{-t} t^{z-1} \, dt,$$

i.e.

$$\Gamma(z+1) = z\Gamma(z). \tag{119}$$

We have proved this equality only for the right side of the real axis. However, if two analytic functions coincide on a certain line, then they coincide everywhere [18] and, consequently, we can take it that the formula (119) is established for all values of z. Let n be a certain positive integer. Applying formula (119) several times we obtain a more general formula which is valid for all complex z's:

$$\Gamma(z+n) = (z+n-1)(z+n-2)\ldots(z+1)z\Gamma(z). \tag{120}$$

Let us now suppose that z lies within the section $(0, 1)$ of the real axis, and return to the fundamental formula (111); we replace the variable of the integration t by u, where $t = u^2$. We thus obtain the following results:

$$\Gamma(z) = 2\int_0^\infty e^{-u^2} u^{2z-1}\,du.$$

Substituting $1 - z$ for z we can write:

$$\Gamma(1-z) = 2\int_0^\infty e^{-v^2} v^{1-2z}\,dv.$$

Hence, by multiplying these equations together, we obtain:

$$\Gamma(z)\Gamma(1-z) = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\left(\frac{u}{v}\right)^{2z-1}du\,dv. \tag{121}$$

The integral on the right-hand side can be treated as a double integral in the plane (u, v), where the domain of integration is the first quadrant, i.e. that part of the plane where $u > 0$ and $v > 0$. We now introduce polar coordinates in place of u and v:

$$u = \varrho\cos\varphi; \quad v = \varrho\sin\varphi.$$

Formula (121) can be rewritten in the form:

$$\Gamma(z)\Gamma(1-z) = 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-\varrho^2}\cot^{2z-1}\varphi\varrho\,d\varrho\,d\varphi,$$

where we integrate with respect to ϱ from 0 to $+\infty$ and with respect to φ from 0 to $\pi/2$, i.e.

$$\Gamma(z)\Gamma(1-z) = 4\int_0^{\frac{\pi}{2}}\cot^{2z-1}\varphi\,d\varphi\int_0^\infty e^{-\varrho^2}\varrho\,d\varrho.$$

It can easily be seen that

$$\int_0^\infty e^{-\varrho^2}\varrho\,d\varrho = \frac{1}{2},$$

and consequently

$$\Gamma(z)\Gamma(1-z) = 2\int_0^{\frac{\pi}{2}}\cot^{2z-1}\varphi\,d\varphi.$$

We now replace φ by a new variable, according to the formula

$$\varphi = \operatorname{arc\,cot} \sqrt{x}; \quad d\varphi = \frac{-dx}{2\sqrt{x}(1+x)}$$

The above result can, in this case, be rewritten as follows:

$$\Gamma(z)\,\Gamma(1-z) = \int_0^\infty \frac{x^{z-1}}{1+x}\,dx.$$

But, as we know from [62], the integral on the right-hand side is equal to $\pi/\sin \pi z$, and consequently we obtain the following formula:

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{122}$$

We have proved this formula only for the section (0, 1) of the rea axis. But, as before, using the process of analytic continuation, we can show that it is valid for all z's.

Formula (120) enables us to transform the evaluation of $\Gamma(z)$ for all real values of z to values of $\Gamma(z)$ on the section (0, 1). Formula (122) makes it possible to transfer the section (0, 1) to the section (0, 1/2). Assuming $z = 1/2$ in the formula (122) we obtain:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\,dt = \sqrt{\pi}. \tag{123}$$

72. Euler's integral of the first class. Euler's integral of the first class is an integral of the following form:

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx. \tag{124}$$

As in the case of the integral (111) we suppose that the real parts of p and q are greater than zero and also that:

$$x^{p-1}(1-x)^{q-1} = e^{(p-1)\log x + (q-1)\log(1-x)},$$

where real values of the logarithms are taken.

Replacing x by a new variable t, according to the formula $t = 1 - x$ we obtain in place of (124):

$$B(p, q) = \int_0^1 t^{q-1}(1-t)^{p-1}\,dt,$$

i.e.

$$B(p, q) = B(q, p). \tag{125}$$

We will introduce yet another formula which explains the fundamental property of the function $B(p, q)$. Integrating by parts we can write:

$$\int_0^1 x^{p-1}(1-x)^q \,dx = \left[\frac{x^p(1-x)^q}{p}\right]_{x=0}^{x=1} + \frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\,dx.$$

As a result of the assumptions we made with regard to p and q we can say that the term outside the integral is zero and that the above formula gives us the following property of the function $B(p, q)$:

$$B(p, q+1) = \frac{q}{p} B(p+1, q). \tag{126}$$

We will now establish the connection between the function $B(p, q)$ and the function (111). Applying the same transformation as in the previous section we can represent the product $\Gamma(p)\,\Gamma(q)$ in the form:

$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)} u^{2p-1} v^{2q-1}\,du\,dv$$

and introducing polar coordinates we obtain

$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty e^{-\varrho^2}\varrho^{2(p+q)-1}\,d\varrho \int_0^{\frac{\pi}{2}} \cos^{2p-1}\varphi \sin^{2q-1}\varphi\,d\varphi.$$

Replacing ϱ by a new variable t according to the formula $\varrho = \sqrt{t}$, we can write

$$\int_0^\infty e^{-\varrho^2}\varrho^{2(p+q)-1}\,d\varrho = \frac{1}{2}\int_0^\infty e^{-t} t^{p+q-1}\,dt = \frac{1}{2}\Gamma(p+q),$$

and consequently:

$$\Gamma(p)\,\Gamma(q) = 2\Gamma(p+q)\int_0^{\frac{\pi}{2}} \cos^{2p-1}\varphi \sin^{2q-1}\varphi\,d\varphi.$$

If φ is now replaced by a new variable of integration x where $x = \cos^2\varphi$, then the above relationship gives

$$\Gamma(p)\,\Gamma(q) = \Gamma(p+q)\int_0^1 x^{p-1}(1-x)^{q-1}\,dx,$$

and this gives a formula where $B(p, q)$ is expressed as a function of $\Gamma(z)$:

$$B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}. \tag{127}$$

73. The infinite product of the function $[\Gamma(z)]^{-1}$. Let us return to the fundamental definition of the function $\Gamma(z)$ as given by the formula (111) and assume, for the sake of simplicity, that $z > 0$. The factor e^{-t} is, as we already know, the limit as $n \to \infty$ of the expression [I, 38]

$$e^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n.$$

Replacing the interval $(0, +\infty)$ by the finite section $(0, n)$ we obtain the integral:

$$P_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt. \tag{128}$$

It is to be expected that as n increases indefinitely the integral will tend to an integral which appears on the right-hand side of formula (111). At the end of this section we shall prove this theorem in detail but for the moment we shall use its results.

Replacing t by a new variable τ where $t = n\tau$. we can rewrite (128) in the form:

$$P_n(z) = n^z \int_0^1 (1 - \tau)^n \tau^{z-1}\, d\tau. \tag{129}$$

We now suppose that n tends to $+\infty$ by taking positive integral values. Integrating by parts we obtain:

$$\int_0^1 (1 - \tau)^n \tau^{z-1}\, d\tau = \left[\frac{1}{z}\, \tau^z\, (1 - \tau)^n\right]_{\tau=0}^{\tau=1} + \frac{n}{z} \int_0^1 (1 - \tau)^{n-1} \tau^z\, d\tau,$$

or, bearing in mind that the term outside the integral vanishes $(z > 0)$:

$$\int_0^1 (1 - \tau)^n \tau^{z-1}\, d\tau = \frac{n}{z} \int_0^1 (1 - \tau)^{n-1} \tau^z\, d\tau = \frac{n}{z(z+1)} \int_0^1 (1 - \tau)^{n-1}\, d\tau^{z+1}.$$

Similarly, continuing to integrate by parts we obtain

$$\int_0^1 (1 - \tau)^n \tau^{z-1}\, d\tau = \frac{n(n-1)}{z(z+1)} \int_0^1 (1 - \tau)^{n-2} \tau^{z+1}\, d\tau,$$

and, in general, we obtain the following expression for the integral (129)

$$n^z \int_0^1 (1 - \tau)^n \tau^{z-1}\, d\tau = \frac{1 \cdot 2 \ldots n}{z(z+1) \ldots (z+n)}\, n^z.$$

As n increases indefinitely the limit of this expression will be $\Gamma(z)$, i.e.

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\ldots n}{z(z+1)\ldots(z+n)}\, n^z, \tag{130}$$

or

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z(z+1)\ldots(z+n)}{1\cdot 2\ldots n}\, n^{-z} \quad (n^{-z} = e^{-z\log n}). \tag{131}$$

To alter the above expression slightly we multiply and divide it by $e^{z(1+1/2+\ldots+1/n)}$. Having done this we can rewrite formula (131) as follows:

$$\frac{1}{\Gamma(z)} = \lim \left\{ e^{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)z}\, z\,\frac{z+1}{1}\times \right.$$

$$\left. \times \frac{z+2}{2}\ldots\frac{z+n}{n}\, e^{-z\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)} \right\}$$

or

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \left\{ e^{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)z}\, z \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}} \right\}. \tag{132}$$

As the integer n increases indefinitely the above finite product becomes an infinite product

$$\prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}. \tag{133}$$

The above infinite product is constructed in exactly the same way as Weierstrass's infinite product [69] with in this case $a_k = -k$ and the series

$$\sum_{k=1}^{\infty} \frac{1}{k^m}$$

converging when $m = 2$. Hence on the right side of (132) the last factor tends to a definite limit (133). We will now show that the variable

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n \tag{134}$$

also tends to a definite limit. To prove this it is sufficient to show that the variable

$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{(n-1)} - \log n = u_n - \frac{1}{n} \tag{135}$$

has a definite limit. The variable u_n will obviously have the samt limit. Consider the branch of the hyperbola $y = 1/x$ in the firse quadrant. The number $1/k$ will be the ordinate of this branch when

$x = k$. Evidently $\log n$ is equal to the surface area bounded by our hyperbola, the OX axis and the ordinates $x = 1$ and $x = n$ and the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}$$

represents the sum of the surface areas of outgoing rectangles with unit bases on the X axis (Fig. 64). It follows directly that the difference (135) increases together with n. On the other hand this difference must be less than the difference of the surface areas of the outgoing and incoming rectangles and this latter difference is equal to $(1 - 1/n)$. Hence v_n will be an increasing bounded variable and, consequently, it has a limit.

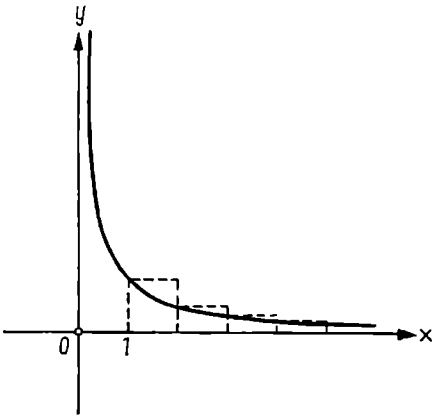

FIG. 64

This limit C is usually known as Euler's constant. It is equal, with an accuracy to the seventh decimal place, to

$$C = 0.5772157\ldots \qquad (136)$$

Formula (132) finally gives us the limit

$$\frac{1}{\Gamma(z)} = e^{Cz} z \prod_{k=1}^{\infty} \left(1 + \frac{k}{z}\right) e^{-\frac{k}{z}}. \qquad (137)$$

On the right-hand side of the above formula is an integral function of z with simple zeros $z = 0, -1, -2, \ldots$ We established formula (137) only for the positive part of the real axis. As a result of the fundamental principle of analytic continuation we can say that it is valid for all values of z and we thus see that *the function* $1/\Gamma(z)$ *is an integral function and formula* (137) *represents it in the form of an infinite product.*

We have thus proved that $1/\Gamma(z)$ is an integral function and it follows directly that the function $\Gamma(z)$ does not vanish anywhere, i.e. it has no zeros at all.

Using the infinite product (137) we can easily prove the formula (122) in [71]. In fact, formula (137) gives us directly

$$\frac{1}{\Gamma(z)\,\Gamma(-z)} = -z^2 \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right),$$

or, from (93) in [67]:

$$\frac{1}{\Gamma(z)\,\Gamma(-z)} = -\frac{z \sin \pi z}{\pi}.$$

Also formula (119) gives us the following if in it we replace z by $(-z)$:

$$\Gamma(-z) = -\frac{\Gamma(1-z)}{z}.$$

Substituting this expression for $\Gamma(-z)$ in the above formula we obtain formula (122):

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

We now have to show that the integral (128) tends to the integral (111) as n increases indefinitely; it is sufficient if we prove this when $z > 0$. Let us first of all, find upper and lower bounds for the difference

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n.$$

It can readily be shown that the function

$$-e^v \left(1 - \frac{v}{n}\right)^n$$

is a primitive for the function

$$e^v \left(1 - \frac{v}{n}\right)^{n-1} \frac{v}{n},$$

and consequently:

$$1 - e^t \left(1 - \frac{t}{n}\right)^n = \int_0^t e^v \left(1 - \frac{v}{n}\right)^{n-1} \frac{v}{n}\,dv.$$

If $0 < t < n$, then the integrand is positive and therefore the same may be said about the left-hand side. Replacing under the integral e^v by e^t and $(1 - v/n)^{n-1}$ by unity we obtain:

$$0 < 1 - e^t \left(1 - \frac{t}{n}\right)^n < e^t \frac{t^2}{2n}$$

or

$$0 < e^{-t} - \left(1 - \frac{t}{n}\right)^n < \frac{t^2}{2n}. \tag{138}$$

Construct the difference:

$$\Gamma(z) - P_n(z) = \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\,dt + \int_n^\infty e^{-t} t^{z-1}\,dt. \tag{139}$$

As n increases indefinitely the second integral on the right-hand side tends to zero because the integral

$$\int_0^\infty e^{-t} t^{z-1}\,dt$$

converges. It remains to be shown that the first integral will also tend to zero as $n \to \infty$. Fix $n = n_0$ so that

$$\int_{n_0}^{\infty} e^{-t} t^{z-1}\, dt < \frac{\varepsilon}{2},$$

where ε is any arbitrarily chosen small whole number.

We can write:

$$\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\, dt =$$

$$= \int_0^{n_0} \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\, dt + \int_{n_0}^{n} \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\, dt$$

and from (138):

$$0 < \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\, dt < \frac{1}{2n} \int_0^{n_0} t^{z+1}\, dt + \int_{n_0}^{n} e^{-t} t^{z-1}\, dt,$$

where in the second integral on the right-hand side we have replaced the difference by a single minuend. The integrand in this integral is positive so that, extending the interval of integration we obtain:

$$0 < \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\, dt < \frac{1}{2n} \int_0^{n_0} t^{z+1}\, dt + \int_{n_0}^{\infty} e^{-t} t^{z-1}\, dt.$$

When n is large the first term is less than $\varepsilon/2$, so that for all sufficiently large n's:

$$0 < \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\, dt < \varepsilon,$$

i.e. owing to the arbitrarily small ε in formula (139) the first term will also tend to zero, i.e. in fact:

$$\Gamma(z) = \lim_{n \to \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt. \tag{140}$$

Notice some other consequences of the formulae we have proved. Taking the logarithmic derivative of both sides of formula (137) we obtain:

$$\frac{d}{dz} \log \Gamma(z) = -C - \frac{1}{z} + z \sum_{k=1}^{\infty} \frac{1}{k(z+k)}. \tag{141}$$

Differentiating both sides:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\Gamma(z)=\sum_{k=0}^{\infty}\frac{1}{(z+k)^2}. \tag{142}$$

Using formula (130) we shall prove the so called doubling formula:

$$2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)=\sqrt{\pi}\,\Gamma(2z). \tag{143}$$

Expressing the functions $\Gamma(z)$ and $\Gamma(z+1/2)$ by formula (130) and the function $\Gamma(2z)$ by the formula, derived from formula (130) by replacing n by $2n$, we obtain:

$$\frac{2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)}=$$

$$=\lim_{n\to\infty}\frac{2^{2z-1}(n!)^2\,2z\,(2z+1)\ldots(2z+2n)}{2n!\,z\left(z+\frac{1}{2}\right)(z+1)\left(z+\frac{3}{2}\right)\ldots(z+n)\left(z+n+\frac{1}{2}\right)}\cdot\frac{n^{2z+\frac{1}{2}}}{(2n)^{2z}}.$$

or

$$\frac{2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)}=\lim_{n\to\infty}\frac{2^{n-1}(n!)^2}{2n!\sqrt{n}}\lim_{n\to\infty}\frac{n}{2z+2n+1}. \tag{144}$$

now

$$\lim_{n\to\infty}\frac{n}{2z+2n+1}=\frac{1}{2},$$

and we can see that the left-hand side of formula (144) is independent of z. Assuming $z=1/2$ we obtain:

$$\frac{2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)}=\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},$$

and from this follows formula (143). In exactly the same way as above the following, more general formula, can be proved:

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right)=$$

$$=(2\pi)^{\frac{1}{2}(m-1)}\,m^{\frac{1}{2}-mz}\,\Gamma(mz). \tag{145}$$

74. The representation of $\Gamma(z)$ by a contour integral. We will now give an expression for $\Gamma(z)$ in the form of a contour integral which holds for all values of z. If z lies to the right of the imaginary axis then

$$\Gamma(z)=\int^{\infty}\mathrm{e}^{-t}\,t^{z-1}\,\mathrm{d}t. \tag{146}$$

Consider the integrand

$$e^{-t}t^{z-1} = e^{-t}\,e^{(z-1)\log t} \tag{147}$$

which is a function of the complex variable t. This function has a branch-point at $t = 0$. Make a cut in the t-plane along the positive part of the real axis t. In the cut plane function (147) will be single-valued; we have assumed that $\log t$ is real on the upper edge of the cut, i.e. we have assumed that $\arg t = 0$ on this edge. Instead of integrating along the upper edge of the real axis, consider a new contour of integration l, illustrated in Fig. 65. This contour starts at $+\infty$, surrounds the origin and returns to $+\infty$. We know from Cauchy's theorem that we can, without changing the value of the integral

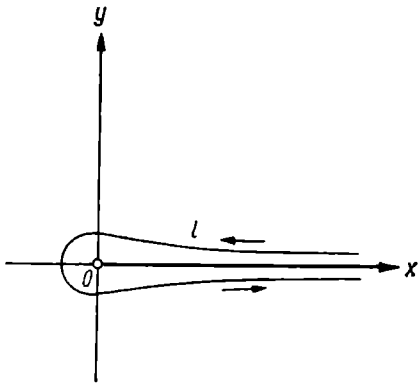

FIG. 65

$$\int_l e^{-t}t^{z-1}\,dt \tag{148}$$

$$(t^{z-1} = e^{(z-1)\log t}),$$

deform the contour l in any arbitrary manner providing the singularity $t=0$ is not altered and both ends of the contour remain at $+\infty$. We shall now explain the connection between the integral (148) and the function $\Gamma(z)$; we will suppose that z lies to the right of the imaginary axis. By deforming the contour l we can obtain a path of integration which consists of the following three parts: (1) the section $(+\infty, \varepsilon)$ of the upper edge of the cut; (2) the circle λ_ε, centre the origin and radius ε and (3) the section $(\varepsilon, +\infty)$ of the lower edge of the cut. On the upper edge, $\log t$ of the integrand (147) is real. In transit to the lower edge $\log t$ acquires the term $2\pi i$, for on the lower edge the integrand will be:

$$e^{(z-1)2\pi i}\,e^{-t+(z-1)\log t},$$

where $\log t$ is real, as before. We thus have:

$$\int_l e^{-t}t^{z-t}\,dt = \int_\infty^\varepsilon e^{-t}t^{z-1}\,dt + e^{(z-1)2\pi i}\int_\varepsilon^\infty e^{-t}t^{z-1}\,dt + \int_{\lambda_\varepsilon} e^{-t}t^{z-1}\,dt, \tag{149}$$

where ε is a certain given positive number. We will show that as $\varepsilon \to 0$ the integral over the circle λ_ε tends to zero. In fact on this

circle the modulus of the factor e^{-t} is bounded independently of z and the bound of the factor t^{z-1} can be calculated

$$|t^{z-1}| = e^{(x-1)\log|t| - y\arg t} = \varepsilon^{x-1} e^{-y\arg t},$$

i.e. it will be infinitely small if $x > 1$, or it will tend to infinity of the order $1/\varepsilon^{1-x}$. Bearing in mind that we are given that $x > 0$ and that the length of the path of integration is $2\pi\varepsilon$ we can readily see that the above integral does, in fact, tend to zero. Hence formula (149) gives us the limit

$$(e^{z2\pi i} - 1) \int_0^\infty e^{-t} t^{z-1}\, dt = \int_l e^{-t} t^{z-1}\, dt$$

or, remembering the definition of $\Gamma(z)$:

$$\int_l e^{-t} t^{z-1}\, dt = (e^{z2\pi i} - 1)\,\Gamma(z). \tag{150}$$

The above formula can also be written as follows:

$$\Gamma(z) = \frac{1}{e^{z2\pi i} - 1} \cdot \int_l e^{-t} t^{z-1}\, dt. \tag{151}$$

The contour l does not pass through the origin and therefore we do not have to consider only those values of z which lie to the right of the imaginary axis. In the same way as for integral (113) in [71] we can show that integral (148) represents an integral function of z. We proved formula (150) only for values of z which lie to the right of the imaginary axis, but as a result of analytic continuation it will also hold in the whole z plane. Formula (151) represents a meromorphic function in the form of a quotient of two integral functions. The denominator $(e^{z2\pi i} - 1)$ vanishes for all positive and negative integral values of z. Whole negative values of z and $z = 0$ give the polarity of $\Gamma(z)$. If z is a positive integral number then the integrand (147) will be a single-valued and regular function of t in the whole plane (i.e. it is an integral function of t), and, according to Cauchy's theorem, its integral along the closed contour l will be equal to zero, i.e. when z is a whole positive number the numerator and denominator on the right-hand side of formula (151) vanish and therefore these values will not be poles of the function $\Gamma(z)$.

Let us replace z by $(1 - z)$ in formula (150):

$$\int_l e^{-t} t^{-z}\, dt = (e^{-z2\pi i} - 1)\,\Gamma(1 - z). \tag{152}$$

Replace t by a new variable of integration τ, assuming that $t = e^{\pi i}\,\tau = -\tau$:

$$\int_l e^{-t}\,t^{-z}\,dt = -\int_{l'} e^{\tau}\,(e^{\pi i}\,\tau)^{-z}\,d\tau = -e^{-z\pi i}\int_{l'} e^{\tau}\,\tau^{-z}\,d\tau\,, \qquad (153)$$

where l' is the contour illustrated in Fig. 66. The τ-plane is obtained from the t-plane by rotation about the origin through an angle $(-\pi)$. The cut along the positive part of the real axis in the t-plane is transformed into a cut along the negative part of the real axis in the τ-plane, where the lower edge of the new cut corresponds to the upper edge of the former cut. On this lower edge of the new cut we take $\arg(e^{\pi i}\,\tau) = 0$, i.e. $\arg\tau = -\pi$. Substituting the expression (153) in formula (152) and multiplying both sides of the equation by $(-e^{\pi z i})$, we have:

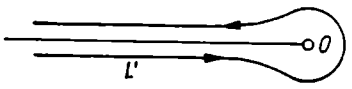

FIG. 66

$$\int_{l'} e^{\tau}\,\tau^{-z}\,d\tau = (e^{\pi z i} - e^{-\pi z i})\,\Gamma(1-z)$$

or

$$\int_{l'} e^{\tau}\,\tau^{-z}\,d\tau = 2i\sin\pi z\,\Gamma(1-z)\,,$$

whence, using formula (122) we obtain the expression $\Gamma(z)^{-1}$ in the form of the contour integral:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{l'} e^{\tau}\,\tau^{-z}\,d\tau\,. \qquad (154)$$

75. Stirling's formula. In this section we shall give an approximate expression for $\log\Gamma(z)$ when z assumes large positive values. As a preliminary we shall prove a formula which establishes the connection between the sum of equidistant values of a certain function and the integral of this function.

Let $f(x)$ be a function which is determined when $x \geqslant 0$ and which has a continuous derivative. Denoting positive integral numbers by n and k, where $k < n$ we can write:

$$f(n) - f(k) = \int_k^n f'(x)\,dx$$

and summing with respect to k from $k = 0$ to $k = n$ we have:

$$(n+1)\,f(n) - \sum_{k=0}^{n} f(k) = \sum_{k=0}^{n}\int_k^n f'(x)\,dx. \qquad (155)$$

In the expanded form the right-hand side can be written as follows:

$$\sum_{k=0}^{n} \int_{k}^{n} f'(x)\,dx = \int_{0}^{n} f'(x)\,dx + \int_{1}^{n} f'(x)\,dx + \int_{2}^{n} f'(x)\,dx + \ldots + \int_{n-1}^{n} f'(x)\,dx + \int_{n}^{n} f'(x)\,dx,$$

where the last term on the right-hand side is obviously equal to zero. If m is a certain positive integral number less than n, then when integrating the above sum in the interval $(m, m+1)$ it will be repeated $(m+1)$ times and we can write formula (155) in the form:

$$(n+1)f(n) - \sum_{k=0}^{n} f(k) = \int_{0}^{n} \{[x]+1\} f'(x)\,dx, \tag{156}$$

where by $[x]$ we denote the integral part of the positive number x, so that $[x] = m$ in the interval $(m, m+1)$ and $[m] = m$. Let us now introduce the function

$$P(x) = [x] - x,$$

which represents minus the fractional part of the number x. If we add unity to x then $[x]$ and x increase by unity and $P(x)$ remains unaltered, i.e. $P(x)$ has a period of unity. The function $P(x)$ is determined when $x \geqslant 0$, but from the law of periodicity when the period is unity, we can naturally extend its definition to include also negative values of x. We know from [II, 142] that the value of the integral of $P(x)$ along any interval of unit length does not depend on the position of this interval. This value gives the so called *mean-value* of our periodic function. In the interval $(0, 1)$ we have $P(x) = -x$ and the mean-value of $P(x)$ is:

$$\int_{0}^{1} P(x)\,dx = -\int_{0}^{1} x\,dx = -\frac{1}{2}.$$

Let us construct a new function with a unit period

$$P_1(x) = [x] - x + \frac{1}{2}, \tag{157}$$

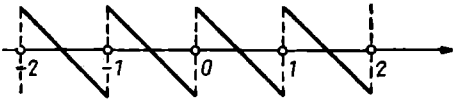

FIG. 67

the mean-value of which is zero. The graph of $P_1(x)$ is illustrated in Fig. 67. Let us replace $[x]$ in the integrand of formula (156) by its expression from formula (157):

$$(n+1)f(n) - \sum_{k=0}^{n} f(k) = \int_{0}^{n} \left\{x + \frac{1}{2} + P_1(x)\right\} f'(x)\,dx. \tag{158}$$

We evidently have:

$$\int_0^n \frac{1}{2} f'(x)\,dx = \frac{1}{2}\left[f(n) - f(0)\right]$$

and, integrating by parts, we have:

$$\int_0^n x f'(x)\,dx = n f(n) - \int_0^n f(x)\,dx.$$

Substituting this in formula (158) we obtain:

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac{1}{2}\left[f(n) + f(0)\right] - \int_0^n P_1(x) f'(x)\,dx, \tag{159}$$

which establishes the connection between the sum of the equidistant values $f(k)$ of the function $f(x)$ and the integral of this function.

Let us choose the function $f(x)$ as follows:

$$f(x) = \log(z + x),$$

where z is a certain positive number and the values of the logarithm are real. Substituting into formula (159) we obtain:

$$\sum_{k=0}^{n}{}' \log(z+k) = \left(z + n + \frac{1}{2}\right)\log(z+n) - \left(z - \frac{1}{2}\right)\log z - n - \int_0^1 \frac{P_1(x)}{z+x}\,dx.$$

Put $z = 1$ in this formula and subtract term by term the new formula thus obtained from the one above. Also subtract $(z-1)\log n$ from both sides of this equation. We then have:

$$\sum_{k=0}^{n} \log\frac{z+k}{1+k} - (z-1)\log n = (z-1)\log\left(1 + \frac{z}{n}\right) + \frac{1}{2}\log\left(1 + \frac{z-1}{1+n}\right) +$$

$$+ (1+n)\log\left(1 + \frac{z-1}{1+n}\right) - \left(z - \frac{1}{2}\right)\log z - \int_0^n \frac{P_1(x)}{z+x}\,dx + \int_0^n \frac{P_1(x)}{1+x}\,dx.$$

As n tends to infinity the first two terms on the right-hand side tend to zero and the third term has the limit [I, 38]:

$$\lim_{n\to\infty} (1+n)\log\left(1 + \frac{z-1}{1+n}\right) = \lim_{n\to\infty} \log\left(1 + \frac{z-1}{1+n}\right)^{1+n} = \log e^{z-1} = z - 1.$$

We can therefore write:

$$\lim_{n\to\infty} \log\left[\frac{z(z+1)\dots(z+n)}{1\cdot 2\dots n}\, n^{-z}\cdot\frac{n}{n+1}\right] =$$

$$= (z-1) + \left(\frac{1}{2} - z\right)\log z - \int_0^\infty \frac{P_1(x)}{z+x}\,dx + \int_0^\infty \frac{P_1(x)}{1+x}\,dx$$

or [73]:

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + 1 - \int_0^\infty \frac{P_1(x)}{1+x}\,dx + \int_0^\infty \frac{P_1(x)}{z+x}\,dx. \tag{160}$$

Now consider the function:

$$Q(x) = \int_0^x P_1(x)\,dx. \tag{161}$$

Owing to the fact that the mean-value of $P_1(x)$ is zero, the function $Q(x)$ is a continuous periodic function with a unit period and $Q(0) = 0$. The modulus of this function is thus bounded. If $0 < x < 1$ then $[x] = 0$ and formula (157) gives:

$$Q(x) = \int_0^x \left(\frac{1}{2} - x\right) dx = \frac{x}{2} - \frac{x^2}{2} \qquad (0 \leqslant x \leqslant 1).$$

It follows directly from this that

$$0 < Q(x) < \frac{1}{8}. \tag{162}$$

Integrating by parts we obtain:

$$\int_0^\infty \frac{P_1(x)}{z+x}\,dx = \int_0^\infty \frac{Q'(x)}{z+x}\,dx =$$

$$= \int_0^\infty \frac{Q(x)}{(z+x)^2}\,dx + \left[\frac{Q(x)}{z+x}\right]_{x=0}^{x=\infty} = \int_0^\infty \frac{Q(x)}{(z+x)^2}\,dx, \tag{163}$$

where the term outside the integral vanishes when $x = \infty$.

These considerations show us, by the way, that the above integrals have a meaning [cf. II, 83]. Replacing x by a new variable of integration t where $x = zt$ we obtain:

$$\int_0^\infty \frac{P_1(x)}{z+x}\,dx = \frac{1}{z}\int_0^\infty \frac{Q(zt)}{(1+t)^2}\,dt. \tag{164}$$

Also from (162):

$$\left|\int_0^\infty \frac{P_1(x)}{z+x}\,dx\right| < \frac{1}{z}\int_0^\infty \frac{Q(zt)}{(1+t)^2}\,dt < \frac{1}{8z}\int_0^\infty \frac{dt}{(1+t)^2} = \frac{1}{8z}.$$

This shows that the integral (164) tends to zero as the positive number z increases indefinitely; moreover the product of this integral and z is bounded. This is usually written as follows:

$$\int_0^\infty \frac{P_1(x)}{z+x}\,dx = O\left(\frac{1}{z}\right).$$

Formula (160) can also be written in the form:

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + C + O\left(\frac{1}{z}\right), \tag{165}$$

or

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + C + \omega(z), \tag{166}$$

where

$$|\omega(z)| < \frac{1}{8z}, \tag{167}$$

and C denotes the constant:

$$C = 1 - \int_0^\infty \frac{P_1(x)}{1+x}\, dx.$$

Let us now determine the value of this constant. For this purpose we shall use the so called *Wallis formula* which expresses $\pi/2$ as the limit of a certain fraction:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^2 \cdot 4^2 \ldots (2n-2)^2 \cdot 2n}{1^2 \cdot 3^2 \ldots (2n-1)^2}. \tag{168}$$

We will prove this theorem at the end of this section so as not to interrupt the continuity of presentation.

Formula (168) can be rewritten in the form:

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty} \frac{2^{2n-\frac{1}{2}} (n!)^2 n^{-\frac{1}{2}}}{(2n)!},$$

from where, by taking logarithms and remembering that $m! = \Gamma(m+1)$ for whole positive values of m we obtain:

$$\lim_{n\to\infty} \left[2 \log \Gamma(n+1) - \log \Gamma(2n+1) + \left(2n - \frac{1}{2}\right) \log 2 - \frac{1}{2} \log n\right] = \log \sqrt{\frac{\pi}{2}}.$$

Using formula (165) we can rewrite this equation in the form:

$$\lim_{n\to\infty} \left[(2n+1) \log (n+1) - \left(2n + \frac{1}{2}\right) \log (2n+1) - 1 + C + \right.$$
$$\left. + \left(2n - \frac{1}{2}\right) \log 2 - \frac{1}{2} \log n\right] = \log \sqrt{\frac{\pi}{2}},$$

or

$$\lim_{n\to\infty} \left\{2n [\log (n+1) + \log 2 - \log (2n+1)] + \right.$$
$$\left. + \left[\log (n+1) - \frac{1}{2} \log (2n+1) - \frac{1}{2} \log n\right] + C - 1 - \frac{1}{2} \log 2\right\} = \log \sqrt{\frac{\pi}{2}},$$

or

$$\lim_{n\to\infty} \left\{\log \left(1 + \frac{1}{2n+1}\right)^{2n} + \frac{1}{2} \log \frac{(n+1)^2}{n(2n+1)} + C - 1 - \frac{1}{2} \log 2\right\} = \log \sqrt{\frac{\pi}{2}}.$$

The first term in the figured brackets tends to $\log e (= 1)$ and the second to $(-\ 1/2 \log 2)$ so that we obtain the equation

$$1 - \frac{1}{2}\log 2 + C - 1 - \frac{1}{2}\log 2 = \log \sqrt{\frac{\pi}{2}},$$

whence $C = \log \sqrt{2\pi}$. Substituting this into formula (165) we obtain Stirling's formula:

$$\log \Gamma(z) = \log \sqrt{2\pi} + \left(z - \frac{1}{2}\right)\log z - z + w(z), \tag{169}$$

or, rejecting the logarithms:

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}}\, e^{-z}\, \varepsilon(z), \tag{170}$$

where the factor $\varepsilon(z) = e^{\omega(z)}$ tends to zero as z increases indefinitely. If z is equal to a whole positive number m, then multiplying both sides of the equation by m we obtain:

$$m! = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \varepsilon_m, \tag{171}$$

where $\varepsilon_m \to 1$ as m increases.

We already know that the function $\Gamma(z)$ has no zeros and that $\log \Gamma(z)$ is a regular single-valued function in the z plane cut along the negative part of the real axis. If this cut were to be isolated by a sector, apex the origin, which can be as small as we please, then formula (169) can be applied to the remainder of the plane. The proof is exactly the same as the proof of formula (169) when $z > 0$. In this case we should take those values of $\log z$ and $\log \Gamma(z)$ in the above cut plane which are real when $z > 0$.

Wallis's Formula. We shall now prove Wallis's formula which we used above. We obtained the following formula in [I, 100]:

$$\int_0^{\frac{\pi}{2}} \sin^{2k} x\, dx = \frac{(2k-1)(2k-3)\ldots 3\cdot 1}{2k(2k-2)\ldots 4\cdot 2}\cdot\frac{\pi}{2},$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\, dx = \frac{2k(2k-2)\ldots 4\cdot 2}{(2k+1)(2k-1)\ldots 5\cdot 3}.$$

Bearing in mind that as n increases the value of $\sin^n x$ decreases we can write:

$$\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\, dx < \int_0^{\frac{\pi}{2}} \sin^{2k} x\, dx < \int_0^{\frac{\pi}{2}} \sin^{2k-1} x\, dx,$$

i.e.

$$\frac{2k(2k-2)\ldots 4\cdot 2}{(2k+1)\cdot(2k-1)\ldots 5\cdot 3} < \frac{(2k-1)\cdot(2k-3)\ldots 3\cdot 1}{2k\cdot(2k-2)\ldots 4\cdot 2}\frac{\pi}{2} < \frac{(2k-2)(2k-4)\ldots 4\cdot 2}{(2k-1)(2k-3)\ldots 5\cdot 3},$$

whence, replacing k by n:

$$\frac{\pi}{2} > \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1},$$

$$\frac{\pi}{2} < \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1}.$$

Putting

$$P_n = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1},$$

we can write

$$P_n \cdot \frac{2n}{2n+1} < \frac{\pi}{2} < P_n.$$

As $n \to +\infty$ the fraction on the left-hand side tends to unity and consequently

$$\lim P_n = \frac{\pi}{2},$$

which gives Wallis's formula.

76. Euler's summation formula. Let us return to formula (159). Integrating the last integral on the right-hand side by parts several times over, we can write it in an expanded form. By using formula (161) we determined the function $Q(x)$ with a unit period which was such that $Q'(x) = P_1(x)$. If we add a constant term to $Q(x)$ we can make the mean-value of this function, and of the function $P_1(x)$, zero. Changing the sign of the function obtained we get the function $P_2(x)$ with a unit period and mean-value zero, so that $P_2'(x) = -P_1(x)$. We know that $P_1(x) = -x + 1/2$ where $0 < x < 1$, so that

$$P_2(x) = \frac{x^2}{2} - \frac{x}{2} + C \qquad (0 \leqslant x < 1),$$

and determining C from the condition

$$\int_0^1 P_2(x)\,dx = 0,$$

we finally obtain

$$P_2(x) = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12} \qquad (0 \leqslant x < 1).$$

In this case $P_2(0) = P_2(1) = 1/12$, and by periodic repetition $P_2(x)$ gives a continuous periodic function; hence the above formula holds in the whole closed interval $0 \leqslant x \leqslant 1$. Furthermore we can similarly determine the function $P_3(x)$ with a unit period and mean-value zero, so that $P_3'(x) = P_2(x)$. We obtain the following expression for this function in the main interval (0, 1):

$$P_3(x) = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}.$$

Continuing this further we can construct the functions $P_n(x)$ with unit periods and mean-values equal to zero so that

$$P'_{2m}(x) = -P_{2m-1}(x); \quad P'_{2m+1}(x) = P_{2m}(x). \tag{172}$$

All these periodic functions can be expanded into Fourier series, in which the constant terms will be zero, since the mean-values of the functions are zero. Figure 67 shows that $P_1(x)$ is an odd function. Determining its Fourier coefficients by the usual Fourier rule we obtain:

$$P_1(x) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}.$$

Similarly we have for the next function $P_2(x)$:

$$P_2(x) = \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{2n^2\pi^2}.$$

Notice that the above series can be obtained directly from the series $P_1(x)$ by integration and change of sign; this corresponds to the relationship $P'_2(x) = -P_1(x)$. The series for $P_2(x)$ is uniformly convergent for all real values of x. Bearing in mind the relationship (172) we can also obtain the Fourier series for the succeeding functions $P_n(x)$ by successive integration; the constant terms of the Fourier series must be assumed to be zero.

We thus have:

$$P_{2m}(x) = \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{2^{2m-1} n^{2m} \pi^{2m}}; \quad P_{2m+1}(x) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{2^{2m} n^{2m+1} \pi^{2m+1}}. \tag{173}$$

These formulae, by the way, give us:

$$P_{2m}(0) = \frac{1}{2^{2m-1} \pi^{2m}} \sum_{n=1}^{\infty} \frac{1}{n^{2m}}; \quad P_{2m+1}(0) = 0.$$

For future convenience we write:

$$P_{2m}(0) = \frac{1}{2^{2m-1} \pi^{2m}} \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{Bm}{(2m)!}, \tag{174}$$

where B_m are the so called *Bernoulli numbers*.

Let us return to formula (159). Integrating by parts and remembering that

$$P_{2m}(0) = P_{2m}(n) = \frac{B_m}{(2m)!}; \quad P_{2m+1}(0) = P_{2m+1}(n) = 0,$$

we obtain:

$$-\int_0^n P_1(x) f'(x)\,dx = \int_0^n P_2'(x) f'(x)\,dx =$$

$$= \frac{B_1}{2!}[f'(n) - f'(0)] - \int_0^n P_2(x) f''(x)\,dx =$$

$$= \frac{B_1}{2!}[f'(n) - f'(0)] - \int_0^n P_3'(x) f''(x)\,dx =$$

$$= \frac{B_1}{2!}[f'(n) - f'(0)] + \int_0^n P_3(x) f'''(x)\,dx =$$

$$= \frac{B_1}{2!}[f'(n) - f'(0)] - \int_0^n P_4'(x) f'''(x)\,dx =$$

$$= \frac{B_1}{2!}[f'(n) - f'(0)] - \frac{B_2}{4!}[f'''(n) - f'''(0)] + \int_0^n P_4(x) f^{(iv)}(x)\,dx,$$

continuing this further we obtain Euler's summation formula:

$$\sum_{k=1}^{n} f(k) = \int_0^n f(x)\,dx + \frac{1}{2}[f(0) + f(n)] + \frac{B_1}{2!}[f'(n) - f'(0)] - \frac{B_2}{4!}[f'''(n) -$$

$$- f'''(0)] + \ldots + (-1)^m \frac{B_{m+1}}{(2m+2)!}[f^{(2m+1)}(n) - f^{(2m+1)}(0)] +$$

$$+ (-1)^m \int_0^n P_{2m+3}(x) f^{(2m+3)}(x)\,dx. \qquad (175)$$

In these evaluations we have naturally assumed that when $x \geqslant 0$, $f(x)$ has continuous derivatives up to the $(2m+3)$th order inclusively.

The last term on the right-hand side gives the final term of Euler's formula. Formula (174) shows clearly that the numbers B_n grow quickly as n increases and the infinite series, which corresponds to Euler's formula, is usually divergent. Nevertheless, it is still sometimes convenient to use formula (175) for the approximate evaluation of the sum on the left-hand side of the formula.

Rewrite formula (160) replacing C by the expression we obtained above:

$$\log \Gamma(z) = \log \sqrt{2\pi} + \left(z - \frac{1}{2}\right)\log z - z + \int_0^\infty \frac{P_1(x)}{z+x}\,dx.$$

Integrating the above integral by parts, as before, and remembering formula (174) and also that $P_n(x)$ remains bounded for all real values of x we have,

when $z > 0$:

$$\log \Gamma(z) = \log \sqrt{2\pi} + \left(z - \frac{1}{2}\right) \log z - z + \frac{B_1}{1 \cdot 2} \cdot \frac{1}{z} - \frac{B_2}{3 \cdot 4} \cdot \frac{1}{z^3} + \\ + \frac{B_3}{5 \cdot 6} \cdot \frac{1}{z^5} - \ldots + (-1)^{m-1} \frac{B_m}{(2m-1)\,2m} \cdot \frac{1}{z^{2m-1}} + \\ + (-1)^{m-1} (2m)! \int_0^\infty \frac{P_{2m+1}(x)}{(z+x)^{2m+1}}\, dx.$$

In exactly the same way as in the previous paragraph we can show that the last integral multiplied by z^{2m+1} remains bounded as $z \to +\infty$, i.e.

$$\int_0^\infty \frac{P_{2m+1}(x)}{(z+x)^{2m+1}}\, dx = O\left(\frac{1}{z^{2m+1}}\right),$$

and the above formula can be rewritten in the form

$$\log \Gamma(z) = \log \sqrt{2\pi} + \left(z - \frac{1}{2}\right) \log z - z + \frac{B_1}{1 \cdot 2} \cdot \frac{1}{z} - \frac{B_2}{3 \cdot 4} \cdot \frac{1}{z^3} + \ldots + \\ + (-1)^{m-1} \frac{B_m}{(2m-1)\,2m} \cdot \frac{1}{z^{2m-1}} + O\left(\frac{1}{z^{2m+1}}\right). \qquad (176)$$

If we reject the last term and write the corresponding infinite series then this series will be divergent for all values of z. If, however, we fix m then the last term, as $z \to +\infty$, will be an infinitesimally small quantity of a higher power, viz. of the order $1/z^{2m+1}$, than the power of the remaining terms, which are of the order $1/z^{2m-1}$.

Formula (176), like formula (169), holds in the z plane from which an arbitrarily small but fixed sector is cut out, the bisector of which is directed along the negative part of the real axis. If z is positive then the last term can be approximated to with greater accuracy and the formula given below holds:

$$\log \Gamma(z) = \log \sqrt{2\pi} + \left(z - \frac{1}{2}\right) \log z - z + \frac{B_1}{1 \cdot 2} \cdot \frac{1}{z} - \frac{B_2}{3 \cdot 4} \cdot \frac{1}{z^3} + \ldots + \\ + (-1)^{m-1} \frac{B_m}{(2m-1)\,2m} \cdot \frac{1}{z^{2m-1}} + \theta_m (-1)^m \frac{B_{m+1}}{(2m+1)(2m+2)} \cdot \frac{1}{z^{2m+1}}, \quad (176_1)$$

where $0 < \theta_m < 1$. We shall not prove this formula here.

77. Bernoulli numbers. Let us define the Bernoulli numbers by the equations:

$$B_m = \frac{(2m)!}{2^{2m-1} \pi^{2m}} \sum_{n=1}^{\infty} \frac{1}{n^{2m}}. \qquad (177)$$

We shall show, for the moment, that these numbers can be defined in an elementary way and that they are all rational numbers.

Write out the expansion of $\cot z$ into partial fractions [65]:

$$\cot z = \frac{1}{z} + \sum_{k=-\infty}^{+\infty} \left(\frac{1}{z - k\pi} + \frac{1}{k\pi} \right),$$

or

$$\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2\pi^2},$$

or, changing to exponential functions according to Euler's formula:

$$i \frac{e^{zi} + e^{-zi}}{e^{zi} - e^{-zi}} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2\pi^2}.$$

Let as suppose that $z = u/2i$:

$$\frac{e^{\frac{u}{2}} + e^{-\frac{u}{2}}}{e^{\frac{u}{2}} - e^{-\frac{u}{2}}} = \frac{2}{u} + 4 \sum_{k=1}^{\infty} \frac{u}{4k^2\pi^2 + u^2},$$

i.e.

$$\frac{e^u + 1}{e^u - 1} = \frac{2}{u} + 4 \sum_{k=1}^{\infty} \frac{u}{4k^2\pi^2 + u^2},$$

or

$$\frac{2}{e^u - 1} + 1 = \frac{2}{u} + 4 \sum_{k=1}^{\infty} \frac{u}{4k^2\pi^2 + u^2}.$$

The formula above may be rewritten in the form:

$$\frac{u}{e^u - 1} - 1 + \frac{u}{2} = 2u^2 \sum_{k=1}^{\infty} \frac{1}{4k^2\pi^2 + u^2}. \tag{178}$$

We can write:

$$\frac{u^2}{4k^2\pi^2 + u^2} = -\sum_{p=1}^{\infty} \left(-\frac{u^2}{4k^2\pi^2} \right)^p \qquad (|u| < 2k\pi).$$

Substituting into formula (178) we obtain:

$$\frac{u}{e^u - 1} - 1 + \frac{u}{2} = -2 \sum_{k=1}^{\infty} \left[\sum_{p=1}^{\infty} \left(-\frac{u^2}{4k^2\pi^2} \right)^p \right].$$

Applying the lemma from the theorem of Weierstrass about the addition of power series [14] we can represent the right-hand side in the form of a series in whole positive powers of u where $|u| < 2\pi$

$$\frac{u}{e^u - 1} - 1 + \frac{u}{2} = 2 \left[\frac{s_2 u^2}{(2\pi)^2} - \frac{s_4 u^4}{(2\pi)^4} + \frac{s_6 u^6}{(2\pi)^6} - \dots \right],$$

where we have assumed for briefness that:

$$s_p = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

Remembering formula (177) we can write:

$$\frac{u}{e^u - 1} = 1 - \frac{u}{2} + \sum_{m=1}^{\infty} (-1)^{m-1} B_m \frac{u^{2m}}{(2m)!}. \tag{179}$$

The function on the left-hand side has singularities in the neighbourhood of the origin: $u = \pm 2\pi i$, so that the above power series has a circle of convergence $|u| < 2\pi$. Dividing u by the series

$$e^u - 1 = \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots,$$

we can obtain the successive Bernoulli numbers B_m. Below we give the first few of these numbers:

$$B_1 = \frac{1}{6}; \quad B_2 = \frac{1}{30}; \quad B_3 = \frac{1}{42}; \quad B_4 = \frac{1}{30}; \quad B_5 = \frac{5}{66}; \quad B_6 = \frac{691}{2730}.$$

78. Method of the steepest descent. In the following few sections we shall explain a method of approximate evaluation of definite contour integrals. As a preliminary let us explain certain problems connected with the variation of the real and imaginary parts of a regular function. Suppose that we are given the following function in a domain B:

$$f(z) = u(x, y) + v(x, y)\, i.$$

At every point of B at which the derivative $f'(z)$ is not zero there will be a direction l along which $u(x, y)$ changes most rapidly. This direction l is given by the direction of the vector grad $u(x, y)$ and the derivative, taken in this direction (and in the opposite direction), has the greatest absolute value. This derivative of $u(x, y)$ in the direction n, perpendicular to l, must be zero [II, 108]. The plane of the directions n is defined by lines at the level $u(x, y) = \text{const.}$ and the orthogonal plane l by the family of trajectories which are perpendicular to the level lines, i.e. by a family $v(x, y) = \text{const}$ [29]. We can therefore say that at every point where $f'(z)$ is not zero $u(x, y)$ varies most steeply along the line $v(x, y) = \text{const}$. Notice that in this case $\partial u/\partial l$ along the above line is not zero. If it should happen that at a certain point not only $\partial u/\partial n$ but also $\partial u/\partial l$ is zero then at this point the derivative of u in any direction would be zero and it would follow that the derivative $f'(z)$ at this point would also vanish.

Let us now investigate the position of our line in the neighbourhood of the point z_0 at which $f'(z_0) = 0$. In this neighbourhood we have:

$$f(z) - f(z_0) = (z - z_0)^p \left[b_0 + b_1 (z - z_0) + \dots\right] \quad (p \geqslant 2;\ b_0 \neq 0). \tag{180}$$

Putting

$$b_\nu = r_\nu e^{i\beta\nu};\ z - z_0 = \varrho e^{i\omega} \quad (r_0 \neq 0) \tag{181}$$

and equating the real and imaginary parts of the difference $f(z) - f(z_0)$ to zero, we obtain the following equations for the lines $u(x, y) = \text{const.}$ and $v(x, y) = \text{const.}$ in the neighbourhood of z_0:

$$\Phi_1(\varrho, \omega) = r_0 \cos(\beta_0 + p\omega) + r_1\varrho \cos[\beta_1 + (p+1)\omega] + r_2\varrho^2 \cos[\beta_2 + (p+2)\omega] + \ldots = 0, \tag{182}$$

$$\Phi_2(\varrho, \omega) = r_0 \sin(\beta_0 + p\omega) + r_1\varrho \sin[\beta_1 + (p+1)\omega] + r_2\varrho^2 \sin[\beta_2 + (p+2)\omega] + \ldots = 0. \tag{183}$$

Consider the equation (182). When $\varrho = 0$ we obtain:

$$\cos(\beta_0 + p\omega) = 0,$$

i.e.

$$\beta_0 + p\omega = (2m+1)\frac{\pi}{2},$$

where m is any integer. Assuming $m = 0, 1, \ldots, 2p-1$, we obtain all the different solutions of the equation (182) with respect to ω when $\varrho = 0$:

$$\omega_m = -\frac{\beta_0}{p} + \frac{2m+1}{2p}\pi \quad (m = 0, 1, 2, \ldots, 2p-1). \tag{184}$$

It is not difficult to see that

$$\left.\frac{\partial \Phi_1}{\partial \omega}\right|_{\varrho=0,\ \omega=\omega_m} \neq 0,$$

and, consequently, it follows from the theorem of undefined functions [I, 159] that equation (182) has $2p$ solutions for ω which are continuous with respect to ϱ and which tend to ω_m when $\varrho = 0$, i.e. equation (182) corresponds to $2p$ lines, which radiate from the point z_0 and which have at that point tangents with amplitudes ω_m. But $\omega_{m+p} = \omega_m + \pi$ and we can have p lines which pass through the point z_0 and have definite tangents at that point. These lines will divide the neighbourhood of the point z_0 into $2p$ curvilinear sectors with equal angles π/p at the apex. Inside these sectors, in the neighbourhood of z_0, we have alternatively $\Phi_1(\varrho, \omega) < 0$ and $\Phi_1(\varrho, \omega) > 0$, viz.

$$\text{when } \frac{\pi}{2} + m\pi < \beta_0 + p\omega < \frac{\pi}{2} + (m+1)\pi:$$

$$\Phi_1(\varrho, \omega) \begin{cases} < 0, \text{ if } m \text{ is even.} \\ > 0, \text{ if } m \text{ is odd.} \end{cases}$$

This is directly due to the fact that the sign on the left-hand side of equation (182) for the given ω, which is other than (184), and for ϱ sufficiently close to zero, is determined by the sign of the first term.

Similarly equation (183) determines p lines which pass through the point z_0, where the tangents to these lines serve as bisectors of the angles defined by the tangents to the lines (182).

The point z_0 we shall call a saddle-point, the sectors $\Phi_1(\varrho, \omega) < 0$ negative and the sectors $\Phi_1(\varrho, \omega) > 0$ positive sectors.

Let us now consider an integral of the following type:

$$I_n = \int_l (z - z_0)^{a-1} F(z) [\varphi(z)]^n \, dz = \int_l (z - z_0)^{a-1} F(z) \, e^{nf(z)} \, dz, \qquad (185)$$

where $F(z)$, $\varphi(z)$ and $f(z) = \log \varphi(z)$ are regular at the point z_0; $F(z_0)$ and $\varphi(z_0)$ are not zero, and n is the greatest positive number. Suppose that the contour l originates at the saddle-point z_0 and ends at a certain point z_1 and that it lines in the negative sector. When $| e^{nf(z)} | = e^{nu(x,y)}$ it has its maximum at the point z_0 and for larger n's this maximum is abrupt. It is therefore to be expected that the principal value of integral (185) is due to integration along the contour l near the point z_0 and it is convenient to select this section to be along the line $v(x, y) = \text{const.}$ along which $u(x, y)$ has its steepest descent. Instead of taking the line itself we can take a small section of the tangent to this line. We must remember, that the contour can be deformed in accordance with Cauchy's theorem. Thus integral (185) can be broken up into two terms: an integral along the small section l' near the point z_0 and an integral along the remaining part of the section l'' up to the point z_1. The upper bound of the integral along l'' is found from its modulus, and the integral along l', from which the principal part of I_n is obtained, is evaluated approximately, and the inequality of error should be given. The integral along l' is usually evaluated approximately by expanding the integrated into a Taylor's series and finding the upper bound of the last term of this series. In the following sectino we shall apply the above system to the evaluation of an integral and, restricting ourselves to relatively approximate upper bounds, we shall isolate the principal part of I_n whilst for the remainder we shall obtain an inequality of a certain order with respect to a small quantity $1/n$.

Let us now make certain general remarks with reference to the evalution of integrals of the type (185). The contour l can pass through the saddle-point and from one negative sector into another. In this case, as before, the principal part of I_n will be determined by the integral over the small section near the point z_0, and the contour must either pass along the line $v(x, y) = \text{const.}$ or along the tangent to this line. If the contour l lies in the positive sector, then the main part of I_n will be obtained by integration over the small section near the point z_1, and this contour should pass along the line of the steepest descent of $u(x, y)$ [the line $v(x, y) = \text{const}$]. If the integrand is many-valued then by integrating along the lines of the steepest descent we must bear in mind any cuts made because of the many-valuedness of the function and a section of the path of integration should be directed along these cuts; in making cuts we must take into account the above general concept of stationary points and the steepest descent of $u(x, y)$. If, there are several saddle-points in the domain through which the contour l passes, then a comparison of the modulus of the integrand domain at these saddle-points should be made and the path of integration should be selected with these general considerations in mind. We shall explain this by a number of examples. Let us now try to evaluate the integral along the contour l in the negative sector and, for the moment, we are only interested in low orders like that of the small quantity $1/n$. Without loss

of generality we can assume that $z_0 = 0$. Also, denoting $f(z)$ by the form $f(z) = f(z_0) + [f(z) - f(z_0)]$ and taking $e^{f(z_0)}$ outside the integral we can assume $f(z_0) = 0$, i.e. $\varphi(z_0) = 1$.

79. Isolation of the principal part of an integral. Let us now consider the integral

$$I_n = \int_0^{z_1} z^{a-1} F(z) [\varphi(z)]^n \, dz = \int_0^{z_1} z^{a-1} F(z) e^{nf(z)} \, dz, \tag{186}$$

where the path of integration goes from the saddle-point $z = 0$ to the point z_1 in the negative sector with m even ($m = 2l$). We suppose that the functions $F(z)$ and $\varphi(z)$ are regular in a certain domain in which the path of integration lies. Suppose that near the point $z = 0$

$$F(z) = a_0 + a_1 z + a_2 z^2 + \ldots; \quad f(z) = \log \varphi(z) = z^p (b_0 + b_1 z + b_2 z^2 + \ldots), \tag{187}$$

where a_0 and b_0 are not zero. If the integral is to exist at the lower limit $z = 0$ we must assume that the real part of the number a, which we denote by $R(a)$, is positive. According to Cauchy's theorem [5] we can deform the path of integration in the neighbourhood of the origin and direct it from $z = 0$ along the tangent to the line of steepest descent, i.e. along the bisector of the above sector which corresponds to $m = 2l$, to the point $z = \varrho_0 e^{i\omega_0'}$ and from there to the point $z = z_1$. Along the second part of the path, $\max |\varphi(z)| < 1 - \eta$, where η is a certain positive number which depends on the choice of ϱ_0. In future we shall select ϱ_0 independently of n, for along the second section of the path the modulus of the integral (186) will not exceed the product $M(1 - \eta)^n$, where M is a certain constant, which is independent of n, so that

$$I_n = \int_0^{\varrho_0 e^{i\omega_0'}} z^{a-1} F(z) \varphi(z)^n \, dz + O[(1 - \eta)^n], \tag{188}$$

where by $0[(1 - \eta)^n]$ we denote a number which tends to zero as n increases or, more accurately, it is such that the quotient obtained by dividing it by $(1 - \eta)^n$ remains bounded as $n \to +\infty$. In future we shall denote by the symbol $0(a_n)$ a number which is such that the quotient obtained by dividing this number by a_n remains bounded as $n \to +\infty$. The amplitude ω_0' of the bisector of the sector with the number $m = 2l$ is expressed by the formula:

$$\omega_0' = -\frac{\beta_0}{p} + \frac{2l+1}{p}\pi. \tag{189}$$

Let σ be a number which is less than the radius of convergence of the series (187). We select, in any case, $\varrho_0 < \sigma$. The series $\sum_{\nu=0}^{\infty} b_\nu z^\nu$ and the differentiated series $\sum_{\nu=1}^{\infty} \nu b_\nu z^{\nu-1}$ have the same radii of convergence as the second of the series (187).

Applying the inequality to every modulus of the coefficients of the power series [14] we can write

$$|a_\nu| < \frac{M}{\sigma^\nu}; \quad |b_\mu| < \frac{M}{\mu\sigma^\mu} \begin{pmatrix} \nu = 0, 1, 2, \ldots \\ \mu = 1, 2, 3, \ldots \end{pmatrix}, \tag{190}$$

where M is a constant. For further evaluations we shall introduce a new concept.

We shall say that the power series $\sum_{\nu=0}^{\infty} g_\nu z^\nu$ (or a corresponding function) is *dominating a power series* (or function) $\sum_{\nu=0}^{\infty} h_\nu z^\nu$, providing the g_ν are positive numbers and $|h_\nu| \leqslant g_\nu$. In this case we obviously have the inequality

$$\left|\sum_{\nu=0}^{\infty} h_\nu z^\nu\right| < \sum_{\nu=0}^{\infty} g_\nu |z|^\nu,$$

and we are naturally assuming that the series converge.

Let us write the product $F(z)\,\varphi(z)^n$ in the form:

$$F(z)\,\varphi(z)^n = (a_0 + a_1 z + \ldots)\, e^{n(b_0 z^p + b_1 z^{p+1} + \ldots)} =$$
$$= a_0 e^{n b_0 z^p} + e^{n b_0 z^p}\left[(a_0 + a_1 z + \ldots)\, e^{n z^p (b_1 z + b_2 z^2 + \ldots)} - a_0\right]. \qquad (191)$$

The difference in the square brackets can be expanded into a power series without a constant term:

$$\psi(z) = (a_0 + a_1 z + \ldots)\, e^{n z^p (b_1 z + b_2 z^2 + \ldots)} - a_0 = c_1 z + c_2 z^2 + \ldots \qquad (192)$$

Bearing in mind that e^z can be expanded into a power series with positive coefficients we obtain a dominating series for the series (192), if we substitute the series in the index of e and in the factor in front of e by the dominating series

$$\left(M + M\frac{z}{\sigma} + M\frac{z^2}{\sigma^2} + \ldots\right) e^{n z^p \left(\frac{M}{1}\cdot\frac{z}{\sigma} + \frac{M}{2}\cdot\frac{z^2}{\sigma^2} + \ldots\right)} - M,$$

or

$$M\left(1 - \frac{z}{\sigma}\right)^{-1} e^{-Mnz^p \log\left(1 - \frac{z}{\sigma}\right)} - M = M\left[\left(1 - \frac{z}{\sigma}\right)^{-1 - Mnz^p} - 1\right], \qquad (193)$$

and in this last series there will also be no constant term. Denoting, for the sake of briefness $nz^p = z'$, we can write the dominating series (or, still better) the dominating function in the form:

$$M\frac{z}{\sigma}\left[\frac{1 + Mz'}{1!} + \frac{(1 + Mz')(2 + Mz')}{2!}\left(\frac{z}{\sigma}\right) + \right.$$
$$\left. + \frac{(1 + Mz')(2 + Mz')(3 + Mz')}{3!}\left(\frac{z}{\sigma}\right)^2 + \ldots\right].$$

On taking $(1 + Mz')$ outside the bracket we can write the dominating function for (192) in the form:

$$(1 + Mz')\, M\frac{z}{\sigma}\left[1 + \left(1 + \frac{Mz'}{2}\right)\left(\frac{z}{\sigma}\right) + \left(1 + \frac{Mz'}{2}\right)\left(1 + \frac{Mz'}{3}\right)\left(\frac{z}{\sigma}\right)^2 + \ldots\right],$$

and, finally, decreasing the denominators, we obtain the dominating function:

$$(1 + Mz')\, M\frac{z}{\sigma}\left[1 + \left(1 + \frac{Mz'}{1}\right)\left(\frac{z}{\sigma}\right) + \left(1 + \frac{Mz'}{1}\right)\left(1 + \frac{Mz'}{2}\right)\left(\frac{z}{\sigma}\right)^2 + \ldots\right]$$
$$= (1 + Mz')\, M\frac{z}{\sigma}\left(1 - \frac{z}{\sigma}\right)^{-1 - Mz'}. \qquad (194)$$

We thus have the inequality:

$$|\psi(z)| < (1 + Mn|z|^p)\, M \frac{|z|}{\sigma}\left(1 - \frac{|z|}{\sigma}\right)^{-1-Mn|z|^p}. \tag{195}$$

According to formula (191) integral (188) can be broken up into two integrals:

$$I_n = a_0 \int_0^{\varrho_0 e^{i\omega_0'}} z^{\alpha-1} e^{nb_0 z^p}\, dz + \int_0^{\varrho_0 e^{i\omega_0'}} z^{\alpha-1} e^{nb_0 z^p}\, \psi(z)\, dz + O[(1-\eta)^n].$$

We now replace z by a new variable of integration t, according to the formula:

$$z = e^{i\omega_0'} \sqrt[p]{\frac{t}{nr_0}},$$

where, from (189):

$$nz^p = -\frac{t}{b_0}.$$

We obtain:

$$I_n = A_n + B_n + O[(1-\eta)^n], \tag{196}$$

where

$$\left.\begin{aligned} A_n &= \frac{1}{p}\, e^{i\omega_0 \alpha} \left(\frac{1}{nr_0}\right)^{\frac{\alpha}{p}} a_0 \int_0^{nr_0\varrho_0^p} e^{-t}\, t^{\frac{\alpha}{p}-1}\, dt, \\ B_n &= \frac{1}{p}\, e^{i\omega_0' \alpha} \left(\frac{1}{nr_0}\right)^{\frac{\alpha}{p}} \int_0^{nr_0\varrho_0^p} e^{-t}\, t^{\frac{\alpha}{p}-1}\, \psi(z)\, dt. \end{aligned}\right\} \tag{197}$$

Taking into account the fact that $|z| = (t/nr_0)^{1/p}$, we obtain from (195):

$$|\psi(z)| < \left(1 + \frac{Mt}{r_0}\right) M \frac{1}{\sigma} \sqrt[p]{\frac{t}{nr_0}} \left(1 - \frac{|z|}{\sigma}\right)^{-1-\frac{Mt}{r_0}}.$$

We took $\varrho_0 < \sigma$ and therefore along the path of integration we have $|z| = (t/nr_0)^{1/p} < \varrho_0 < \sigma$ so that, replacing $|z|$ by the greater quantity ϱ_0 we have:

$$|\psi(z)| < \left(1 + \frac{Mt}{r_0}\right) M \frac{1}{\sigma} \sqrt[p]{\frac{t}{nr_0}} \left(1 - \frac{\varrho_0}{\sigma}\right)^{-1-\frac{Mt}{r_0}}.$$

Bearing in mind that for $q > 0$ and a complex γ we have $|q|^\gamma = q^{R(\gamma)}$, where $R(\gamma)$ is the real part of γ, we obtain the following inequality for B_n:

$$|B_n| < \frac{M}{p} |e^{i\omega_0' \alpha}| \frac{\left(1 - \frac{\varrho_0}{\sigma}\right)^{-1}}{\sigma} \left(\frac{1}{nr_0}\right)^{\frac{R(\alpha)+1}{p}} \times$$

$$\times \int_0^{nr_0\varrho_0^p} e^{-t} \left(1 - \frac{\varrho_0}{\sigma}\right)^{-\frac{Mt}{r_0}\frac{R(\alpha)+1}{p}-1} t \left(1 + \frac{Mt}{r_0}\right) dt.$$

If we also subject ϱ_0 to the following condition apart from the condition $\varrho_0 < \sigma$

$$a = e\left(1 - \frac{\varrho_0}{\sigma}\right)^{\frac{M}{r_0}} > 1,$$

we can take a fixed ϱ_0, which satisfies both these conditions. In the last integral the integrand will contain a factor a^{-t} and we can integrate to $(+\infty)$ The integral will be convergent and the value of the integral of a positive function will increase owing to the extension of the interval of integration. The integral will no longer depend on n and we obtain the inequality:

$$|B_n| \leqslant M_1 \left(\frac{1}{n}\right)^{\frac{R(a)+1}{p}},$$

where M_1 is a certain constant which does not depend on n, i.e.

$$B_n = O\left[\left(\frac{1}{n}\right)^{\frac{R(a)+1}{p}}\right].$$

The quantity $0[(1-\eta)^n]$ having been divided by $(1-\eta)^n$ remains bounded as $n \to +\infty$. It remains the more bounded for being divided by $(1/n)^{(R(a)+1)/p}$, for the ratio $(1-\eta)^n : (1/n)^{(R(a)+1)/p} \to 0$ as $n \to \infty$, i.e.

$$B_n + O\left[(1-\eta)^n\right] = O\left[\left(\frac{1}{n}\right)^{\frac{R(a)+1}{p}}\right],$$

and formula (196) can be rewritten in the form:

$$I_n = A_n + O\left[\left(\frac{1}{n}\right)^{\frac{R(a)+1}{p}}\right]. \tag{198}$$

Let us now consider A_n. We can write:

$$A_n = \frac{a_0}{p} e^{i\omega_0' a} \left(\frac{1}{nr_0}\right)^{\frac{a}{p}} \int_0^{\infty} e^{-t} t^{\frac{a}{p}-1} dt - \frac{a_0}{p} e^{i\omega_0' a} \left(\frac{1}{nr_0}\right)^{\frac{a}{p}} \int_{nr_0\varrho_0^p}^{\infty} e^{-t} t^{\frac{a}{p}-1} dt,$$

or have [71]:

$$A_n = \frac{a_0}{p} e^{i\omega_0' a} \left(\frac{1}{nr_0}\right)^{\frac{a}{p}} \Gamma\left(\frac{a}{p}\right) + C_n, \tag{199}$$

where

$$C_n = -\frac{a_0}{p} e^{i\omega_0' a} \left(\frac{1}{nr_0}\right)^{\frac{a}{p}} \int_{nr_0\varrho_0^p}^{\infty} e^{-t} t^{\frac{a}{p}-1} dt,$$

and consequently:

$$|C_n| \leqslant \frac{|a_0|}{p} |e^{i\omega_0' a}| \left(\frac{1}{nr_0}\right)^{\frac{R(a)}{p}} \int_{nr_0\varrho_0^p}^{\infty} e^{-t} t^{\frac{R(a)}{p}-1} dt.$$

For large positive values of t the function

$$e^{-t}\, t^{\frac{R(a)}{p}+1}$$

decreases and therefore, isolating the factor t^{-2} under the integral and replacing the remainder of the integrand by its value at the lower limit of integration we can say that for large n the value of the integral is less than:

$$e^{-nr_0\varrho_0^p}\,(nr_0\varrho_0^p)^{\frac{R(a)}{p}+1}\int\limits_{nr_0\varrho_0^p}^{\infty}\frac{dt}{t^2}=e^{-nr_0\varrho_0^p}\,(nr_0\varrho_0^p)^{\frac{R(a)}{p}}.$$

This gives us the inequality for C_n:

$$|C_n|\leqslant M_2\, e^{-nr_0\varrho_0^p},$$

where the constant M_2 does not depend on n, i.e.

$$C_n=O\left(e^{-nr_0\varrho_0^p}\right).$$

Bearing in mind that, as n increases, the exponential function $e^{-nr_0}\,\varrho_0^p$ decreases faster than any negative power of n we can write:

$$C_n+O\left[\left(\frac{1}{n}\right)^{\frac{R(a)+1}{p}}\right]=O\left[\left(\frac{1}{n}\right)^{\frac{R(a)+1}{p}}\right].$$

The formulae (198) and (199) give us finally:

$$I_n=\frac{a_0}{p}\;e^{i\omega_0' a}\left(\frac{1}{nr_0}\right)^{\frac{a}{p}}\Gamma\left(\frac{a}{p}\right)+O\left[\left(\frac{1}{n}\right)^{\frac{R(a)+1}{p}}\right]. \tag{200}$$

In this formula the principal term is of order $(1/n)^{R(a)/p}$ and the last term is of a smaller order still.

If more accurate inequalities are required several more terms can be isolated from I_n in the order of increasing powers of $(1/n)$. This brings us to the following general formula which we do not intend to prove here:

$$I_n=\frac{1}{p}\sum_{\nu=0}^{m-1}{}'\;d_\nu\left(\frac{1}{nr_0}\right)^{\frac{a+\nu}{p}}+O\left[\left(\frac{1}{n}\right)^{\frac{R(a)+m}{p}}\right]. \tag{201}$$

In this formula

$$d_\nu=e^{i\omega_0'(a+\nu)}\sum_{\mu=0}^{\nu}\frac{g_{\nu,\mu}}{(-b_0)^\mu}\,\Gamma\left(\frac{a+\nu}{p}+\mu\right),$$

$g_{\nu,0}=a_\nu$ and $g_{\nu,\mu}$ are coefficients of z^ν in the expansion:

$$\frac{1}{\mu!}\,(a_0+a_1z+\ldots)\,(b_1z+b_2z^2+\ldots)^\mu.$$

We shall consider the simplest path of integration from the point $z=0$ to the point $z=z_1$; at the beginning of the path, i.e. at the point $z=0$ the modulus of

$\varphi(z)$ has its greatest value. Consider now the integral:

$$I'_n = \int_{z_1}^{z_2} z^{\alpha-1} F(z) \varphi(z)^n \, dz, \tag{202}$$

where z_1 and z_2 lie in sectors where $|\varphi(z)| < 1$. The path takes us from z_1 to $z = 0$, surrounds this point along the arc of a small circle, centre at $z = 0$ and proceeds to $z = z_2$. If $R(\alpha) > 0$ then, as the radius of the above circle tends to zero, the integral along the arc of the circle will also tend to zero and we can simply integrate from $z = z_1$ to $z = 0$ along the sector with the number $m = 2l_1$; and from $z = 0$ to $z = z^2$ along the sector with the number $m = 2l^2$ we thus obtain

$$I'_n = I_{n,l_2} - I_{n,l_1},$$

where I_{n,l_2} and I_{n,l_1} are integrals of the former kind over contours which lie in the above sectors. Hence:

$$I'_n = \frac{a_0}{p} \left(\frac{1}{n r_0}\right)^{\frac{\alpha}{p}} [e^{i\omega'_2 \alpha} - e^{i\omega'_1 \alpha}] \Gamma\left(\frac{\alpha}{p}\right) + O\left[\left(\frac{1}{n}\right)^{\frac{(\alpha)+1}{p}}\right], \tag{203}$$

where

$$\omega'_1 = -\frac{\beta_0}{p} + \frac{2l_1 + 1}{p}\pi; \qquad \omega'_2 = -\frac{\beta_0}{p} + \frac{2l_2 + 1}{p}\pi.$$

It can be shown that formula (203) is valid also when $R(\alpha) \leqslant 0$.

Example. Consider

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx.$$

On substituting $x = ny$ we obtain:

$$\frac{\Gamma(n+1)}{n^{n+1}} = \int_0^\infty (y e^{-y})^n \, dy.$$

The function ye^{-y} has a maximum when $y = 1$. Substituting $y = 1 + z$:

$$\frac{e^n \Gamma(n+1)}{n^{n+1}} = \int_{-1}^\infty [(1+z) e^{-z}]^n \, dz.$$

Divide the interval of integration into two parts: $(-1, +1)$ and $(+1, +\infty)$. For the second interval we have:

$$\int_1^\infty [(1+z) e^{-z}]^n \, dz = \int_1^\infty [(1+z) e^{-z}]^{n-1} (1+z) e^{-z} \, dz <$$

$$< \int_1^\infty \left(\frac{2}{e}\right)^{n-1} (1+z) e^{-z} \, dz = \left(\frac{2}{e}\right)^{n-1} \int_1^\infty (1+z) e^{-z} \, dz,$$

for, when $z > 1$

$$(1+z) e^{-z} < \frac{2}{e}.$$

Therefore

$$\int_1^\infty [(1+z)\,e^{-z}]^n\,dz = O\left[\left(\frac{2}{e}\right)^n\right]. \tag{204}$$

The integral

$$\int_{-1}^{1} [(1+z)\,e^{-z}]^n\,dz \tag{205}$$

has the form we considered above, but in this case

$$a_0 = 1; \quad a = 1; \quad F(z) = 1 \text{ and } \log\varphi(z) = \log(1+z) - z,$$

from which it follows that $p = 2$ and $b_0 = 1/2$, i.e. $r_0 = 1/2$. It can also be seen that $\omega_1' = \pi$ and $\omega_2' = 0$. We finally have from (203):

$$\frac{e^n\,\Gamma(n+1)}{n^{n+1}} = \frac{1}{2}\left(\frac{2}{n}\right)^{\frac{1}{2}} 2\Gamma\left(\frac{1}{2}\right) + O\left(\frac{1}{n}\right) + O\left[\left(\frac{2}{e}\right)^n\right],$$

or, taking into account that $(2/e)^n$ decreases more rapidly than $1/n$ and that $\Gamma(1/2) = \sqrt{\pi}$, we obtain:

$$\frac{e^n\,\Gamma(n+1)}{n^{n+1}} = \frac{\sqrt{2\pi}}{n^{\frac{1}{2}}} + O\left(\frac{1}{n}\right),$$

or

$$\Gamma(n+1) = n\Gamma(n) = \sqrt{2\pi}n^{n+\frac{1}{2}}\,e^{-n}\left[1 + n^{\frac{1}{2}}\,O\left(\frac{1}{n}\right)\right] =$$

$$= \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right],$$

or

$$\Gamma(n) = \sqrt{2\pi}n^{n-\frac{1}{2}}\,e^{-n}\left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right].$$

The last term $0(1/\sqrt{n})$ should, in fact, be $0(1/n)$.

To obtain this result the integral (205) must be divided in two, viz. from $z = 0$ to $z = 1$ and from $z = -1$ to $z = 0$, and formula (201) should be applied to each of these integrals when $m = 2$. At the same time terms corresponding to $\nu = 1$ in these integrals will cancel each other and we shall obtain the same principal term and last term $0(1/n)$.

All the details of the above method, the proof of the general formula (201) and examples can be found in Perron's paper "*Ueber die näherungsweise Berechnung von Funktionen grosser Zahlen*" (News of the Bavarian Academy, 1917).

One of the first works in this field was a work by P. A. Nekrasov.

80. Examples. 1. Consider the integral

$$I = \int_{-\infty}^{\frac{1}{2}} \frac{1}{z+ia}\, e^{n(z^3-z^2)}\, dz\,, \tag{206}$$

where a is a small positive number and n is a large positive number. The function $f(z) = z^3 - z^2$ has its maximum when $z = 0$ and the real axis is the line of the steepest descent of $v(x, y) = 0$.

Bearing this in mind let us represent I in the form:

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{z+ia}\, e^{n(z^3-z^2)}\, dz + \omega\,, \tag{207}$$

where

$$\omega = \int_{-\infty}^{-\frac{1}{2}} \frac{1}{z+ia}\, e^{n(z^3-z^2)}\, dz.$$

Noting that $z^3 - z^2 < -z^2$ when $z < 0$ we obtain:

$$|\omega| < \frac{1}{\sqrt{a^2+\frac{1}{4}}} \int_{-\infty}^{-\frac{1}{2}} e^{-nz^2}\, dz = \frac{e^{-\frac{n}{4}}}{\sqrt{a^2+\frac{1}{4}}} \int_{-\infty}^{-\frac{1}{2}} e^{-n\left(z^2-\frac{1}{4}\right)}\, dz =$$

$$= \frac{e^{-\frac{n}{4}}}{\sqrt{a^2+\frac{1}{4}}} \int_{-\infty}^{-\frac{1}{2}} e^{-n\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\, dz.$$

Replacing $z - 1/2$ by (-1) and introducing a new variable of integration $t = -(z + 1/2)$ we obtain:

$$|\omega| < \frac{e^{-\frac{n}{4}}}{\sqrt{a^2+\frac{1}{4}}} \int_0^{\infty} e^{-nt}\, dt = \frac{e^{-\frac{n}{4}}}{n\sqrt{a^2+\frac{1}{4}}} < \frac{2e^{-\frac{n}{4}}}{n}. \tag{208}$$

To evaluate the first term on the right-hand side of formula (207) we suppose that $e^{nz^3} = 1 + \Delta$, where

$$\Delta = \frac{nz^3}{1!} + \frac{(nz^3)^2}{2!} + \dots,$$

hence

$$|\Delta| \leqslant n|z|^3 \left(1 + \frac{n|z|^3}{2!} + \frac{(n|z|^3)^2}{3!} \dots +\right) < n|z|^3 e^{n|z|^3}. \tag{209}$$

We obtain:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{z+ia} e^{n(z^2-z^2)}\,dz = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-nz^2}}{z+ia}\,dz + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\Delta e^{-nz^2}}{z+ia}\,dz. \tag{210}$$

Also

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-nz^2}}{z+ia}\,dz = \int_{-\infty}^{+\infty} \frac{e^{-nz^2}}{z+ia}\,dz + \omega_1, \tag{211}$$

where

$$|\omega_1| < \frac{2}{\sqrt{a^2+\frac{1}{4}}} \int_{\frac{1}{2}}^{\infty} e^{-nz^2}\,dz$$

and

$$\int_{\frac{1}{2}}^{\infty} e^{-nz^2}\,dz = e^{-\frac{n}{4}} \int_{\frac{1}{2}}^{\infty} e^{-n\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}\,dz,$$

or, replacing $z + 1/2$ by unity:

$$\int_{\frac{1}{2}}^{\infty} e^{-nz^2}\,dz < e^{-\frac{n}{4}} \int_{\frac{1}{2}}^{\infty} e^{-n\left(z-\frac{1}{2}\right)}\,dz = \frac{e^{-\frac{n}{4}}}{n},$$

whence, finally,

$$|\omega_1| < \frac{2e^{-\frac{n}{4}}}{n\sqrt{a^2+\frac{1}{4}}} < \frac{4}{n} e^{-\frac{n}{4}}. \tag{212}$$

Consider the first term on the right-hand side of formula (211). Separating the real and imaginary parts of the integrand and noting that the integral of the real part is zero since it is odd, we obtain:

$$\int_{-\infty}^{+\infty} \frac{e^{-nz^2}}{z+ia}\,dz = -i\int_{-\infty}^{+\infty} \frac{e^{-\beta^2 t^2}}{t^2+1}\,dt \qquad (\beta = a\sqrt{n}).$$

On assuming that $\beta^2 = \gamma$ and differentiating the integral

$$I(\gamma) = \int_{-\infty}^{+\infty} \frac{e^{-\gamma t^2}}{t^2+1}\,dt$$

with respect to the parameter γ we obtain:

$$\frac{dI(\gamma)}{d\gamma} = I(\gamma) - \sqrt{\frac{\pi}{\gamma}}.$$

On integrating this equation, bearing in mind that $I(\gamma)=0$ when $\gamma=+\infty$, we obtain:

$$I(\beta^2)=2\sqrt{\pi}\,e^{\beta^2}\int_{\beta}^{\infty}e^{-x^2}\,dx,$$

and, finally,

$$\int_{-\infty}^{+\infty}\frac{e^{-nz^2}}{z+ia}\,dz=-i2\sqrt{\pi}\,e^{a^2n}\int_{a\sqrt{n}}^{\infty}e^{-x^2}\,dx. \tag{213}$$

The integral on the right-hand side (the incomplete Laplace integral) is tabulated.

We come lastly to the upper bound of the second term on the right-hand side of formula (120). We shall carry out this investigation in two different ways.

On recalling (209) and the fact that $|z|^3<|z|^2/2$ when $|z|<1/2$ we obtain

$$|\Delta|\,e^{-nz^2}<n|z|^3e^{-\frac{n}{2}z^2},$$

whence

$$\left|\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\Delta e^{-nz^2}}{z+ia}\,dz\right|<\frac{2n}{a}\int_{0}^{\frac{1}{2}}z^3e^{-n\frac{z^2}{2}}\,dz<\frac{2n}{a}\int_{0}^{\infty}z^3e^{-n\frac{z^2}{2}}\,dz=\frac{4}{an}. \tag{214}$$

On the other hand, taking into account the inequality:

$$\left|\frac{z}{z+ia}\right|=\frac{|z|}{\sqrt{|z|^2+a^2}}<\frac{1}{\sqrt{1+4a^2}}\quad\text{when}\quad|z|<\frac{1}{2},$$

we obtain

$$\left|\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\Delta e^{-nz^2}}{z+ia}\,dz\right|<n\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{|z|^3}{|z+ia|}e^{-\frac{n}{2}z^2}\,dz<\frac{2n}{\sqrt{1+4a^2}}\int_{0}^{\infty}x^2e^{-\frac{n}{2}x^2}\,dx,$$

whence

$$\left|\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\Delta e^{-nz^2}}{z+ia}\,dz\right|<\frac{\sqrt{2\pi}}{\sqrt{n}}. \tag{214$_1$}$$

The principal part of (213) is of the order $1/a\sqrt{n}$; if a is not a small number then, using formulae (211), (212), (213) and (214), we can write:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{e^{-nz^2}}{z+ia}\,dz=-i2\sqrt{\pi}\,e^{a^2n}\int_{a\sqrt{n}}^{\infty}e^{-x^2}\,dx+\omega', \tag{215}$$

where

$$|\omega'| < \frac{6}{n} e^{-\frac{n}{4}} + \frac{4}{an}. \tag{216}$$

When a takes small positive values we can write:

$$|\omega'| < \frac{6}{n} e^{-\frac{n}{4}} + \frac{\sqrt{2\pi}}{\sqrt{n}}. \tag{216$_1$}$$

2. When considering the asymptotic representation of Hankel's functions we find if necessary to evaluate approximately the integral

$$I = \int_{-a-\varepsilon}^{-a+\varepsilon} e^{nf(z)}\, dz, \tag{217}$$

in which

$$f(z) = \sinh z - \xi z, \tag{218}$$

where the parameter $\xi > 1$ and n is a large positive number.

a in (217) is the positive zero of the equation $f'(z) = 0$, i.e. $\cosh a = \xi$.

The number $\varepsilon > 0$ we shall consider right from the beginning to be less than unity and later on we shall subject it to even more restrictive conditions.

It can readily be shown that the path of integration in (217) coincides with the major part of the line of steepest descent of the integrand. For this reason the evaluation of the integral involves the rational transformation of the integrand and the evaluation of elementary integrals.

We shall consider two methods by means of which integral (217) can be evaluated. In the first method our main object will be to find the principal term and to obtain a relatively simple assessment of error. In doing this we shall disregard certain essential properties of the integrand.

In the second method we shall take these properties into consideration and consequently the result obtained will be more accurate.

First method. Expand the function (128) into a power series in where $x = z + a$. We obtain

$$f(z) = f(-a) - \frac{\sinh a}{2!} x^2 + \frac{\cosh a}{3!} x^3 - \frac{\sinh a}{4!} x^4 + \dots \tag{219}$$

or

$$f(z) = f(-a) - \frac{\sinh a}{2} x^2 [1 + R], \tag{220}$$

where R satisfies the following inequality:

$$|R| \leqslant \frac{2 \cosh a}{\sinh a} |x| \left(\frac{1}{3!} + \frac{1}{4!} + \dots\right) \leqslant \frac{\cosh a}{2 \sinh a} |x| \tag{221}$$

if $|x| < 1$.

We represent the integrand (217) in the following form:

$$e^{nf(z)} = e^{nf(-a)} e^{-n\frac{\sinh a}{2} x^2} (1 + \delta_0), \tag{222}$$

where

$$\delta_0 = e^{-n\frac{\sinh\alpha}{2}Rx^3} - 1. \tag{223}$$

If we use the inequality

$$|e^y - 1| \leqslant |y|\, e^{|y|}$$

and take (221) into account then we obtain for δ_0 the following simple inequality:

$$|\delta_0| < \frac{n\cosh\alpha}{4}|x|^3 e^{n\frac{\sinh\alpha}{2}|R|x^3}. \tag{224}$$

Substituting (222) in (217) we obtain:

$$I = e^{nf(-\alpha)}\left[\int_{-\varepsilon}^{\varepsilon} e^{-n\frac{\sinh\alpha}{2}x^3}\,dx + \int_{-\varepsilon}^{-\varepsilon}\delta_0 e^{-n\frac{\sinh\alpha}{2}x^3}\,dx\right]. \tag{225}$$

Subject ε and n to the following two conditions:

$$n\frac{\sinh\alpha}{2}\varepsilon^2 = N \gg 1 \tag{226}$$

and

$$\frac{\cosh\alpha}{\sinh\alpha}\varepsilon \leqslant 1. \tag{226_1}$$

In a more accurate method of finding the upper bound the condition (226) is preserved but condition (226_1) is replaced by a stricter condition.

If condition (226_1) is satisfied then when $|x| \leqslant \varepsilon$, R satisfies the inequality $|R| < 1/2$ and from (224):

$$\left|\delta_0 e^{-n\frac{\sinh\alpha}{2}x^3}\right| < \frac{n\cosh\alpha}{4}|x|^3 e^{-n\frac{\sinh\alpha}{4}x^3} \qquad (|x| < \varepsilon). \tag{224_1}$$

If condition (226) is satisfied then the integrand in (225) becomes exceedingly small near the ends of the interval of integration. For this reason the transition to an infinite interval of integration has no essential influence on the result. Carrying out the necessary calculations we have

$$\int_{-\varepsilon}^{\varepsilon} e^{-n\frac{\sinh\alpha}{2}x^3}\,dx = \int_{-\infty}^{+\infty} e^{-n\frac{\sinh\alpha}{2}x^3}\,dx + \Delta_1 = \sqrt{\frac{2\pi}{n\sinh\alpha}} + \Delta_1, \tag{227}$$

where

$$\Delta_1 = -2\int_{\varepsilon}^{\infty} e^{-n\frac{\sinh\alpha}{2}x^3}\,dx,$$

and

$$|\Delta_1| = 2e^{-N}\int_{\varepsilon}^{\infty} e^{-n\frac{\sinh\alpha}{2}(x^3-\varepsilon^3)}\,dx,$$

whence, taking into account that $x^2 - \varepsilon^2 > (x - \varepsilon)^2$ when $x > \varepsilon$

$$|\Delta_1| < 2e^{-N} \int_0^\infty e^{-n\frac{\sinh a}{2}y^2}\, dy = \sqrt{\frac{2\pi}{n \sinh a}}\, e^{-N}. \tag{228}$$

Further, on taking into account (224_1) we find:

$$\left| \int_{-\varepsilon}^{+\varepsilon} \delta_0 e^{-n\frac{\sinh a}{2}x^2}\, dx \right| < \frac{n \cosh a}{4} 2 \int_0^\infty x^3 e^{-n\frac{\sinh a}{4}x^2}\, dx,$$

whence

$$\left| \int_{-\varepsilon}^{\varepsilon} \delta_0 e^{-n\frac{\sinh a}{2}x^2}\, dx \right| < \frac{2 \cosh a}{\pi \sinh a} \cdot \frac{2\pi}{n \sinh a}. \tag{229}$$

Using formulae (225), (227), (228) and (229) we obtain the following expression for the required integral:

$$I = e^{nf(-a)} \sqrt{\frac{2\pi}{n \sinh a}} (1 + \omega), \tag{230}$$

where

$$|\omega| < e^{-N} + \frac{2 \cosh a}{\pi \sinh a} \sqrt{\frac{2\pi}{n \sinh a}}. \tag{231}$$

If conditions (226) and (226_1) are satisfied then $|\omega|$ is found to be appreciably less than unity.

The above method of evaluation is very rough but it is also very simple as far as calculations are concerned. The disadvantage of this method lies in the fact that it does not take into account the change of sign in the expansion (219) and the presence of odd powers in this expansion.

We shall now remove these disadvantages.

Second method. Write the expansion (219) in the form:

$$f(z) = f(-a) - \frac{\sinh a}{2} x^2 + R_1 - R_2, \tag{232}$$

where

$$R_1 = \frac{\cosh a}{3!} x^3 \left(1 + \frac{x^2}{4 \cdot 5} + \frac{x^4}{4 \cdot 5 \cdot 6 \cdot 7} + \ldots\right)$$

$$R_2 = \frac{\sinh a}{4!} x^4 \left(1 + \frac{x^2}{5 \cdot 6} + \frac{x^4}{5 \cdot 6 \cdot 7 \cdot 8} + \ldots\right). \tag{232_1}$$

To start with suppose that in integral (217) ε is only subject to the condition $0 < \varepsilon < 1$. But even so we can say that R_1 and R_2 differ very little from the first terms of their respective expansions. Substitute the value of the integrand

$$e^{nf(z)} = e^{nf(-a)} e^{-n\frac{\sinh a}{2}x^2} e^{-nR_2} \left(1 + \frac{nR_1}{1!} + \frac{n^2R_1^2}{2!} + \ldots\right)$$

in integral (217). Owing to the fact that R_1 is odd we obtain:

$$I = e^{nf(-a)} \int_{-\varepsilon}^{+\varepsilon} e^{-n\frac{\sinh a}{2}x^4} e^{-nR_2} \left(1 + \frac{n^2R_1^2}{2!} + \frac{n^4R_1^4}{4!} + \ldots\right) dx. \tag{233}$$

Consider the expression:

$$e^{-nR_2}\left(1 + \frac{n^2R_1^2}{2!} + \frac{n^4R_1^4}{4!} + \ldots\right) =$$

$$= \left(1 - \frac{nR_2}{1!} + \frac{n^2R_2^2}{2!} - \ldots\right)\left(1 + \frac{n^2R_1^2}{2!} + \frac{n^4R_1^4}{4!} + \ldots\right). \tag{234}$$

Select the positive ε in such a way that the inequalities given below should be satisfied simultaneously:

$$\varepsilon < 1; \quad n^2R_1^2 < 5; \quad nR_2 < 1. \tag{235}$$

Using formula (232_1) we can see that these inequalities will be satisfied providing

$$\varepsilon < 1 \quad \text{and} \quad \varepsilon < \left(\frac{12}{n\cosh a}\right)^{\frac{1}{3}}. \tag{235_1}$$

If this is so then the terms of the series with alternate signs

$$S_1 = 1 - \frac{nR_2}{1!} + \frac{n_2R_2^2}{2!} - \ldots$$

will rapidly decrease. Hence

$$S_1 = 1 - nR_2 + a_1, \tag{236}$$

where

$$0 < a_1 < \frac{n^2R_2^2}{2}.$$

We find from (232_1):

$$1 - nR_2 = 1 - n\frac{\sinh a}{4!}x^4 - n\frac{\sinh a}{6!}x^6\left(1 + \frac{x^2}{7\cdot 8} + \frac{x^4}{7\cdot 8\cdot 9\cdot 10} + \ldots\right).$$

Comparing this with (236) we obtain:

$$S_1 = 1 - n\frac{\sinh a}{4!}x^4 + \delta_1, \tag{237}$$

where

$$|\delta_1| = \left|a_1 - \frac{n\sinh a}{6!}x^6\left(1 + \frac{x^2}{7\cdot 8} + \ldots\right)\right|$$

which is, in any case, smaller than the greater of the two quantities:

$$\frac{56}{55}\cdot\frac{n\sinh a}{6!}x^6 \quad \text{and} \quad \left(\frac{30}{29}\right)\frac{n^2\sinh^2 a}{2\cdot(4!)^2}x^6.$$

It can be shown that the latter quantity gives the greater of the errors shown below. We can therefore take it that

$$|\delta_1| < \left(\frac{30}{29}\right)^2 \frac{n^2 \sinh^2 a}{2 \cdot (4!)^2} x^8. \tag{238}$$

For the second series, from (234):

$$S_2 = 1 + \frac{n^2 R_1^2}{2!} + \frac{n^4 R_1^4}{4!} + \ldots$$

we have:

$$S_2 = 1 + \frac{n^2 R_1^2}{2} + \alpha_2, \tag{236_1}$$

where

$$0 < \alpha_2 < \frac{6}{5} \frac{n^4 R_1^4}{4!}.$$

We find from (232_1):

$$1 + \frac{n^2 R_1^2}{2} = 1 + \frac{n^2 \cosh^2 a}{2 \cdot (3!)^2} x^6 \left(1 + \frac{x^2}{4 \cdot 5} + \ldots\right)^2 =$$

$$= 1 + \frac{n^2 \cosh^2 a}{2 \cdot (3!)^2} x^6 (1 + 2r + r^2),$$

where

$$r = \frac{x^2}{4 \cdot 5} \left(1 + \frac{x^2}{6 \cdot 7} + \frac{x^4}{6 \cdot 7 \cdot 8 \cdot 9} + \ldots\right).$$

It is evident that $r < (42/41)\,(x^2/4.5)$ and

$$2r + r^2 < \frac{42}{41} \cdot \frac{x^2}{10} + \left(\frac{42}{41}\right)^2 \cdot \frac{x^4}{(20)^2} < \frac{11}{10} \cdot \frac{x^2}{10}.$$

We therefore obtain:

$$S_2 = 1 + \frac{n^2 R_1^2}{2} + \alpha_2 = 1 + \frac{n^2 \cosh^2 a}{2 \cdot (3!)^2} x^6 + \delta_2, \tag{237_1}$$

where $\delta_2 > 0$ and

$$\delta_2 = \alpha_2 + (2r + r^2) \frac{n^2 \cosh^2 a}{2 \cdot 6^2} x^6 <$$

$$< \left(\frac{20}{19}\right)^4 \frac{n^4 \cosh^4 a}{5 \cdot 6^3 \cdot 4!} x^{12} + \frac{11}{10} \cdot \frac{n^2 \cosh^2 a}{20 \cdot 6^2} x^8. \tag{238_1}$$

Bearing in mind the inequalities (235) and multiplying expressions (237) and (237_1) together we obtain for (232) the following formula:

$$e^{-nR_3} S_2 = 1 - n \frac{\sinh a}{4!} x^4 + \frac{n^2 \cosh^2 a}{2 \cdot (3!)^2} x^6 + \delta, \tag{239}$$

where

$$|\delta| < 5\,|\delta_1| + \delta_2, \tag{239_1}$$

and $|\delta_1|$ and $|\delta_2|$ satisfy the inequalities (238) and (238_1). Let us now substitute (239) in (233). We have:

$$I = e^{nf(-a)}\left\{\int_{-\infty}^{+\infty}\left(1 - n\frac{\sinh a}{4!}x^4 + n^2\frac{\cosh^2 a}{2\cdot 6^2}x^6\right)e^{-n\frac{\sinh a}{2}x^2}\,dx + \Delta_0 + \Delta_1\right\}, \tag{240}$$

where

$$\left.\begin{aligned}\Delta_0 &= 2\int_{\varepsilon}^{\infty}\left(1 - n\frac{\sinh a}{4!}x^4 + n^2\frac{\cosh^2 a}{2\cdot 6^2}x^6\right)e^{-n\frac{\sinh a}{2}x^2}\,dx,\\ \Delta_1 &= 2\int_0^{\varepsilon}\delta e^{-n\frac{\sinh a}{2}x^2}\,dx.\end{aligned}\right\} \tag{241}$$

The integrals in (240) have to be evaluated and the error (241) must be assessed. Subject the numbers n and ε which satisfy (235_1) to yet another most important condition:

$$\frac{n\sinh a}{2}\varepsilon^2 = N \gg 1. \tag{242}$$

When this is done the assessment of error is obtained in an elementary form and this gives us the formulae:

$$|\Delta_0| < \left(\frac{2}{n\sinh a}\right)^{\frac{1}{2}} e^{-N}\left[1 + \frac{3N^{\frac{5}{2}}\cosh^2 a}{18\,n\sinh^3 a}\right], \tag{243}$$

$$|\Delta_1| < \sqrt{\pi}\left(\frac{2}{n\sinh a}\right)^{\frac{5}{2}}\left(\frac{1}{8} + \frac{\cosh^2 a}{25\sinh^2 a} + \frac{\cosh^4 a}{8\sinh^2 a}\right), \tag{243_1}$$

where we assume in (243) that $N \geqslant 8$.

Evaluating the fundamental integral in (240) we arrive at the following final formula:

$$I = e^{nf(-a)}\sqrt{\pi}\left(\frac{2}{n\sinh a}\right)^{\frac{1}{2}}\left[1 - \frac{1}{8}\left(1 - \frac{5\cosh^2\sigma}{3\sinh^2 a}\right)\frac{1}{n\sinh a} + \omega'\right], \tag{244}$$

where

$$|\omega'| \leqslant \frac{e^{-N}}{\sqrt{\pi}}\left(1 + \frac{N^{\frac{5}{2}}\cosh^2 a}{6n\sinh^3 a}\right) + \left(\frac{2}{n\sinh a}\right)^2\left(\frac{1}{8} + \frac{\cosh^2 a}{25\sinh^2 a} + \frac{\cosh^4 a}{8\sinh^4 a}\right). \tag{245}$$

We obtained (244) and (235) on the assumption that the conditions (242) and (243_1) are satisfied. Suppose in what follows that

$$\varepsilon = \left(\frac{12}{n\cosh a}\right)^{\frac{1}{3}}. \tag{246}$$

We find

$$n \sinh a = \frac{N^3 \cosh^2 a}{18 \sinh^2 a} \tag{247}$$

and

$$\frac{N^{\frac{5}{2}} \cosh^2 a}{6n \sinh^3 a} = \frac{3}{\sqrt{N}}. \tag{247_1}$$

It can be seen from (245) that if we take $N = 8$ then the terms in the second line of (254) will take values which exceed the value of the term containing the factor e^{-n}.

Therefore, if we agree to take $N \geqslant 8$ the formula giving the error can be rewritten in a simpler form:

$$|\omega| < 2\left(\frac{2}{n \sinh a}\right)^2 \left(\frac{1}{8} + \frac{\cosh^2 a}{25 \sinh^2 a} + \frac{\cosh^4 a}{8 \sinh^4 a}\right). \tag{248}$$

This inequality will be valid providing that $N \geqslant 8$ and this can be written as follows:

$$n^{\frac{1}{3}} \sinh a \geqslant \frac{8}{\sqrt[3]{18}} \cosh^{\frac{2}{3}} a$$

or

$$n^{\frac{1}{3}} \sinh a > 3 \cosh^{\frac{2}{3}} a. \tag{249}$$

In the chapter devoted to Bessel functions we shall make use of the results obtained above but substitute

$$\cosh a = \frac{p}{z} \text{ and } n = z,$$

where p is a symbol and z is the argument of Bessel's function.

In this case $z \sinh a = \sqrt{p^2 - z^2}$ and the condition (249) acquires the form:

$$\sqrt{p^2 - z^2} \geqslant 3p^{\frac{2}{3}}. \tag{249_1}$$

If we take into account the fact that that $p > \sqrt{p^2 - z^2}$ then it becomes evident that condition (249_1) can be satisfied only when $p > p_0$, where $p_0 = 3p_0^{2/3}$, i.e. $p_0 = 27$. When making more accurate calculations this borderline can be somewhat lowered.

CHAPTER IV

FUNCTIONS OF SEVERAL VARIABLES AND MATRIX FUNCTIONS

81. Regular functions of several variables. The theory of analytic functions of several variables is similar in its fundamental concepts to the theory of functions of a single variable. However, when developed further it has several characteristic differences. We shall only deal with fundamental concepts of this theory and consider power series of several variables in greater detail. To simplify arguments we shall take two independent variables. The definitions and proofs will also be valid for a greater number of variables.

Let z_1 and z_2 be two complex variables and

$$f(z_1, z_2) \tag{1}$$

be a function of these variables. Let us suppose that the variable z_1 varies in a certain domain B_1 and the variable z_2 in a certain domain B_2. If, at the same time, function (1) is a single-valued and continuous function of its two arguments and, if for any arbitrary values of the variables in the above domains, the relationships

$$\frac{f(z_1 + \Delta z_1, z_2) - f(z_1, z_2)}{\Delta z_1} \text{ and } \frac{f(z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{\Delta z_2}$$

tend to a definite limit as the complex increments Δz_1 and Δz_2 tend to zero, then function (1) is said to be *regular or holomorphic as z_1 and z_2 vary* in the domains B_1 and B_2. The limits of the above functions give the individual derivatives of function (1) with respect to the variables z_1 and z_2:

$$\frac{\partial f(z_1, z_2)}{\partial z_1} \text{ and } \frac{\partial f(z_1, z_2)}{\partial z_2}.$$

82. The double integral and Cauchy's formula. Let l_1 and l_2 be two contours which lie in the domains B_1 and B_2 respectively. We construct a double integral which is obtained by the successive evaluation of

The variables z_1 and z_2 appear under the integral as parameters. Differentiating with respect to these parameters we can see that our function $f(z_1, z_2)$ has derivatives of all orders and we obtain expressions for these derivatives in the form of double integrals:

$$\frac{\partial^{p+q} f(z_1, z_2)}{\partial z_1^p \partial z_2^q} = -\frac{p!\, q!}{4\pi^2} \int_{l_1} dz_1 \int_{l_2} \frac{f(z_1', z_2')}{(z_1' - z_1)^{p+1} (z_2' - z_2)^{q+1}} dz_2'. \tag{3}$$

As for functions of one complex variable the principle of the modulus follows from Cauchy's formula: if the function $f(z_1, z_2)$ is regular in the closed domains B_1 and B_2 and $|f(z_1', z_2')| \leqslant M$, when z_1' belongs to l_1 and z_2' belongs to l_2, then $|f(z_1, z_2)| \leqslant M$ when z_1 belongs to the closed domain B_1 and z_2 to the closed domain B_2.

The theorem of Weierstrass is proved in exactly the same way as for functions of one complex variable: if the terms of the series

$$\sum_{k=1}^{\infty} \varphi_k(z_1, z_2)$$

are regular functions in the closed domains B_1 and B_2 and the series converges uniformly in these domains, then the sum of the series is a regular function in the above domains and the series can be differentiated term by term with respect to z_1 and z_2 as often as we please, providing z_1 lies inside B_1 and z_2 inside B_2.

The series remains uniformly convergent in any closed domains B_1' and B_2', which lie inside B_1 and B_2.

All the above statements can be extended to include functions of several independent variables. Below we shall only consider power series.

83. Power series. A power series of two independent variables z_1 and z_2 with centres at b_1 and b_2 can be written as follows:

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{pq} (z_1 - b_1)^p (z_2 - b_2)^q, \tag{4}$$

where the variables of summation p and q are independent of each other; they take all positive integral values, starting with zero. Series (4) is a double series.

We dealt with such series earlier in [I, 142] when the terms of the series were real.

Let us suppose that the series, constructed from the moduli of the terms of the series

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|a_{pq}||z_1-b_1|^p|z_2-b_2|^q. \tag{5}$$

converges.

In such a case, as we know from [11], the series, constructed from real and imaginary parts of series (4) will be absolutely convergent and the sums of these double series with real terms will not depend on the order of summation. Consequently in this case, i.e. when series (5) converges, series (4) will also converge and its sum will be fully determined for any order of summation. In future we shall only consider the case when series (5) converges, i.e. when series (4) converges absolutely.

Abel's theorem can readily be established in the same way as in [13]. Suppose that the series (4) is absolutely convergent when $z_1 = a_1$ and $z_2 = a_2$. It follows directly that the moduli of the terms of this series remain bounded for the above values of the independent variables, i.e. a number M exists such that for any arbitrary numbers p and q the inequality holds:

$$|a_{pq}||a_1-b_1|^p|a_2-b_2|^q < M$$

i.e.

$$|a_{pq}| < \frac{M}{|a_1-b_1|^p|a_2-b_2|^q}. \tag{6}$$

Consider now two circles K_1 and K_2 of the variables z_1 and z_2:

$$|z_1-b_1| < |a_1-b_1|; \quad |z_2-b_2| < |a_2-b_2|. \tag{7}$$

The first of these circles includes points z_1 which are nearer to b_1 than to a_1 and the second includes points z_2 which are nearer to b_2 than to a_2.

Take a certain point z_1 in the circle K_1 and a certain point z_2 in the circle K_2, i.e.

$$|z_1-b_1| = q_1|a_1-b_1| \text{ and } |z_2-b_2| = q_2|a_2-b_2|,$$

where $0 < q_1$ and $q_2 < 1$. At the same time, using (6), we obtain the following inequality for the moduli of the terms of series (4):

$$|a_{pq}||z_1-b_1|^p|z_2-b_2|^q < Mq_1^pq_2^q. \tag{8}$$

It can readily be seen that the double series with positive terms

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}Mq_1^pq_2^q$$

will be convergent. In fact this series is obtained by multiplying two series with positive terms [I, 138]:

$$M(1 + q_1 + q_1^2 + \ldots) \text{ and } (1 + q_2 + q_2^2 + \ldots),$$

and its sum is

$$\frac{M}{(1 - q_1)(1 - q_2)}.$$

Hence in the case under consideration, series (5) will be convergent and series (4) absolutely convergent. It also follows from the inequality (8) that series (4) will be uniformly convergent in the circles K_1' and K_2' with centres at b_1 and b_2 and radii ϱ_1 and ϱ_2, which are less than the radii of K_1 and K_2. In this proof we did not use the absolute convergence of series (4) when $z_1 = a_1$ and $z_2 = a_2$, but we used the inequality

$$|a_{pq}(a_1 - b_1)^p (a_2 - b_2)^q| \leqslant M,$$

i.e. we used the fact that the terms of this series are bounded when $z_1 = a_1$ and $z_2 = a_2$.

We thus arrive at the following conclusion: *if the moduli of all terms of series (4) are bounded when $z_1 = a_1$ and $z_2 = a_2$ by one and the same number, then series (4) will be absolutely convergent inside the circles (7) and uniformly convergent inside the circles:*

$$|z_1 - b_1| \leqslant (1 - \varepsilon)|a_1 - b_1|;\ |z_2 - b_2| \leqslant (1 - \varepsilon)|a_2 - b_2|,$$

where ε is any small fixed positive number.

Notice that if, when $z_1 = a_1$ and $z_2 = a_2$, series (4) converges (not necessarily absolutely) for any arbitrary order of summation, then its terms tend to zero as they move away from the origin; hence their moduli are bounded by one and the same number and inside the circles (7) the series will be absolutely convergent.

What was said above brings us to the concept of the radius of convergence of series (4) in exactly the same way as in [13].

In this case there exist two positive numbers R_1 and R_2 which show that series (4) converges absolutely when $|z_1 - b_1| < R_1$ and $|z_2 - b_2| < R_2$, and that it diverges when $|z_1 - b_1| > R_1$ and $|z_2 - b_2| > R_2$. Notice that in this case, the region of absolute convergence of series (4) is simultaneously determined by two radii of convergence R_1 and R_2 and these radii, generally speaking, cannot be determined separately, for the value of one depends on the other. If R_1 is made smaller then it is very probable that R_2 can be made

bigger. In other words in this case we can speak only of *combined radii of convergence* R_1 and R_2 or, which comes to the same thing, of *combined circles of convergence.* Consider, for example, the power series

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(p+q)!}{p!\,q!}z_1^p z_2^q. \tag{9}$$

Series (5) will become

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(p+q)!}{p!\,q!}|z_1|^p|z_2|^q. \tag{10}$$

Group together all terms of this series in which the sum $p+q$ is equal to a given number s. According to the binomial theorem the sum of these terms will be

$$(|z_1|+|z_2|)^s,$$

and series (10) can be rewritten in the form

$$\sum_{s=0}^{\infty}(|z_1|+|z_2|)^s,$$

from which follows directly that it will converge when, and only when $|z_1|+|z_2|<1$. Hence for series (9) the combined radii of convergence are determined by the equation $R_1+R_2=1$. If we take, for example, $R_1=\theta$, where $0<\theta<1$, we shall have $R_2=1-\theta$.

Consider as a second example the series

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}z_1^p z_2^q.$$

It can readily be seen that here the necessary and sufficient condition for convergence is expressed by the inequalities $|z_1|<1$ and $|z_2|<1$, i.e. in this case $R_1=1$ and $R_2=1$ and the radii of convergence are determined separately.

Consider series (4). As a result of uniform convergence and the theorem of Weierstrass the sum of series (4) inside the combined circles of convergence is a regular function $f(z_1, z_2)$ of two variables. As in [13], we can differentiate series (4) with respect to both variables as often as we please inside the circles of convergence. This differentiation does not alter the circles of convergence.

Differentiating several times and assuming subsequently that $z_1 = b_1$ and $z_2 = b_2$ we obtain, as in [14], the following expressions for the coefficients of the terms of the series:

$$a_{pq} = \frac{1}{p!\,q!} \frac{\partial^{p+q}(z_1, z_2)}{\partial z_1^p \partial z_2^q}\Big|_{z_1=b_1;\ z_2=b_2}, \tag{11}$$

i.e. series (4) is the Taylor's series for the function $f(z_1, z_2)$.

If R_1 and R_2 are the combined radii of convergence of series (4), then this series will be absolutely and uniformly convergent when $|z_1 - b_1| \leqslant R_1 - \varepsilon$ and $|z_2 - b_2| \leqslant R_2 - \varepsilon$, where ε is any small positive fixed number. At the same time, from (3) and (11), we have the following inequality for the coefficients of the series:

$$|a_{pq}| < \frac{M}{(R_1 - \varepsilon)^p (R_2 - \varepsilon)^q}, \tag{12}$$

where M is a positive constant, the value of which must depend on the choice of ε.

Substituting the coefficients a_{pq} in series (4) by positive numbers which exceed the moduli $|a_{pq}|$, we obtain a power series

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{M}{R_1'^p R_2'^q} (z_1 - b_1)^p (z_2 - b_2)^q \quad (R_1' = R_1 - \varepsilon;\ R_2' = R_2 - \varepsilon), \tag{13}$$

usually known as *dominating or major for series* (4). It can readily be seen that the sum of series (13) is equal to

$$\frac{M}{\left(1 - \frac{z_1 - b_1}{R_1'}\right)\left(1 - \frac{z_2 - b_2}{R_2'}\right)} \tag{14}$$

and this latter function is known as the *dominating function of series* (4). Coefficients in the expansion of powers of $(z_1 - b_1)$ and $(z_2 - b_2)$ are positive and greater than the moduli of the coefficients of a_{pq}.

The result [14] can readily be generalized to include two variables. Let $f(z_1, z_2)$ be a function, regular in the circles $|z_1 - b_1| \leqslant R_1$ and $|z_2 - b_2| \leqslant R_2$ with centres at b_1 and b_2 and let l_1 and l_2 be the circumferences of these circles. On fixing any two points z_1 and z_2 inside the above circles we can obtain Cauchy's formula

$$f(z_1, z_2) = -\frac{1}{4\pi^2} \int_{l_1} dz_1' \int_{l_2} \frac{f(z_1', z_2')}{(z_1' - z_1)(z_2' - z_2)} dz_2'. \tag{15}$$

Consider the rational fraction

$$\frac{1}{(z_1' - z_1)(z_2' - z_2)}.$$

As in [14] we can expand it into a series of powers of the differences $(z_1 - b_1)$ and $(z_2 - b_2)$

$$\frac{1}{(z_1' - z_1)(z_2' - z_2)} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(z_1 - b_1)^p (z_2 - b_2)^q}{(z_1' - b_1)^{p+1} (z_2' - b_2)^{q+1}};$$

this converges uniformly with respect to z_1' and z_2', providing these points lie on the circumferences l_1 and l_2. Substituting this expression in formula (15) and integrating term by term, we can obtain our function $f(z_1, z_2)$ in the above circles in the form of a power series

$$f(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{pq} (z_1 - b_1)^p (z_2 - b_2)^q. \tag{16}$$

The coefficients of this series are determined by the formulae

$$a_{pq} = -\frac{1}{4\pi^2} \int_{l_1} dz_1' \int_{l_2} \frac{f(z_1', z_2')}{(z_1' - b_1)^{p+1} (z_2' - b_2)^{q+1}} dz_2' =$$

$$= \frac{1}{p!\, q!} \frac{\partial^{p+q} f(z_1, z_2)}{\partial z_1^p \partial z_2^q} \Big|_{z_1 = b_1;\ z_2 = b_2}. \tag{17}$$

Thus any function $f(z_1, z_2)$ which is regular inside two circles can be expanded inside these circles into a power series. As in [14] it can readily be seen that this expansion is *unique*, for the coefficients must be determined by formula (11).

In series (4) we can group together terms which are equal with respect to the differences $(z_1 - b_1)$ and $(z_2 - b_2)$, i.e. we can write series (4) in the form

$$\sum_{s=0}^{\infty} \sum_{p+q=s} a_{pq} (z_1 - b_1)^p (z_2 - b_2)^q, \tag{18}$$

where the inside finite sum includes those values of p and q the sum of which is s. Formula (18) gives the function $f(z_1, z_2)$ inside the circles of convergence in the form homogeneous polynomials with respect to $(z_1 - b_1)$ and $(z_2 - b_2)$. Suppose now, conversely, that series (18) is given in the form of homogeneous polynomials and that it converges uniformly in certain circles $|z_1 - b_1| \leqslant R_1$ and $|z_2 - b_2| \leqslant R_2$. According to the theorem of Weierstrass the sum of this series will be a regular function of $f(z_1, z_2)$ in these circles.

We can also differentiate our series (18) as often as we please with respect to both variables. Differentiating and assuming subsequently that $z_1 = b_1$ and $z_2 = b_2$ we obtain formula (11) for the coefficients a_{pq}, i.e. these coefficients will be the coefficients of

the Taylor's series for the function $f(z_1, z_2)$ and we can rewrite series (18) in the form of the double series (4); this series will converge absolutely and uniformly inside the above circles. We can therefore say that if a series given by homogeneous polynomials converges uniformly inside certain circles then this series can be simply rewritten in the form of a double series, which is the usual power series and which converges absolutely inside the above circles.

If we separate the real and imaginary parts of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then in a four dimensional space with coordinates (x_1, y_1, x_2, y_2) the region of uniform convergence of series (18) given by homogeneous polynomials, can be wider than that for series (4).

In example (9) series (18) has the form:

$$\sum_{s=0}^{\infty} (z_1 + z_2)^s,$$

and the domain of its uniform convergence is given by the inequality

$$|z_1 + z_2| < 1,$$

i.e.

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 < 1. \tag{19}$$

For series (9) itself we must have $R_1 + R_2 = 1$; its domain of convergence is determined by the inequality:

$$|z_1| + |z_2| < 1,$$

i.e.

$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} < 1,$$

or

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} < 1. \tag{20}$$

The inequality (19) determines a wider domain than (20), i.e. if the numbers x_k and y_k satisfy (20) they will also satisfy (19) and vice versa. In fact, from the inequality

$$(x_1x_2 + y_1y_2)^2 \leqslant (x_1^2 + y_1^2)(x_2^2 + y_2^2)$$

it follows directly that the left-hand side of inequality (19) is less than or equal to the left-hand side of inequality (20).

All the above arguments can be extended to include n variables, when the domain of absolute and uniform convergence of the series will be a set of n circles.

84. Analytic continuation. A function of two variables $f(z_1, z_2)$, determined by a power series (4) inside the circles of convergence, can be regular in a wider domain; as for a function of a single variable the question of analytic continuation of this function arises. Here, as in the case of a single variable [18], the fundamental proposition that if two functions are regular in certain domains, they coincide in these domains, if at any two points $z_1 = b_1$ and $z_2 = b_2$ in these domains, the values of the above functions and of all their derivatives are the same.

Let us now consider once again the function $f(z_1, z_2)$ which is given by a power series. Let $z_1 = c_1$ and $z_2 = c_2$ be two points belonging to the circles of convergence. Using series (4) we can determine the values of the derivatives

$$\left.\frac{\partial^{p+q} f(z_1, z_2)}{\partial z_1^p \partial z_2^q}\right|_{z_1=c_1;\ z_2=c_2}$$

and construct the Taylor's series for the function $f(z_1, z_2)$ in integral powers of the differences $(z_1 - c_1)$ and $(z_2 - c_2)$

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a'_{pq}(z_1 - c_1)^p (z_2 - c_2)^q. \tag{21}$$

It can readily be shown that such a reconstruction of the power series is equivalent to making the following assumptions in series (4):

$$(z_1 - b_1)^p = [(z_1 - c_1) + (c_1 - b_1)]^p,$$
$$(z_2 - b_2)^q = [(z_2 - c_2) + (c_2 - b_2)]^q.$$

We expand the brackets using Newton's binomial theorem and group together terms with the same powers of $(z_1 - c_1)$ and $(z_2 - c_2)$. Series (21) will certainly converge and its sum inside the circles with centres at c_1 and c_2, which belong to the circles of convergence of series (4), will be $f(z_1, z_2)$. It may happen, however, that these new circles fall outside the old circles of convergence. In that case we obtain our function $f(z_1, z_2)$ in a wider domain, i.e. we extend the domain in which our regular function can exist. Applying the above process of *analytic continuation by means of circles* several times we can, in certain cases, extend the domain in which our regular function can exist; the whole set of values which we obtain in this process will give us the analytic function obtained from its element as given by series (4). We shall not consider here in greater detail

analytic continuation with regard to singularities. All that has been said holds for any number of independent variables. Notice that the paths L_1 and L_2, along which z_1 and z_2 vary in the course of the analytic continuation of $f(z_1, z_2)$, do not determine the result of this continuation. It is important to know exactly the way in which z_1 and z_2 move with respect to each other along L_1 and L_2. This is all we are going to say about the general theory of functions of several complex variables. At present this branch of the theory of functions has developed widely. A more detailed description of fundamental facts in this field can be found in the *Course of Mathematical Analysis* by Hurse. There is also a special book by B. A. Fuchs *The Theory of Analytical Functions of Several Complex Variables* (1948), where extensive bibliographical references are given.

85. Matrix functions. Preliminary propositions. We shall now consider the case when the argument of the function consists of one or several matrices; we shall start with a single matrix. Earlier in [III$_1$, 44] we considered the simplest cases, viz. the polynomial and the rational function of one matrix. Before going on to consider more complicated functions we shall establish some fundamental propositions. In future we shall denote by n the order of the matrix.

Suppose that we are given an infinite sequence of matrices

$$X_1, X_2 \dots$$

We say that the limit of this sequence is the matrix X if for any arbitrary values of the symbols i and k:

$$\lim_{m\to\infty} \{X_m\}_{ik} = \{X\}_{ik}, \tag{22}$$

i.e. the elements of the matrix X_m have as their limits the corresponding elements of the matrix X. At the same time we shall always assume that the matrices under consideration are of the same order.

We shall now introduce certain new symbols which will be useful in future. The symbol $\| a \|$ will denote a matrix, all elements of which are equal to the number a. The symbol $| X |$ will denote a matrix the elements of which are equal to the moduli of the elements of the matrix X, i.e.

$$\{| X |\}_{ik} = |\{X\}_{ik}|. \tag{23}$$

If a certain matrix Y has positive elements which are greater than the elements of the matrix $|X|$, we shall write this down in the form of an inequality

$$|X| < Y.$$

In other words, this inequality is equivalent to the following system of n^2 inequalities:

$$|\{X\}_{ik}| < \{Y\}_{ik} \qquad (i, k = 1, 2, \ldots, n).$$

Consider the infinite series the terms of which are the matrices:

$$Z_1 + Z_2 + \ldots$$

This series is said to be convergent if the sum of its first n terms (matrices) tends to a definite limiting matrix Z. Such a matrix is the sum of the series

$$Z = Z_1 + Z_2 + \ldots \tag{24}$$

This equation (24) is evidently equivalent to the following n^2 equations:

$$\{Z\}_{ik} = \{Z_1\}_{ik} + \{Z_2\}_{ik} + \ldots \qquad (i, k = 1, 2, \ldots, n). \tag{25}$$

Let us suppose that in the neighbourhood of the matrix A lie all the matrices X which satisfy the condition

$$|X - A| < \|\varrho\|, \tag{26}$$

where ϱ is a given positive number. The inequality (26) is equivalent to the following n^2 inequalities:

$$|\{X - A\}_{ik}| < \varrho.$$

Of fundamental importance in the determination of functions of matrices in future will be power series of these matrices and we shall now consider these series.

86. Power series of one matrix. A power series of a single matrix has the form:

$$a_0 + a_1(X - a) + a_2(X - a)^2 + \ldots, \tag{27}$$

where a_k and a are given numbers. To simplify our notation we shall assume in future that $a = 0$. Instead of series (27) we shall then have a series

$$a_0 + a_1 X + a_2 X^2 + \ldots \tag{28}$$

In accordance with the multiplication law of matrices we have:

$$\{X^2\}_{ik} = \sum_{s=1}^{n} \{X\}_{is} \{X\}_{sk}$$

and, in general

$$\{X^m\}_{ik} = \sum_{j_1, j_2, \ldots, j_{m-1}} \{X\}_{ij_1} \{X\}_{j_1 j_2} \cdots \{X\}_{j_{m-1} j_{m-1}} \{X\}_{j_{m-1} k},$$

where the summation includes all symbols j_k independently, from 1 to n. Hence the elements of the matrix given by the sum of the series (28) can be expressed by the series

$$a_0 \delta_{ik} + \sum_{m=1}^{\infty} a_m \sum_{j_1, j_2, \ldots j_{m-1}}' \{X\}_{ij_1} \{X\}_{j_1 j_2} \cdots \{X\}_{j_{m-1} k}, \tag{29}$$

where δ_{ik} denotes a number determined by the formula

$$\delta_{ik} = \begin{cases} 0 \text{ when } i \neq k \\ 1 \text{ when } i = k. \end{cases} \tag{30}$$

The last circumstance is directly due to the fact that the constant term of series (28) is the number a_0, i.e. it is a diagonal matrix, all diagonal elements of which are equal to a_0. Formula (29) shows that series (28) is equivalent to n^2 power series of a special kind with n^2 variables $\{X\}_{ik}$. Notice that when $m = 1$ the terms of the sum in formula (29) have the form $a_1 \{X\}_{ik}$ and the inside sum disappears.

Consider now the problem of the convergence of series (28). We shall consider, first of all, its absolute convergence, i.e. together with series (28) we shall also consider the series

$$|a_0| + |a_1| |X| + |a_2| |X|^2 + \ldots \tag{31}$$

or its corresponding n^2 series

$$|a_0| \delta_{ik} + \sum_{m=1}^{\infty} |a_m| \sum_{j_1, j_2, \ldots, j_{m-1}} \{|X|\}_{ij_1} \{|X|\}_{j_1 j_2} \cdots \{|X|\}_{j_{m-1} k}. \tag{32}$$

If these series converge then series (29) will converge all the more, i.e. the convergence of series (31) guarantees the convergence of series (28) and in this case series (28) will be absolutely convergent. In agreement with the definition of the matrix $|X|$ we have:

$$\{|X|\}_{ik} = |\{X\}_{ik}|,$$

i.e. expression (32) is obtained from (29) by replacing all numbers by their moduli.

We shall now explain the sufficient condition for absolute convergence of series (28). Construct a power series for the complex variable

$$a_0 + a_1 z + a_2 z^2 + \dots, \tag{33}$$

and let the radius of convergence of this series be equal to $n\varrho$, where n is the order of our matrix and ϱ is a positive number. We know from [14] that we have in this case the following inequality for the coefficients of series (33):

$$|a_m| \leqslant \frac{M}{(n\varrho - \varepsilon)^m}, \tag{34}$$

where ε is any small fixed positive number and M is a positive number which depends on the choice of ε. Let us now take the matrix $\| b \|$, where b is a certain number, and determine its positive integral powers

$$\{ \| b \|^2 \}_{ik} = bb + bb + \dots + bb = nb^2, \qquad \| b \|^2 = \| nb^2 \|,$$

and generally

$$\| b \|^m = \| n^{m-1} b^m \|. \tag{35}$$

Let us now suppose that $b = \varrho_1 > 0$ and take a certain matrix X which satisfies the condition $|X| < \| \varrho_1 \|$. In this case we obviously have:

$$|X|^m < \| \varrho_1 \|^m, \text{ i.e. } |X|^m < \| n^{m-1} \varrho_1^m \|.$$

As a result of the inequality (34)

$$|a_m| |X^m| < \frac{M}{n} \left\| \left(\frac{n\varrho_1}{n\varrho - \varepsilon} \right)^m \right\|.$$

If $\varrho_1 < \varrho$, then by taking ε sufficiently small we have

$$0 < \frac{n\varrho_1}{n\varrho - \varepsilon} < 1,$$

and in this case series (31) will be convergent and series (28) will be absolutely convergent. If the radius of convergence of series (33) is infinite then it is said that the sum of this series is an integral function of z. It follows from what was said above that in this case series (28) will also be absolutely convergent for any matrix X. We thus obtain the following theorem.

THEOREM. *If the radius of convergence of series* (33) *is equal to* $n\varrho$, *then series* (28) *will be absolutely convergent for all matrices situated in the neighbourhood of the origin.*

$$|X| < \| \varrho \|. \tag{36}$$

If series (33) *determines an integral function, then series* (28) *will be absolutely convergent for all matrices.*

When considering the absolute convergency of series (28) in the domain (36) we can say that the sum of this series $f(X)$ will be a regular function in the above domain.

Consider, for example, the exponential function of the matrix:

$$e^X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \dots \qquad (37_1)$$

The corresponding power series (33) has an infinite radius of convergence and, consequently, series (37_1) will be absolutely convergent for any matrix X, or, as is generally said, it will be an integral function of this matrix.

Consider also the exponential function for any base

$$a^X = e^{X \log a} = 1 + \frac{X \log a}{1!} + \frac{X^2 \log^2 a}{2!} + \dots, \qquad (37_2)$$

where $\log a$ is a certain fixed value of the logarithm of the complex number a. Function (37_2) is also an integral function of the matrix X. We shall now explain the uniqueness of an expansion into a power series. Suppose that we are given two power series

$$\sum_{m=0}^{\infty} a_m X^m \text{ and } \sum_{m=0}^{\infty} b_m X^m,$$

each one of which is absolutely convergent in the neighbourhood of (36); we also suppose that in this neighbourhood the sums of the series coincide, i.e.

$$\sum_{m=0}^{\infty} a_m X^m = \sum_{m=0}^{\infty} a'_m X^m.$$

We shall now prove that in this case the coefficients a'_m must coincide with the coefficients a_m. Notice that condition (36) is also satisfied by the diagonal matrices

$$X = z = [z, z, \dots, z],$$

in which $|z| < \varrho$. Hence the above assumption gives:

$$\sum_{m=0}^{\infty} a_m z^m = \sum_{m=0}^{\infty} a'_m z^m \qquad (|z| < \varrho).$$

We know that the expansion of a function of a complex variable into a power series is unique in any particular circle and, consequently, $a'_m = a_m$. We thus obtain the following theorem.

UNIQUENESS THEOREM. *If two power series, which are absolutely convergent in a certain neighbourhood* (36), *have in this neighbourhood the same sum, then all coefficients of these two series will be the same.*

If we use the self-explanatory formula

$$(SXS^{-1})^k = SX^kS^{-1},$$

then for the function $f(X)$, given by the power series (28) or (27) we have, as in [III$_1$, 44], the equation:

$$f(SXS^{-1}) = Sf(X)S^{-1}.$$

87. Multiplication of power series. Conversion of power series. Suppose that we are given two power series

$$f_1(X) = \sum_{m=0}^{\infty} a_m X^m \text{ and } f_2(X) = \sum_{m=0}^{\infty} b_m X^m,$$

which are absolutely convergent in the neighbourhood (36). We can construct a new matrix, which is obtained by multiplying their sums

$$Y = f_2(X) f_1(X).$$

The elements of this matrix will be determined by the formulae:

$$\{Y\}_{ik} = \sum_{s=1}^{n} \{f_2(X)\}_{is} \{f_1(X)\}_{sk}, \tag{38}$$

where

$$\{f_1(X)\}_{sk} = a_0\delta_{sk} + \sum_{m=1}^{\infty} a_m \sum_{j_1, \ldots, j_{m-1}} \{X\}_{sj_1} \{X\}_{j_1j_2} \cdots \{X\}_{j_{m-1}k},$$

$$\{f_2(X)\}_{is} = b_0\delta_{is} + \sum_{m=1}^{\infty} b_m \sum_{j_1, \ldots, j_{m-1}} \{X\}_{ij_1} \{X\}_{j_1j_2} \cdots \{X\}_{j_{m-1}s}.$$

Owing to the absolute convergence of the above two series we can multiply them term by term, so that for elements of the matrix Y we have, from (38):

$$\{Y\}_{ik} = a_0b_0\delta_{ik} + \sum_{m=1}^{\infty} (a_0b_m + a_1b_{m-1} + \cdots + a_mb_0) \sum_{j_1, \ldots, j_{m-1}} \{X\}_{ij_1} \{X\}_{j_1j_2} \cdots \{X\}_{j_{m-1}k},$$

and the matrix itself may be written in the form:

$$Y = a_0 b_0 \delta_{ik} + \sum_{m=1}^{\infty} (a_0 b_m + a_1 b_{m-1} + \ldots + a_m b_0) X^m.$$

It follows from this that *absolutely convergent power series of matrices can be multiplied like the power series of numerical variables and the product does not depend on the relative position of factors.*

We shall now construct a function which is the inverse of the given function $f(X)$, determined by a certain power series

$$Y = f(X) = a_0 + a_1 X + a_2 X^2 + \ldots, \tag{39}$$

and we assume that in this series the coefficient a_1 is not zero.

Consider the power series of the usual complex variable

$$w = a_0 + a_1 z + a_2 z^2 + \ldots \tag{40}$$

We know that there is only one power series which satisfies the condition $a_1 \neq 0$:

$$z = c_1 (w - a_0) + c_2 (w - a_0)^2 + \ldots \tag{41}$$

which determines the inverse function of function (40) in a certain neighbourhood $| w - a_0 | < n\varrho$. If we substitute series (41) in the right-hand side of formula (40) we obtain:

$$w = a_0 + a_1 \sum_{k=1}^{\infty} c_k (w - a_0)^k + a_2 \left[\sum_{k=1}^{\infty} c_k (w - a_0)^k \right]^2 + \ldots;$$

if we now raise the series to their corresponding powers according to the multiplication law for series, and isolate similar terms by grouping together terms with equal powers of $(w - a_0)$, we arrive at the identity $w = w$. If in all the above calculations we replace z by the matrix X and w by the matrix Y, then all calculations involving power series of matrices in powers of the differences $(Y - a_0)$ will be the same as those in operations with power series of numerical variables $(w - a_0)$ and, consequently, the results will be the same, i.e. *when $a_1 \neq 0$ the power series* (39), *which is determined in the neighbourhood $X = 0$, permits a unique conversion of the kind*

$$X = \sum_{k=1}^{\infty} c_k (Y - a_0)^k, \tag{42}$$

and this latter series will be absolutely convergent in the neighbourhood

$$| Y - a_0 | < \| \varrho \|. \tag{43}$$

This neighbourhood must be determined by the radius of convergence of series (41).

The coincidence of the above formal operations between power series containing $(Y - a_0)^k$ and power series containing $(w - a_0)^k$, is due to the fact that the above matrix series contains numerical elements and one power matrix $Y - a_0$. In fact numbers can be commuted in any matrix and powers of one and the same matrix can also be commuted. This gives us the above coincidence of formal operations. For example, for any whole positive k we can apply to the term

$$(Y - a_0)^k$$

Newton's binomial expansion. But, in general, this formula can no longer be applied to the term

$$(U_1 + U_2)^k,$$

where U_1 and U_2 are different matrices.

We shall apply the above remarks to the series

$$w = e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots$$

The inversion of this series gives $\log w$ and gives us, as we know, the power series

$$\log w = \log\left[1 + (w - 1)\right] = \frac{w-1}{1} - \frac{(w-1)^2}{2} + \dots,$$

which converges in the circle $|w - 1| < 1$.

Thus the inversion of the exponential function

$$Y = e^X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \dots$$

enables us to determine the logarithm of the matrix in the form of the power series

$$\log Y = \frac{Y-1}{1} - \frac{(Y-1)^2}{2} + \dots, \tag{44}$$

which converges absolutely in the region

$$|Y - 1| < \left\| \frac{1}{n} \right\|. \tag{45}$$

The equation of the matrix

$$e^X = Y \tag{46}$$

with respect to X has an infinite number of solutions for a given Y. Series (44) gives one of the solutions of this equation, viz. it gives that solution which is a regular function of Y in the neighbourhood of a unitary matrix and becomes a zero matrix when $Y = 1$. The problem of the remaining solutions of the equation, both in the neighbourhood of a unitary matrix and outside that neighbourhood is connected with the analytic continuation of series (44) or, which comes to the same thing, with the analytic continuation of n^2 power series, to which series (44) is equivalent. We shall deal with this problem later.

Let us now determine the power function of a matrix. It can be determined by means of the logarithm of a matrix as follows:

$$X^a = e^{a \log X}. \tag{47}$$

If we have a numerical variable z

$$z^a = e^{a \log z},$$

then substituting $a \log z$ in the expansion of the exponential function

$$e^{a \log z} = 1 + \frac{a \log z}{1} + \frac{a^2 \log^2 z}{2!} + \dots$$

and substituting

$$\log z = \log [1 + (z-1)] = \frac{z-1}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \dots,$$

we obtain a power series in the form:

$$z^a = [1 + (z-1)]^a = 1 + \frac{a}{1!}(z-1) + \frac{a(a-1)}{2!}(z-1)^2 + \dots,$$

which converges when $|z - 1| < 1$. Bearing in mind the coincidence of formal operations in power series with one matrix described above we obtain:

$$X^a = e^{a \log X} = 1 + \frac{a}{1!}(X-1) + \frac{a(a-1)}{2!}(X-1)^2 + \dots, \tag{48}$$

and this expansion will be absolutely convergent in the region

$$|X - 1| < \left\| \frac{1}{n} \right\|. \tag{49}$$

88. Further investigations of convergence. We have already mentioned above that the power series (28) is equivalent to the n^2 series (29) of the variables $\{X\}_{ik}$: consider the inner sum of series (29):

$$\sum_{j_1, \dots, j_{m-1}} \{X\}_{ij_1} \{X\}_{j_1 j_2} \dots \{X\}_{j_{m-1} k}. \tag{50}$$

Grouping together similar terms in the above sum we can obtain series (29) in the form of the usual power series of n^2 variables $\{X\}_{ik}$. If we replace all terms in the sum (50) by their moduli and the numbers a_m by $|a_m|$, then this is evidently equivalent to substituting all terms by their moduli in the above power series. It follows that *if the series* (29) *are converted into the usual power series of the variables* $\{X\}_{ik}$ *then the absolute convergence of these series is equivalent to the absolute convergence of series* (32), i.e. *it is equivalent to the absolute convergence of series* (28).

In general, the convergence of series (28) in the widest sense of this word implies the existence of a limit in the form of a succession of matrices

$$a_0 + \sum_{m=1}^{l} a_m X^m \tag{51}$$

where l increases indefinitely. The addition of a term $m = l + 1$ to the sum (51) is equivalent to adding to the sums

$$a_0\delta_{ik} + \sum_{m=1}^{l} a_m \sum_{j_1, \ldots, j_{m-1}} \{X\}_{ij_1}\{X\}_{j_1j_2}\cdots\{X\}_{j_{m-1}k} \qquad (i,\ k = 1,\ 2,\ \ldots,\ n) \tag{52}$$

a homogeneous polynomial with respect to $\{X\}_{ik}$:

$$a_{l+1} \sum_{j_1, \ldots, j_l} \{X\}_{ij_1}\{X\}_{j_1j_2}\cdots\{X\}_{j_lk} \tag{53}$$

of the power $(l + 1)$.

Hence the convergence of series (28) in the above sense is equivalent to the convergence of the n^2 series (29) in which all terms of the type (53) are collected together. We shall consider, first of all, the convergence of series (28) in a special domain, determined by the inequality

$$|X| < A, \tag{54}$$

where A is a given matrix with positive elements. The inequality (54) is equivalent to n^2 inequalities

$$|\{X\}_{ik}| < \{A\}_{ik}, \tag{55}$$

which determine n^2 concentric circles with centre the origin, for the complex variables $\{X\}_{ik}$. We can therefore assume that series (28) converges in the domain (54). Let θ be any positive number less

than unity. It is given that series (28) will converge when $X = \theta A$, i.e. n^2 equations of the type will converge

$$a_0\delta_{ik} + \sum_{m=1}^{\infty} \theta^m a_m \sum_{j_1, \ldots, j_{m-1}} \{A\}_{ij_1}\{A\}_{j_1j_2} \ldots \{A\}_{j_{m-1}k}.$$

The latter series can be regarded as power series of θ and we can therefore say that their convergence will be absolute, i.e. the following series will also converge:

$$|a_0\delta_{ik}| + \sum_{m=1}^{\infty} \theta^m |a_m| \sum_{j_1, \ldots, j_{m-1}} \{A\}_{ij_1}\{A\}_{j_1j_2} \ldots \{A\}_{j_{m-1}k},$$

all terms of which are positive. We thus see that the series (29) will be absolutely convergent for the matrix θA. They will be all the more convergent for all matrices which satisfy the condition $|X| < \theta A$. Remembering that we can choose θ as near to unity as we like, we can say that the series (29) will be absolutely convergent for all matrices in the domain (54). At the same time the series (28) will also be absolutely convergent. We thus have the following theorem.

THEOREM. *If series* (28) *converges in a domain of the type* (54) *then it will converge absolutely in this domain or, in other words, the* n^2 *power series* (29) *will converge absolutely in the concentric circles* (55).

Until now we investigated the convergence of a power series in special domains which were determined by the inequality (54) or by the inequality (36), which is a particular case of inequality (54). We shall now consider the general case of convergence of a power series and assume that the matrix X can be converted into the purely diagonal form as we did for unitary matrices and also for matrices, all the characteristic zeros of which were different. Our condition may also be formulated as follows: we shall only consider matrices with simple elements. Such matrices can be written in the form [III$_1$, 27]:

$$X = S[\lambda_1, \lambda_2, \ldots, \lambda_n]S^{-1}, \tag{56}$$

where S is a certain matrix the determinant of which is not zero and λ_l are the characteristic zeros of the matrix X. To simplify notation we shall introduce symbols for segments of the series:

$$f_l(X) = a_0 + \sum_{m=1}^{l} a_m X^m; \quad f_l(z) = a_0 + \sum_{m=1}^{l} a_m z^m,$$

and the sums of the series we shall denote, as before, by

$$f(X) \text{ and } f(z).$$

Substituting expressions (56) in $f_1(X)$ we have [III$_1$, 44]:

$$f_l(X) = a_0 + S\Big(\sum_{m=1}^{l} a_m [\lambda_1^m,\ \lambda_2^m,\ \ldots,\ \lambda_n^m]\Big) S^{-1}$$

or

$$f_l(X) = S\,[f_l(\lambda_1);\ f_l(\lambda_2),\ \ldots;\ f_l(\lambda_n)]\,S^{-1}. \tag{57}$$

If all characteristic zeros λ_l lie inside the circle of convergence of series (33) then expression (57) will have a definite limit, viz.

$$f(X) = S\,[f(\lambda_1),\ f(\lambda_2),\ \ldots,\ f(\lambda_n)]\,S^{-1}, \tag{58}$$

and, consequently, the series (28) will converge. Let us now suppose that at least one of the characteristic zeros λ_1 lies outside the circle of convergence of series (33) and we will show that (57) cannot tend to a definite limit. In fact, we can rewrite equation (57) in the following form:

$$[f_l(\lambda_1),\ f_l(\lambda_2),\ \ldots,\ f_l(\lambda_n)] = S^{-1} f_l(X)\, S.$$

If $f_1(X)$ tends to a limit, then the left-hand side of the above equation would also have a limit, i.e. all elements of the diagonal matrix on the left would have a definite limit. However, this cannot be so for the element $f_l(\lambda_1)$, since λ_1 lies outside the circle of convergence of the series (33). We thus obtain the following theorem.

THEOREM. *The power series* (28) *converges if all the characteristic zeros of the matrix* X *lie inside the circle of convergence of the series* (33) *and it diverges if at least one of these zeros lies outside the above circle.*

We proved this theorem when the matrix X had simple elements, i.e. when it was given in the form (56). The proof can be extended to include the general case but we do not intend to do this here.

Let us now consider the general case of absolute convergence, i.e. the convergence of the series (31).

Bearing in mind that a power series of the usual complex variable is absolutely convergent inside its circle of convergence we can say that the radius of convergence of the series

$$\sum_{m=0}^{\infty} |a_m|\, z^m$$

will coincide with the radius of convergence of the series (33). Applying the theorem which we have just proved to the series (31) we obtain the following theorem for absolute convergence.

THEOREM. *The series* (28) *will converge absolutely if all characteristic zeros of the matrix* $|X|$ *lie inside the circle of convergence of the series* (33) *and it will not converge absolutely when at least one of these characteristic zeros lies outside the above circle.*

It follows from what was said at the beginning of this paragraph that the absolute convergence of the series (28) implies that this series will also be convergent in the general sense. Using this circumstance it can readily be shown that the maximum modulus of the characteristic zeros of the matrix $|X|$ is not less than the maximum modulus of the characteristic zeros of the matrix X. In fact, let ϱ_1 be the maximum modulus of the characteristic zeros of the matrix $|X|$ and ϱ_2 be have a similar meaning X. If we suppose that $\varrho_2 > \varrho_1$, we will show that this brings us to a contradiction. Select in series (33) the coefficients a_m in such a way that this series has a radius of convergence equal to ϱ, where ϱ satisfies the condition $\varrho_2 > \varrho > \varrho_1$. Such will be, for example, the power series obtained by expanding the fraction

$$\frac{1}{1-\frac{z}{\varrho}}.$$

As a result of the above theorems series (31) will, in this case, converge and series (28) diverge; this contradicts the fact that absolute convergence also implies ordinary convergence.

Let us turn to formula (58). It shows that when the matrix X has characteristic zeros λ_i and all its elements are simple then the matrix $f(X)$, determined by a convergent power series, will have characteristic zeros $f(\lambda_i)$ and its elements will also be simple. This property, with a certain clause, can be extended to include elements which are not simple, viz. the following proposition holds: *if the elements of the matrix* X *are*

$$(\lambda-\lambda_1)^{p_1},\ (\lambda-\lambda_2)^{p_2},\ \ldots,\ (\lambda-\lambda_s)^{p_s},$$

then the elements of the matrix $f(X)$, *given by a power series, will be*

$$[\lambda-f(\lambda_1)]^{p_1},\ [\lambda-f(\lambda_2)]^{p_2},\ \ldots,\ [\lambda-f(\lambda_s)]^{p_s},$$

providing the derivatives $f'(\lambda_k)$ *do not vanish.* Formula (58) may also be used for the analytic continuation of the function $f(X)$, when given by a power series. Suppose that this series converges absolutely in a domain of the type (54); take a certain matrix X_0 in this domain, and continuously vary the elements of this matrix by a definite law. The characteristic zeros λ_i will vary continuously at the same time.

Suppose that the elements of the matrix S which appears in formula (56), are doing likewise. The analytic continuation of the matrix $f(X)$ thus involves, according to formula (58), the analytic continuation of the function $f(\lambda)$ of one complex variable.

The analytic continuation described above is very inconvenient owing to the fact that formula (58) contains the matrix S, which has no definite connection with the given matrix X. In fact we already saw on the example of Hermitian matrices that the matrix S can be selected in different ways. In certain cases the continuation described above will not coincide with the analytic continuation of n^2 series (29). Below we shall explain in greater detail the problem of analytic continuation for which we shall use a new important formula. To derive some auxiliary propositions for the proof of this formula we have to deal first with some simple formulae connected with interpolation.

89. Interpolation polynomials. The fundamental and simplest problem of interpolation consists of the following: we are required to construct a polynomial expression with a power not higher than $(n-1)$, which would take given values at n points in the plane of the complex variable. Suppose that at the points z_k $(k = 1, 2, \ldots, n)$ it takes the value w_k. Notice, first of all, that there can only be one such polynomial. In fact, we know [I, 185] that two polynomials with powers not higher than $(n-1)$ are identical if their values at n different points coincide. The solution of the problem of interpolation can be given by the following simple formula:

$$P_{n-1}(z) = \sum_{k=1}^{n} \frac{(z-z_1)(z-z_2)\ldots(z-z_{k-1})(z-z_{k+1})\ldots(z-z_n)}{(z_k-z_1)(z_k-z_2)\ldots(z_k-z_{k-1})(z_k-z_{k+1})\ldots(z-z_n)} w_k. \quad (59)$$

It can be seen directly that the expression on the right-hand side is a polynomial of z with a power not higher than $(n-1)$. If we put, for example, $z = z_1$, then on the right-hand side all terms, except the first, vanish and the fraction in the first term will obviously be unity, i.e. $P_{n-1}(z_1) = w_1$ and, similarly, $P_{n-1}(z_k) = w_k$.

If $f(z)$ is a regular function in a certain domain and the points z_k belong to this domain then the formula

$$P_{n-1}(z) = \sum_{k=1}^{n} \frac{(z-z_1)\ldots(z-z_{k-1})(z-z_{k+1})\ldots(z-z_n)}{(z_k-z_1)\ldots(z_k-z_{k-1})(z-z_{k+1})\ldots(z_k-z_n)} f(z_k) \quad (60)$$

gives that unique polynomial with a power not higher than $(n-1)$, the values of which at points z_k coincide with values of the function

$f(z)$. This polynomial is usually known as Lagrange's interpolation polynomial for points z_k and formula (60) is known as Lagrange's interpolation formula.

The general polynomial of order $(n-1)$

$$a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}$$

has n parameters a_s. In Lagrange's formula these parameters are determined from n conditions, viz. from the conditions that at the points z_k the values of the polynomial should be equal to $f(z_k)$. Let us now formulate the problem in a more general way. Suppose again that $f(z)$ is regular in a certain domain and that we are given j points $z_1, z_2, \ldots, z_j$ inside this domain; we are required to construct a polynomial of order not higher than $(n-1)$, the values of which at the points z_k must coincide with the values of all its derivatives up to the degree $(p_k - 1)$, with analogous values for the function $f(z)$, i.e. in this case we have for the polynomial $P(z)$ the following condition

$$P(z_k) = f(z_k); \ldots; \quad P^{(p_k-1)}(z_k) = f^{(p_k-1)}(z_k) \quad (k = 1, 2, \ldots, j),$$

where we assume that $p_1 + p_2 + \ldots + p_j = n$, so that the total number of conditions will again be equal to n. It can readily be shown, as above, that such a polynomial must be unique. In fact, if there were two such polynomials, then their difference would be a polynomial of order not higher than $(n-1)$ and z_k zeros of multiplicity p_k, i.e. we would find that this polynomial with a power not higher than $(n-1)$ would have n zeros. Hence even with this new wider formulation the interpolation problem can only have one solution. We shall give the method by which the required interpolation polynomial can be constructed. Construct for this purpose the polynomial of degree n:

$$p(z) = (z - z_1)^{p_1} (z - z_2)^{p_2} \ldots (z - z_j)^{p_j}$$

and the function

$$\varphi(z) = \frac{f(z)}{p(z)}. \tag{61}$$

This latter function has poles at points z_k of order not greater than p_k. The sum of the infinite parts of this function with respect to the above poles can be represented by a certain fraction in which the degree of the numerator is lower than the degree of the denominator and where the denominator has the form

$$(z - z_1)^{q_1} (z - z_2)^{q_2} \ldots (z - z_j)^{q_j},$$

where the integers q_k are not greater than the integers p_k. Multiplying the numerator and denominator by one and the same factor we can reduce the above sum of the infinite parts of the function $\varphi(z)$ to the form:

$$\frac{P_{n-1}(z)}{p(z)},$$

where $P_{n-1}(z)$ is a certain polynomial of degree not higher than $(n-1)$. After this formula (61) can be rewritten in the form:

$$\frac{f(z)}{p(z)} = \frac{P_{n-1}(z)}{p(z)} + \omega(z),$$

where $\omega(z)$ is a certain function which is regular in the whole domain, including the points z_k. Rewrite the above formula as follows:

$$f(z) = P_{n-1}(z) + p(z)\,\omega(z). \tag{62}$$

The second term on the right-hand side in the neighbourhood of the point z_k can be written as a product of $(z - {}^{p_k}z_k)$ and a certain function which is regular at the point z_k, i.e. the second term on the right-hand side vanishes together with the derivatives at the point z_k up to the order $(p_k - 1)$. Thus at these points the value of the polynomial $P_{n-1}(z)$ and the derivatives up to the order $(p_k - 1)$ coincide with the corresponding values of the function $f(z)$, i.e. the polynomial $P_{n-1}(z)$ is the required interpolation polynomial. In future we shall sometimes denote it by $h(z, z_1, \ldots, z_n)$. If all values z_k are different then this is simply Lagrange's polynomial. If, however, there are equal values among the values z_k, e.g. if a certain number z_k appears p_k times then at the point z_k the values of the polynomial and its derivatives up to the order $(p_k - 1)$ coincide with the corresponding values of the function $f(z)$. When $n = 2$ and $z_1 \neq z_2$ we have

$$h(z;\ z_1,\ z_2) = \frac{z - z_2}{z_1 - z_2} f(z_1) + \frac{z - z_1}{z_2 - z_1} f(z_2),$$

and when $z_1 = z_2$

$$h(z;\ z_1,\ z_1) = f(z_1) + \frac{z - z_1}{1} f'(z_1).$$

90. Cayley's identity and Sylvester's formula. Let X be a certain matrix and

$$D(X - \lambda I) = 0 \tag{63}$$

be its characteristic equation where $D(Y)$ denotes the determinant of the matrix Y. Denote the zeros of this equation by $\lambda_1, \lambda_2, \ldots, \lambda_n$. The left-hand side can be written in the form

$$(-1)^n(\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n) = (-1)^n\psi(\lambda), \qquad (64)$$

where the a_k are simply expressed by elements of the matrix X or by zeros of the equation (63). Thus, for example

$$a_1 = -(\lambda_1 + \lambda_2 + \ldots + \lambda_n);\ a_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{n-1}\lambda_n.$$

The expressions a_k are examples of a numerical function or of a matrix, i.e. they represent a function which takes a numerical value for the given matrix X. We have already considered such functions [III$_1$, 27]. Let us recall that $(-1)^n a_n$ is the determinant of the matrix and a_1 is the trace of the matrix equal to the sum of its diagonal elements.

Cayley's identity consists of the following: if in the polynomial $\psi(\lambda) = \lambda_n + a_1\lambda_n^{-1} + \ldots + a_n$ we replace the letter λ by the matrix X, then we obtain a zero matrix as the result, i.e. the following identity holds:

$$\psi(X) = X^n + a_1X^{n-1} + \ldots + a_n = 0. \qquad (65)$$

Let us suppose that the characteristic roots λ_k are different or, more generally, that the matrix X can be represented in the form:

$$X = S[\lambda_1, \lambda_2, \ldots, \lambda_n]S^{-1}.$$

At the same time, as we saw in [III$_1$, 44] we have:

$$\psi(X) = S[\psi(\lambda_1), \psi(\lambda_2), \ldots, \psi(\lambda_n)]S^{-1}.$$

But the zeros λ_k are zeros of the polynomial $\psi(z)$ and therefore:

$$\psi(X) = S[0, 0, \ldots, 0]S^{-1}.$$

In the centre stands a diagonal matrix in which all elements on the main diagonal are zeros. Such a matrix consists of zeros throughout and, in general, the right-hand side of the above equation is also a matrix which consists of zeros, i.e. formula (65) does, in fact, apply. It is not difficult to extend the proof of this identity to include the general case when, by using the method of limit transition, we consider, to start with, matrices with different characteristic zeros.

Consider now a certain function $f(X)$ which is determined in the domain

$$|X| < A \qquad (66)$$

by an absolutely convergent power series

$$f(X) = a_0 + a_1 X + a_2 X^2 + \dots \tag{67}$$

Take a certain matrix X belonging to (66) and assume that the characteristic zeros λ_k are different. We now suppose that in the identity (62)

$$p(z) = (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n) = \psi(z).$$

We then have the identity

$$f(z) = P_{n-1}(z) + \psi(z)\,\omega(z), \tag{68}$$

where $P_{n-1}(z)$ is Lagrange's interpolation polynomial for the points λ_k. Formula (68) obviously remains an identity if we replace the variable z by the matrix X, for in the product on the right-hand side there is only one matrix X, the powers of which commute and therefore a product can be constructed in the same way as before when z is replaced by the matrix X. In this case the polynomial $\psi(X)$ coincides with the polynomial (65) and from Cayley's identity we have

$$f(X) = P_{n-1}(X).$$

In the expanded form we have in the domain (66) for any matrix X with different characteristic zeros:

$$f(X) = \sum_{k=1}^{n} \frac{(X - \lambda_1) \dots (X - \lambda_{k-1})(X - \lambda_{k+1}) \dots (X - \lambda_n)}{(\lambda_k - \lambda_1) \dots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1})(\lambda_k - \lambda_n)} f(\lambda_k). \tag{69}$$

Formula (69), which is known *Sylvester's formula gives the infinite series* (67) *in the form of a polynomial matrix* and the infinite power series enter the formula through the expressions $f(\lambda_k)$, which are ordinary power series of the complex variable.

If among the characteristic zeros λ_k of the matrix X there are similar values then we shall have on the right-hand side of formula (69) not Lagrange's polynomial but the more general interpolation polynomial which we mentioned in the previous section; we have a similar representation for series (67) in the form of a polynomial of the matrix

$$f(X) = h(X;\ \lambda_1,\ \lambda_2,\ \dots,\ \lambda_n). \tag{70}$$

For a matrix of the second order this gives us, when $\lambda_1 \neq \lambda_2$:

$$f(X) = \frac{X - \lambda_2}{\lambda_1 - \lambda_2} f(\lambda_1) + \frac{X - \lambda_1}{\lambda_2 - \lambda_1} f(\lambda_2), \tag{71}$$

and when $\lambda_1 = \lambda_2$:

$$f(X) = f(\lambda_1) + (X - \lambda_1) f'(\lambda_1). \tag{72}$$

Thus, for example, for an exponential matrix of the second order, when $\lambda_1 \neq \lambda_2$, we have:

$$e^X = \frac{X - \lambda_2}{\lambda_1 - \lambda_2} e_1^\lambda + \frac{X - \lambda_1}{\lambda_2 - \lambda_1} e_2^\lambda. \tag{73}$$

Note that the general formula (70) which should be applied when there are similar zeros among the zeros λ_k, can be obtained from formula (69) by the method of limit transitions, when certain groups of zeros λ_k tend to a common value.

91. Analytic continuation. Formula (67) which determines a regular function $f(X)$ in the domain (66) is equivalent to the n^2 power series (29) which are absolutely convergent in the concentric circles:

$$|\{X\}_{ik}| < \{A\}_{ik}.$$

By performing analytic continuation of these n^2 power series we determine the matrix $f(X)$ in a wider domain *and the whole set of matrices obtained as a result of this analytic continuation will determine the analytic function $f(X)$ as given by its original element, viz. by series* (67) *in domain* (66). Consider Sylvester's formula. When the elements of the matrix X vary according to a definite law, its characteristic zeros λ_k will also vary continuously in a definite manner and, according to (69), the analytic continuation of the function $f(X)$ will, in fact, involve the analytic continuation of the function $f(z)$ of one complex variable. If in the course of this analytic continuation some of the zeros λ_k coincide then instead of formula (69) we must turn to formula (70). If the function of the complex variable $f(z)$ remains single-valued after analytic continuation then the only difficulty caused by using Sylvester's formula for analytic continuation is due to those matrices X, the characteristic zeros of which contain values of λ which are singularities of the function $f(z)$. Thus, for example, on analytic continuation of the function which is determined in the neighbourhood of zero by the series

$$f(X) = I + X + X^2 + \dots,$$

those matrices will be singularities, the characteristic zeros of which include at least one which is equal to unity. It can be proved that in the case under consideration the analytic continuation by Sylvester's formula is fully equivalent to the analytic continuation of the above n^2 power series and that it gives all values of the analytic function.

Let us now consider the case when $f(z)$ is an analytic function which becomes many-valued in the course of its analytic continuation. In this case the function $f(z)$, as we saw above, can be single-valued not in

the ordinary plane of the complex variable but on a certain Riemann surface R with several sheets; this shows that our functions are many-valued. When the elements of the matrix X vary continuously its characteristic zeros will do likewise on the above Riemann surface R. If we are determining the values of our analytic functions $f(X)$ for a certain particular value of the matrix X_0 then we must know not only the matrix X_0 but also the process of analytic continuation of the function $f(X)$ which produced this matrix X_0, by starting with a certain matrix in the domain (66), in which the function was given by series (67). In other words, we must know not only the matrix X_0 but also the path of the analytic continuation which produces this matrix. Let us suppose that this path was such that Sylvester's formula was used throughout the analytic continuation, but that when X tends to X_0 two characteristic zeros λ_1 and λ_2 tend to coincide, i.e. they tend to one and the same complex value λ_0, although they lie on different sheets of the Riemann surface R. In the limit, the characteristic zeros λ_1 and λ_2 of the matrix X_0 coincide, but the function $f(z)$ in the neighbourhood of this common value λ_0 will be determined by two different Taylor's series, since the corresponding points lie on different sheets of the Riemann surface R. In general in the limit although $\lambda_1 = \lambda_2$ we have $f(\lambda_1) \neq f(\lambda_2)$. In this case the Lagrange–Sylvester formula is devoid of any meaning and we assume that the matrix X_0 along the above path of analytic continuation is a singularity of the function $f(X)$. It may, of course, sometimes happen that $f(\lambda_1) = f(\lambda_2)$ in the limit and the difference becomes apparent only in certain derivatives, i.e. for a certain s we have $f^{(s)}(\lambda_1) \neq f^{(s)}(\lambda_2)$, although $\lambda_1 = \lambda_2$. In this case by moving the characteristic zero away from the common value λ_0 by as little as we please, we can obtain such a value λ_0 on different sheets, for which $f(\lambda_1) \neq f(\lambda_2)$ although $\lambda_1 = \lambda_2$; we assume, as before, that the matrix X_0 is a singularity of the analytic function $f(X)$. Hence *in the case of a many-valued function certain matrices must be added to singular matrices of the analytic function* $f(X)$ *which are determined by those paths of analytic continuation, in which similar characteristic zeros of the matrix* X *correspond to different analytic elements of the function* $f(z)$.

We will not explain in greater detail the above characteristics of a many-valued analytic function $f(X)$. Let us investigate only one simple special case. Consider the matrix

$$X = S[\lambda_1, \lambda_2, \ldots, \lambda_n] S^{-1},$$

where S is a definite matrix, the determinant of which is not zero and whose zeros λ_k are different. We suppose that this matrix lies in the domain (66) where our function $f(X)$ is determined by series (67). We shall now continuously vary the matrix X by a definite law, viz. when fixing S the zeros λ_k will vary in such a way that they will always be different and will never coincide with the singularities of the function $f(z)$; also, in the limit, all zeros λ_k will tend to the common value λ_0, but will lie on different sheets of the Riemann surface R of the function $f(z)$. Suppose, for simplicity, that on these sheets the values of the function $f(z)$ at the point λ_0 differ in pairs. At the limit we have the matrix

$$X_0 = S\left[\lambda_0,\ \lambda_0,\ \ldots,\ \lambda_0\right] S^{-1} = \lambda_0,$$

i.e. simply the zero λ_0. The values of the function are determined in the original domain (66) by the formula

$$S\left[f(\lambda_1),\ f(\lambda_2),\ \ldots,\ f(\lambda_n)\right] S^{-1}. \tag{74}$$

The problem involves the analytic continuation of $f(\lambda_k)$ for a fixed S. In this case we obtain a definite limit for the function, which must be equal to:

$$S\left[\mu_1,\ \mu_2,\ \ldots,\ \mu_n\right] S^{-1}, \tag{75}$$

where μ_k denotes the value of the analytic function $f(\lambda_0)$ on that sheet of the Riemann surface on which the zero λ_k lies. Notice that the final result (75) depends on the choice of the matrix S. By changing the elements of the matrix S as little as we please we also change the final result (75), which can be readily shown by bearing in mind that $\mu_i \neq \mu_k$ when $i \neq k$. The fixing of the matrix S involves the choice of a definite law of variation of the matrix X in the course of analytic continuation. In doing this we obtain a definite limit for the function $f(X)$ at the singularity $X = X_0$. By slightly altering the elements of S we arrive at a different limit. It also follows from this that series (29) cannot be analytically continued over the point $X = X_0$. This singularity is, of course, connected with the path of analytic continuation which brings us to that point. It can be shown generally that the singularities of $f(X)$ as determined above for the case of the many-valued function $f(z)$, will also be singularities in the analytic continuation of n^2 power series (29) and vice versa. In other words, the use of Sylvester's formula in analytic continuation is equivalent to the analytic continuation of n^2 series (29). Those matrices will be singular

in that continuation among the characteristic zeros of which there are singular points of $f(z)$ and also matrices in which similar characteristic zeros lie on different sheets of the Riemann surface of the function $f(z)$.

92. Examples of many-valued functions. The logarithm of the matrix

$$Y = \log X \tag{76}$$

is, by definition, the solution of the equation

$$e^Y = X. \tag{77}$$

Let us suppose that the matrix X has simple elements:

$$X = S[\lambda_1, \lambda_2, \ldots, \lambda_n] S^{-1}, \tag{78}$$

and that none of the zeros λ_k is zero. It can readily be seen that we obtain the solution of equation (77) by assuming

$$Y = S[\log \lambda_1, \log \lambda_2, \ldots, \log \lambda_n] S^{-1}, \tag{79}$$

for, as we saw above

$$e^Y = S[e^{\log \lambda_1}, e^{\log \lambda_2}, \ldots, e^{\log \lambda_n}] S^{-1} = S[\lambda_1, \lambda_2, \ldots, \lambda_n] S^{-1},$$

i.e. the matrix (79) satisfies equation (77). We can in formula (79) take any value of $\log \lambda_k$ so that we have

$$Y = S[\log \lambda_1 + 2\pi r_1 i, \ldots, \log \lambda_n + 2\pi r_n i] S^{-1}, \tag{80}$$

where $\log \lambda_k$ always denotes the principal value of the logarithm

$$-\pi < \arg \lambda_k \leqslant \pi,$$

and r_k are any integers.

The many-valuedness of formula (80) is due to two causes. In the first place it is due to the freedom of choice of the numbers r_k and, secondly, to a certain arbitrariness of the matrix S, which enters formula (78), when the matrix X is fixed. If, when $\lambda_l = \lambda_k$, we have $r_l = r_k$ then the values of $\log X$ are said to be regular. We will show that a regular value of a logarithm is fully determined by the choice of the integers r_k so that it is wholly independent of the choice of the matrix S. In fact, let $\mu_1, \mu_2, \ldots, \mu_j$ be different characteristic zeros of the matrix X and $r_1, r_2, \ldots, r_j$ be the corresponding integers in formula (80). Construct Lagrange's interpolation polynomial with a degree not higher than $(j-1)$, with the condition

$$P(\mu_k) = \log \mu_k + 2\pi r_k i \qquad (k = 1, 2, \ldots, j).$$

According to formula (78) we have:

$$P(X) = S[P(\lambda_1), \ldots, P(\lambda_n)] S^{-1},$$

or

$$P(X) = S[\log \lambda_1 + 2\pi r_1 i \ldots, \log \lambda_n + 2\pi r_n i] S^{-1},$$

i.e. $P(X) = Y$, from which it follows directly that the assumed value of the logarithm does not depend on the choice of the matrix S, for this latter matrix does not appear in the construction of the polynomial $P(X)$.

Applying Lagrange's formula we have:

$$\log X = \sum_{k=1}^{n} \frac{(X-\lambda_1)\dots(X-\lambda_{k-1})(X-\lambda_{k+1})\dots(X-\lambda_n)}{(\lambda_k-\lambda_1)\dots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})\dots(\lambda_k-\lambda_n)} \log \lambda_k, \tag{81}$$

if all the characteristic zeros are different. Using this formula it can be shown that any matrix, in which at least one characteristic zero is zero, is singular for the function $\log X$.

Let us suppose that it is impossible to represent the matrix X in the form (78), i.e. we suppose that the matrix has multiple elements. Using the arguments of [88] it can be shown that the elements of X are

$$(\lambda-\lambda_1)^{p_1}, \dots, (\lambda-\lambda_m)^{p_m}, \tag{82}$$

and that the elements of the matrix $\log X$, which give the solution of equation (77), will be:

$$(\lambda-\log\lambda_1)^{p_1}, \dots, (\lambda-\log\lambda_m)^{p_m}. \tag{83}$$

If, when the values of λ_k are equal, we take equal values of $\log \lambda_k$ then the corresponding values of $\log X$ are said to be regular. It can also be shown that formula (81) gives all the regular values of the logarithm in the course of analytic continuation and that it gives regular values only.

Consider, for example, the simplest irregular values of a logarithm. Take as the matrix X a certain zero λ, i.e. a diagonal matrix with elements λ. We can write this matrix as follows:

$$X = S[\lambda, \lambda, \dots, \lambda]S^{-1} = S\lambda S^{-1} = \lambda I,$$

where S is any matrix, the determinant of which is not zero. Fixing the numbers r_k in a definite way we obtain the value of the logarithm:

$$\log X = S[\log\lambda + 2\pi r_1 i, \quad \log\lambda + 2\pi r_2 i, \dots, \log\lambda + 2\pi r_n i]S^{-1}$$

or

$$\log X = S[\log\lambda, \log\lambda, \dots \log\lambda]S^{-1} + S[2\pi r_1 i, 2\pi r_2 i, \dots, 2\pi r_n i]S^{-1},$$

and finally:

$$\log X = \log\lambda I + 2\pi i S[r_1, r_2, \dots, r_n]S^{-1}.$$

If the zeros r_k are not equal then the second term will essentially depend on the matrix S, which can be chosen quite arbitrarily.

We saw above that formula (79) gives the solution of equation (77) if the matrix X has the form (78). It can be shown that in this case formula (79) gives all the solutions of equation (77) [S in (79) is chosen arbitrarily].

Consider now the square root of the matrix

$$Y = X^{\frac{1}{2}}$$

as the solution of the equation (84)

$$Y^2 = X.$$

For the matrix X, situated near a unit matrix, we can represent one branch of this many-valued function in the form of the following power series:

$$Y = [I + (X - I)]^{\frac{1}{2}} = I + \tfrac{1}{2}(X - I) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}(X - I)^2 + \dots \qquad (85)$$

This series can be transformed by Sylvester's formula providing the characteristic zeros of the matrix X are different:

$$Y = X^{\frac{1}{2}} = \sum_{k=1}^{n} \frac{(X - \lambda_1) \dots (X - \lambda_{k-1})(X - \lambda_{k+1}) \dots (X - \lambda_n)}{(\lambda_k - \lambda_1) \dots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \dots (\lambda_k - \lambda_n)} \sqrt{\lambda_k}. \qquad (86)$$

To simplify the notation we suppose that the matrix is of second order. Let the matrix X be of the form:

$$| X = S[\lambda_1, \lambda_2] S^{-1} \qquad (\lambda_1 \text{ and } \lambda_2 \neq 0).$$

It can easily be proved that equation (84) has the solution

$$S[\pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}] S^{-1}, \qquad (87)$$

where arbitrary values of the radicals may be taken.

It can be shown that this formula gives all the solutions of equation (84) and that S, as in (79), is chosen arbitrarily.

If we take in formula (87) only regular values, i.e. when λ_1 and λ_2 coincide we take equal values of the radicals only, then it can be shown in the same way as for a logarithm that formula (87) will give a definite solution which depends on the choice of the radical but which is independent of the arbitrary choice of S.

If λ_1 and λ_2 are different, then formula (87) gives, in general, four different solutions of equation (84). On supposing that λ_1 and λ_2 are equal, we have for the matrix X:

$$X = S[\lambda_1, \lambda_1] S^{-1} = \lambda_1 I,$$

where S is any matrix the determinant of which is not zero.

Formula (87) gives

$$X^{\frac{1}{2}} = S[\pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_1}] S^{-1}.$$

If we take equal values of the radicals then this is equivalent to the following:

$$X^{\frac{1}{2}} = \pm\sqrt{\lambda_1}\, I. \qquad (88)$$

Consider now the case when different values of the radicals are taken:

$$X^{\frac{1}{2}} = \sqrt{\lambda_1}\, S[1, -1] S^{-1}, \qquad (89)$$

or

$$X^{\frac{1}{2}} = -\sqrt{\lambda_1}\, S[1, -1] S^{-1}, \qquad (90)$$

where S is any matrix, the determinant of which is not zero. Write out this matrix and its inverse in the expanded form:

$$S = \begin{Vmatrix} s_{11}, & s_{12} \\ s_{21}, & s_{22} \end{Vmatrix}; \quad S^{-1} = \begin{Vmatrix} s_{22}D^{-1}, & -s_{12}D^{-1} \\ -s_{21}D^{-1}, & s_{11}D^{-1} \end{Vmatrix} = D^{-1} \begin{Vmatrix} s_{22}, & -s_{12} \\ -s_{21}, & s_{11} \end{Vmatrix},$$

where

$$D = D(S) = \begin{vmatrix} s_{11}, & s_{12} \\ s_{21}, & s_{22} \end{vmatrix} = s_{11}s_{22} - s_{12}s_{21}.$$

Formula (89) can be rewritten in the following form:

$$X^{\frac{1}{2}} = \sqrt{\lambda_1}D^{-1} \begin{Vmatrix} s_{11}, & s_{12} \\ s_{21}, & s_{22} \end{Vmatrix} [1, \quad -1] \begin{Vmatrix} s_{22}, & -s_{12} \\ -s_{21}, & s_{11} \end{Vmatrix},$$

or

$$X^{\frac{1}{2}} = \sqrt{\lambda_1}\,(s_{11}s_{22} - s_{12}s_{21})^{-1} \begin{Vmatrix} s_{11}s_{22} + s_{12}s_{21}, & -2s_{11}s_{12} \\ 2s_{21}s_{22}, & -(s_{11}s_{22} + s_{12}s_{21}) \end{Vmatrix}. \tag{91}$$

We can thus see that in this case the square root $X^{1/2}$ has an infinite number of values and that these values contain arbitrary elements s_{ik} of the matrix S.

If the elements of X are (82) then the elements $X^{1/2}$ will be:

$$(\lambda - \sqrt{\lambda_1})^{p_1}, \ \ldots, \ (\lambda - \sqrt{\lambda_m})^{p_m}.$$

and if, when the values λ_k are equal, equal values of $\sqrt{\lambda_k}$ are taken then the values of $X^{1/2}$ are said to be regular.

Formula (86) gives all the regular values of $X^{1/2}$ in the course of analytic continuation. We assume that not one of the zeros λ_k is zero, for the point $z = 0$ is a singularity of the function $\sqrt{z}$.

93. Systems of linear equations with constant coefficients. Suppose that we are given a system of linear differential equations with constant coefficients:

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n, \end{aligned}\right\} \tag{92}$$

where x_k are functions of the independent variable t and x_k' the derivatives of these functions.

Let us suppose that the required functions $(x_1, \ldots, x_n)$ are components of a certain vector

$$\mathbf{x}(x_1, x_2, \ldots, x_n).$$

The components x_k are, in this case, functions of t and we determine the differentiation of the vector $\mathbf{x}$ with respect to t, as a new vector

with components $(x_1', \ldots, x_n')$:

$$\frac{d\mathbf{x}}{dt}(x_1', \ldots, x_n').$$

We lastly introduce a matrix A with elements a_{ik}. With this notation we can rewrite the system (92) as follows:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{93}$$

Suppose that there is a solution of this equation which satisfies the original conditions:

$$x_k|_{t=0} = x_k^{(0)} \qquad (k = 1, 2, \ldots, n). \tag{94}$$

These original conditions form a certain vector which we denote by

$$\mathbf{x}^{(0)}(x_1^{(0)}, \ldots, x_n^{(0)}).$$

It can readily be shown that the solution of the system (93) for the given original conditions (94) has the form:

$$\mathbf{x} = \left(I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \ldots\right)\mathbf{x}^{(0)} \tag{95}$$

or, introducing the matrix

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \ldots,$$

we can rewrite the solution (95) as follows:

$$\mathbf{x} = e^{At}\mathbf{x}^{(0)}. \tag{96}$$

In fact, formula (95) gives

$$\mathbf{x} = \mathbf{x}_0 + \frac{t}{1!}A\mathbf{x}_0 + \frac{t^2}{2!}A^2\mathbf{x}_0 + \ldots$$

Differentiating with respect to t we have:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}_0 + \frac{t}{1!}A^2\mathbf{x}_0 + \frac{t^2}{2!}A^3\mathbf{x}_0 + \ldots$$

or

$$\frac{d\mathbf{x}}{dt} = A\left(I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \ldots\right)\mathbf{x}_0,$$

and from (95):

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

At the same time the original conditions must also be satisfied, for when $t = 0$ formula (96) gives $\mathbf{x}\,|_{t=0} = \mathbf{x}^{(0)}$.

We can rewrite the system (92) by using the matrix notation in another form. As a preliminary we shall explain the *fundamental rules of differentiating a matrix*. Suppose that the elements of a certain matrix X are functions of the variable t. We determine the derivative $\mathrm{d}X/\mathrm{d}t$ as a matrix, the elements of which are obtained by differentiating the elements of the matrix X with respect to t, i.e. [III$_1$, 83]

$$\left\{\frac{\mathrm{d}X}{\mathrm{d}t}\right\}_{ik} = \frac{\mathrm{d}\{X\}_{ik}}{\mathrm{d}t}.$$

The usual law for differentiating a sum follows directly from this definition, viz. if X and Y are two matrices the elements of which are functions of t then

$$\frac{\mathrm{d}(X+Y)}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t} + \frac{\mathrm{d}Y}{\mathrm{d}t}. \tag{97}$$

In exactly the same way the formula for differentiating a product can easily be proved

$$\frac{\mathrm{d}}{\mathrm{d}t}(XY) = \frac{\mathrm{d}X}{\mathrm{d}t}Y + X\frac{\mathrm{d}Y}{\mathrm{d}t}, \tag{98}$$

but it must be remembered that, in general, the positions of the factors in the above formula (98) must not be interchanged. From the definition of the product we have:

$$\{XY\}_{ik} = \sum_{s=1}^{n}\{X\}_{is}\{Y\}_{sk},$$

whence

$$\frac{\mathrm{d}\{XY\}_{ik}}{\mathrm{d}t} = \sum_{s=1}^{n}\frac{\mathrm{d}\{X\}_{is}}{\mathrm{d}t}\{Y\}_{sk} + \sum_{s=1}^{n}\{X\}_{is}\frac{\mathrm{d}\{Y\}_{sk}}{\mathrm{d}t},$$

which gives us formula (98) immediately. This formula can readily be generalized to include any number of factors. Thus, for example, for three factors we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(XYZ) = \frac{\mathrm{d}X}{\mathrm{d}t}YZ + X\frac{\mathrm{d}Y}{\mathrm{d}t}Z + XY\frac{\mathrm{d}Z}{\mathrm{d}t}. \tag{99}$$

We shall also introduce a formula for the differentiation of the reciprocal matrix. Suppose that the determinant of the matrix X is not zero so that we have the reciprocal matrix X^{-1} where:

$$XX^{-1} = I.$$

Differentiating this identity with respect to t we obtain:

$$\frac{dX}{dt}X^{-1} + X\frac{dX^{-1}}{dt} = 0,$$

from which follows the law for differentiating the inverse matrix:

$$\frac{dX^{-1}}{dt} = -X^{-1}\frac{dX}{dt}X^{-1}. \tag{100}$$

Let us now return to the system (92). Consider n solutions of this system. They form a square table consisting of n^2 functions:

$$\begin{Vmatrix} x_{11}(t), & x_{12}(t), \ldots, & x_{1n}(t) \\ x_{21}(t), & x_{22}(t), \ldots, & x_{2n}(t) \\ \cdots & \cdots & \cdots \\ x_{n1}(t), & x_{n2}(t), \ldots, & x_{nn}(t) \end{Vmatrix} \tag{101}$$

Where that the first subscript of the function denotes the number of the function and the second subscript the number of the solution into which the function enters, i.e. for example $x_{23}(t)$ denotes an expression for the second function x_2 which enters into a solution with number 3. We therefore have:

$$\frac{dx_{ik}(t)}{dt} = a_{i1}x_{1k}(t) + \ldots + a_{in}x_{nk}(t) \qquad (i,\ k = 1,\ 2, \ldots,\ n),$$

and formula (92) can be rewritten in the form of the following matrix:

$$\frac{dX}{dt} = AX, \tag{102}$$

where X is the matrix (101). Let us recall, once again, that for this system of notation the matrix X gives n solutions of the system (93) since every column of this matrix gives a certain solution of the system (93). In this case the original condition will be the statement that when $t = 0$

$$X|_{t=0} = X^{(0)}, \tag{103}$$

where $X^{(0)}$ is an arbitrarily chosen matrix with constant elements. In the same way as above it can be shown that the solution of the system (102), the original condition of which is (103) has the form

$$X = e^{At}X^{(0)}. \tag{104}$$

Let us suppose that the determinant of the matrix $X^{(0)}$, which gives the original conditions, is not zero. We will show that in this case the determinant of the matrix X will not be zero for any t. Bearing in

mind formula (104) we can see that in order to do so it is sufficient to show that the determinant of the matrix e^{At} is not zero, for the determinant of the product of two matrices is equal to the product of the determinants of these matrices.

It can readily be shown that, in general, the determinant of the exponential matrix e^{Y} is never zero.

In fact, together with the matrix

$$e^{Y} = I + \frac{Y}{1!} + \frac{Y^2}{2!} + \ldots + \frac{Y^n}{n!} + \ldots \tag{105}$$

we construct the matrix

$$e^{-Y} = I - \frac{Y}{1!} + \frac{Y^2}{2!} - \ldots + (-1)^n \frac{Y^n}{n!} + \ldots \tag{106}$$

Multiplying the two series on the right-hand sides of the above formulae we only deal with numbers and powers of one matrix Y, so that the places of all factors can be interchanged.

Thus we formally obtain the same result as would have been obtained had we substituted the variable matrix Y by the variable number z. But, at the same time, as a result of the identity $e^{z}e^{-z} = 1$, the multiplication of the right-hand sides of formulae (105) and (106) will give unity and, consequently, in the case under consideration, we shall also have the following equation:

$$e^{Y}e^{-Y} = I,$$

which holds for any matrix Y. It follows directly from this equation that the matrix e^{-Y} is the inverse of the matrix e^{Y} and that the determinant of the matrix e^{Y} is not zero. Notice that if Y and Z are two different matrices which do not commute, then the product $e^{Y}e^{Z}$ will not, in general, be equal to e^{Y+Z}.

Thus it follows directly from formula (104) and from the proved property of an exponential matrix that if the determinant of the matrix $X^{(s)}$, which gives the original conditions, is not zero then the determinant of the matrix X will not be zero for any t. In this case the matrix X will give n linearly-independent solutions of the system (102). We shall now show that if Y is a matrix which gives any n solutions of the system (102) then it can be expressed in terms of the above matrix X by means of the formula

$$Y = XB, \tag{107}$$

where B is a certain matrix with constant elements. Formula (107) obviously shows that any solution of the system can be expressed linearly by n linearly-independent solutions of the system. To prove formula (107) we notice, first of all, that from the given conditions, Y should satisfy equation (102), i.e.

$$\frac{dY}{dt} = AY. \tag{108}$$

It is also given that the determinant of the matrix X which satisfies equation (102) is not zero and consequently, the inverse matrix X^{-1} exists. From the law for differentiating inverse matrices we have:

$$\frac{dX^{-1}}{dt} = -X^{-1}\frac{dX}{dt}X^{-1}$$

or, bearing in mind formula (102), we obtain:

$$\frac{dX^{-1}}{dt} = -X^{-1}AXX^{-1} = -X^{-1}A. \tag{109}$$

Let us now construct the derivative of the product

$$\frac{d}{dt}(X^{-1}Y) = \frac{dX^{-1}}{dt}Y + X^{-1}\frac{dY}{dt},$$

where, from (108) and (109)

$$\frac{d}{dt}(X^{-1}Y) = -X^{-1}AY + X^{-1}AY,$$

i.e.

$$\frac{d}{dt}(X^{-1}Y) = 0.$$

We thus see that the product $X^{-1}Y$ is a certain matrix B, the elements of which do not depend on t; from this formula (107) follows directly.

94. Functions of several matrices. We shall now explain the fundamental ideas and facts connected with functions of several matrices. Owing to the impossibility of commuting, the theory of functions of several variable matrices is considerably more complicated than the theory of functions of one variable matrix; we shall therefore only consider the most fundamental facts of this theory.

Let us begin with the polynomial. The general form of a homogeneous polynomial of the second order in two matrices will be

$$aX_1^2 + bX_1X_2 + cX_2X_1 + dX_2^2.$$

A homogeneous polynomial of the second order in l variable matrices will have the form

$$\sum_{i,k=1}^{l} a_{ik} X_i X_k,$$

where the summation is carried out with respect to the variables i and k, which independently of each other run through all the integers, from unity to l. A homogeneous polynomial of order m in l variable matrices will have the form:

$$\sum_{j_1, \ldots, j_m=1}^{l} a_{j_1 \ldots j_m} X_{j_1} \ldots X_{j_m}. \tag{110}$$

Here, as before, $a_{j_1 \ldots j_m}$ denote numerical coefficients. In formula (110) each of the variables of summation j_k takes all integral values from 1 to l, so that the above sum will contain l^m terms in all. Let us now consider that particular case when in formula (110) all the coefficients $a_{j_1, \ldots, j_m}$ are unity:

$$\sum_{j_1, \ldots, j_m=1}^{l} X_{j_1} \ldots X_{j_m}. \tag{111}$$

It can readily be shown that the sum (111) represents a power of the sum of the matrix X_{jk}, i.e.

$$(X_1 + \ldots + X_l)^m = \sum_{j_1, \ldots, j_m=1}^{l} X_{j_1} \ldots X_{j_m}. \tag{112}$$

Thus, for example

$$(X_1 + X_2)^2 = (X_1 + X_2)(X_1 + X_2) = X_1^2 + X_1 X_2 + X_2 X_1 + X_2^2.$$

Consider now a power series of l matrices. Such a series can be written as follows:

$$a_0 + \sum_{m=1}^{\infty} \sum_{j_1, \ldots, j_m=1}^{l} a_{j_1, \ldots, j_m} X_{j_1} \ldots X_{j_m}. \tag{113}$$

A full investigation of the convergence of this series is considerably more difficult than for a power series of one matrix and we shall only prove the sufficient condition for absolute convergence of series (113). Notice that series (113), like a series of one matrix, is absolutely convergent, providing the series below is convergent

$$|a_0| + \sum_{m=1}^{\infty} \sum_{j_1, \ldots, j_m=1}^{l} |a_{j_1 \ldots, j_m}| |X_{j_1}| \ldots |X_{j_m}|, \tag{114}$$

where the convergence of this latter series guarantees both the convergence of series (113) and the independence of the sum of this series from the position of its terms. Let us fix the integer m and denote by $a^{(m)}$ the maximum modulus $|a_{j_1,\ldots,j_m}|$, i.e.

$$|a_{j_1,\ldots,j_m}| \leqslant a^{(m)}. \tag{115}$$

We now construct the series of the usual complex variable

$$\sum_{m=1}^{\infty} a^{(m)} z^m, \tag{116}$$

and let $n\varrho$ be its radius of convergence, where n is the order of the matrix. Replacing the coefficients $|a_{j_1,\ldots,j_m}|$ by greater coefficients, viz. by $a^{(m)}$ in series (114), we obtain the series

$$a_0 + \sum_{m=1}^{\infty} a^{(m)} \sum_{j_1,\ldots,j_m=1}^{l} |X_{j_1}| \ldots |X_{j_m}|,$$

which can obviously be rewritten in the form:

$$a_0 + \sum_{m=1}^{\infty} (|X_1| + \ldots + |X_l|)^m. \tag{117}$$

This series can be regarded as a power series of one matrix

$$Z = |X_1| + \ldots + |X_l|, \tag{118}$$

and bearing in mind that the radius of convergence of series (116) is equal to $n\varrho$, we can say [86], that series (117) will converge provided that

$$|X_1| + \ldots + |X_l| < \|\varrho\|.$$

In this case series (114) will be all the more convergent. We thus obtain the following theorem.

THEOREM. *If the positive numbers $a^{(m)}$ are determined from the condition* (115), *and series* (116) *has a radius of convergence equal to $n\varrho$, then the power series* (113) *will be absolutely convergent provided condition* (118) *is satisfied.*

In the particular case when the radius of convergence of the series (116) is equal to infinity, series (113) will be absolutely convergent for any matrix X_k.

Notice also that the function $f(X_1, \ldots, X_l)$, which is defined as the sum of the series (113), satisfies the obvious relationship

$$f(SX_1S^{-1}, \ldots, SX_lS^{-1}) = Sf(X_1, \ldots, X_l)S^{-1},$$

where S is any matrix, the determinant of which is not zero. We had an analogous property before for the analytic function of one variable matrix.

We notice in conclusion, without attempting to give the proof, one characteristic of a power series of several matrices in relation to the uniqueness theorem. Here the uniqueness theorem is formulated as follows: if the equation

$$a_0 + \sum_{m=1}^{\infty} \sum_{j_1, \ldots, j_m=1}^{l} a_{j_1, \ldots, j_m} X_{j_1} \ldots X_{j_m} = \\ = b_0 + \sum_{m=1}^{\infty} \sum_{j_1, \ldots, j_m=1}^{l} b_{j_1, \ldots, j_m} X_{j_1} \ldots X_{j_m}$$

holds for all matrices

$$X_1, \ldots, X_l$$

of any order, which are sufficiently near to a zero matrix, then $b_0 = a_0$ and $b_{j_1, \ldots, j_m} = a_{j_1, \ldots, j_m}$.

Had we omitted the conditions of "*any order*" in our formulation then the theorem would have been false. In a particular case a homogeneous polynomial can be constructed, the coefficients of which are not zero

$$\sum_{j_1, \ldots, j_m=1}^{l} c_{j_1, \ldots, j_m} X_{j_1} \ldots X_{j_m},$$

which makes the identity equal to zero for all matrices of a definite order.

A description of the general theory of analytic functions of matrices and their applications to the theory of systems of linear differential equations was given in the works of the late I. A. Lappo-Danilevskij, printed in the *Journal of the Leningrad Physico-mathematical Society*. At present all papers remaining after the death of I. A. Lappo-Danilevskij are published in the works of the Institute of Mathematics at the Academy of Science, U. S. S. R.

CHAPTER V

LINEAR DIFFERENTIAL EQUATIONS

95. The expansion of a solution into a power series. In Volume II we dealt with linear differential equations of the second order with variable coefficients and, in particular, with the solution of these equations by means of power series. At the time we only considered equations which could be formally satisfied by a certain power series, without proving the convergence of this series. In this chapter we shall give a full and systematic account of linear equations of the second order, the coefficients of which are analytic functions of a complex variable. We therefore assume that the independent variable in a differential equation is a complex variable and that the unknown function and the coefficients are analytic functions.

Let us write a linear differential equation of the second order in the form:

$$w'' + p(z)\,w' + q(z)\,w = 0, \tag{1}$$

where w' and w'' are derivatives of the function w with respect to the variable z.

We are also given the following initial conditions:

$$w\,|_{z=z_0} = c_0; \quad w'\,|_{z=z_0} = c_1. \tag{2}$$

Let us suppose that the coefficients $p(z)$ and $q(z)$ are regular functions in the circle $|z - z_0| < R$. We will show that there is a solution of the equation (1) in this circle (regular function), which satisfies the equations (2). By introducing a new unknown function $u = w'$, we can rewrite equation (1) as a system of two equations of the first order:

$$\frac{du}{dz} = -p(z)\,u - q(z)\,w; \quad \frac{dw}{dz} = u.$$

For the sake of symmetry we shall consider the general case of a system of two linear equations with two dependent variables:

$$\frac{du}{dz} = a(z)\,u + b(z)\,v; \quad \frac{dv}{dz} = c(z)\,u + d(z)\,v \tag{3}$$

and show that this system has a regular solution in the circle $|z - z_0| < R$, which satisfies all initial conditions:

$$u|_{z=z_0} = \alpha; \quad v|_{z=z_0} = \beta, \tag{4}$$

provided the coefficients of the system (3) are regular functions in the above circle.

We shall again use the method of successive approximations which we used earlier in Volume II. The proof will be exactly the same as before. Instead of the system (3) with the initial conditions (4) let us write the equations in the form of integrals:

$$u = \alpha + \int_{z_0}^{z} [a(z)\,u + b(z)\,v]\,dz; \quad v = \beta + \int_{z_0}^{z} [c(z)\,u + d(z)\,v]\,dz. \tag{5}$$

Consider the circle K: $|z - z_0| < R_1$, where R_1 is a positive number, smaller than R. At every point strictly inside this circle, the coefficients are regular functions and, consequently, the following inequalities hold:

$$|a(z)| < M; \quad |b(z)| < M; \quad |c(z)| < M; \quad |d(z)| < M, \tag{6}$$

where M is a fixed positive number. Applying the method of successive approximations we assume:

$$u_0(z) = \alpha; \; v_0(z) = \beta, \tag{7}$$

and generally

$$\left.\begin{aligned} u_{n+1}(z) &= \alpha + \int_{z_0}^{z} [a(z)\,u_n + b(z)\,v_n]\,dz. \\ v_{n+1}(z) &= \beta + \int_{z_0}^{z} [c(z)\,u_n + d(z)\,v_n]\,dz. \end{aligned}\right\} \tag{8}$$

At every stage the functions of z to be integrated are regular and in no case does the value of the integral depend on the path of integration in the circle K. Suppose also that m is a positive number, satisfying the inequalities

$$|\alpha| \leqslant m; \quad |\beta| \leqslant m. \tag{9}$$

To simplify the working we assume that $z_0 = 0$ and integrate from 0 to z along a straight line. In this case

$$z = \varrho e^{i\varphi}; \quad dz = e^{i\varphi}\,d\varrho \qquad (0 \leqslant \varrho \leqslant R_1). \tag{10}$$

The first of the formulae (8), when $n = 0$, gives:

$$u_1(z) - a = \int_0^{\varrho} [a(z)\,\alpha + b(z)\,\beta]\, e^{i\varphi}\, d\varrho.$$

Substituting under the integral all terms by their moduli and using (6) and (7) we obtain the inequality

$$|u_1(z) - u_0(z)| \leqslant 2Mm\varrho \tag{11_1}$$

and, similarly

$$|v_1(z) - v_0(z)| \leqslant 2Mm\varrho. \tag{11_2}$$

The first of the equations (8), when $n = 1$, gives

$$u_2(z) = a + \int_0^z [a(z)\,u_1 + b(z)\,v_1]\, dz,$$

and, subtracting from this the first of the equations (8) when $n = 0$, we obtain:

$$u_2(z) - u_1(z) = \int_0^z [a(z)\,(u_1 - u_0) + b(z)\,(v_1 - v_0)]\, dz.$$

Replacing each term by its modulus and using the inequalities (11_1) and (11_2) we have:

$$|u_2(z) - u_1(z)| \leqslant (2M)^2 m \int_0^{\varrho} \varrho\, d\varrho$$

or

$$|u_2(z) - u_1(z)| \leqslant m\frac{(2M\varrho)^2}{2!},$$

and similarly

$$|v_2(z) - v_1(z)| \leqslant m\frac{(2M\varrho)^2}{2!}.$$

Continuing in this way we obtain the following results:

$$|u_{n+1}(z) - u_n(z)| \leqslant m\frac{(2M\varrho)^{n+1}}{(n+1)!},$$

$$|v_{n+1}(z) - v_n(z)| \leqslant m\frac{(2M\varrho)^{n+1}}{(n+1)!}.$$

It follows that the terms of the series

$$u_0 + [u_1(z) - u_0] + [u_2(z) - u_1(z)] + \cdots \tag{12}$$

in the circle $|z - z_0| < R_1$ have smaller moduli than the positive numbers

$$m\frac{(2M\varrho)^{n+1}}{(n+1)!},$$

which form a convergent series, i.e. the series (12) converges absolutely and uniformly in the circle $| z - z_0 | < R_1$. The sum of the first $(n + 1)$ terms of this series is equal to $u_n(z)$ and, consequently, $u_n(z)$ tends uniformly to a function $u(z)$ in the circle $| z - z_0 | < R_1$. Similarly $v_n(z)$ tends uniformly to a function $v(z)$. According to the theorem of Weierstrass on uniformly convergent series, these limit functions will also be regular in the circle K. Consider now the equations (8). In the first of these the integrand tends uniformly to the limit function

$$a(z)\, u + b(z)\, v.$$

But we know from [I, 146] that as we can integrate any uniformly convergent series term by term, the integral of the limit of a uniformly convergent sequence of functions is the limit of the integral; hence taking the limit in the equations (8) we see that the limit functions u and v satisfy the equations (5). Assuming in these equations that $z = z_0$ we see that our functions satisfy the intial conditions (4); differentiating equations (5) we see that they give the solution of the system (3).

Let us now return to the equation (1), which is a particular case of the system (3). We have shown that in any circle, centre z_0, which lies inside the circle $| z - z_0 | < R$, a solution of the equation can be found which satisfies the conditions (2) for every c_0 and c_1. The functions $p(z)$ and $q(z)$ can be expanded in the circle $| z - z_0 | < R$ into power series

$$p(z) = a_0 + a_1 (z - z_0) + \ldots; \quad q(z) = b_0 + b_1 (z - z_0) + \ldots$$

The solution so obtained is also a regular function and therefore it can be expanded into a power series in which, from (2), the first two coefficients are equal to c_0 and c_1:

$$w = c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + \ldots \tag{13}$$

Substituting this series in equation (1) we equate to zero the coefficients of different powers of z and this gives us, as we saw in [II, 45], equations of the type:

$$2 \cdot 1 c_2 + a_0 c_1 + b_0 c_0 = 0$$

$$3 \cdot 2 c_3 + 2 a_0 c_2 + a_1 c_1 + b_0 c_1 + b_1 c_0 = 0,$$

. . . . . .

whence the coefficients c_k are determined successively. This shows, first of all, that there can only be one solution. It also follows from the

above proof that there must be a solution, i.e. that by substituting the coefficients we obtained above in the series (13) we obtain a series which converges in every circle inside the circle $|z - z_0| < R$, in other words, it converges in the circle $|z - z_0| < R$. We thus obtain the following theorem.

THEOREM 1. *If the coefficients of the equation* (1) *are regular functions in the circle* $|z - z_0| < R$ *then there exists a unique solution of the equation in this circle which satisfies the initial conditions* (2) *for any given* c_0 *and* c_1.

By giving c_0 and c_1 definite numerical values we can construct two solutions w_1 and w_2, which satisfy the initial conditions:

$$w_1|_{z=z_0} = a_1; \quad w_1'|_{z=z_0} = \beta_1,$$
$$w_2|_{z=z_0} = a_2; \quad w_2'|_{z=z_0} = \beta_2.$$

If

$$a_1\beta_2 - a_2\beta_1 \neq 0, \tag{14}$$

then any solution, regular in the circle $|z - z_0| < R$, can be expressed in terms of w_1 and w_2 in the form:

$$w = A_1 w_1 + A_2 w_2. \tag{15}$$

In fact if this solution w is based on the initial conditions (2), then we have the following system of equations for the constants A_1 and A_2:

$$A_1 a_1 + A_2 a_2 = c_0; \quad A_1\beta_1 + A_2\beta_2 = c_1,$$

and this system, from (14), determines the values of A_1 and A_2. The constructed solutions for w_1 and w_2, will be linearly-independent solutions of the equation (1) [II, 24].

Note. The application of the method of succesive approximations in the system (3) gave us the infinite series (12) for the function u. This series will not be a power series but its uniform convergence in the circle $|z - z_0| < R_1$ guarantees the existence of a regular solution in that circle in the form of a power series. We can construct the function $u_n(z)$ and the series (12) in any domain in which the coefficients of the system (3) are regular functions and it can be argued, in the same way as above, that in any such domain the series (12), and the analogous series for v, will be uniformly convergent and will give the solution of the system. The form of these series in certain cases will be explained below.

96. The analytic continuation of the solution. Let us now suppose that the coefficients $p(z)$ and $q(z)$ are regular functions in a domain B of the z plane. Let us take a certain solution of this equation, which satisfies the initial conditions (2) at a point z_0 in B. This solution, as we know from above, will have the form of a convergent power series in a circle, centre z_0, which lies wholly in B (and perhaps also in a larger circle). It will be a power series of the form (13). Let us now take a fixed point z in the circle of convergence of this series and rewrite the series in powers of $(z - z_1)$, as we did in the section on the analytic continuation of a function. We thus obtain a new series

$$\sum_{k=0}^{\infty} d_k(z - z_1)^k . \tag{16}$$

Its sum will coincide with w in the common domain of the circles of convergence of the series (13) and (16). Consequently in this common domain the sum $f(z)$ of the series (16) will be the solution of the equation (1); in other words, when substituting $w = f(z)$ in the left-hand side of equation (1) the latter will vanish in a certain part of the circle of convergence of the series (16). But in that case, as a result of the fundamental principle of analytic continuation, it will also vanish in the part of the circle which belongs to B; the series (16) will thus give a solution of our equation. This solution will be fully determined from its initial conditions at the point z_1,

$$f(z_1) = w|_{z=z_1}; \quad f'(z_1) = w'|_{z=z_1},$$

where w is determined by the initial series (13).

As a result of the theorem proved in the previous section the series (16) is bound to converge in a circle, centre z_1, which belongs to that domain B in which $p(z)$ and $q(z)$ are regular functions. We thus arrive at the following theorem.

THEOREM II. *If the coefficients $p(z)$ and $q(z)$ are regular functions in a domain B, then any solution of the equation in the form of a power series with the centre of convergence in B can be analytically continued in any direction in B and this analytic continuation gives everywhere a solution of the equation* (1).

Let us make some essential additions to the above theorem. Notice that if B is a connected domain, then according to the fundamental principle of analytic continuation [18], w will be a single-valued regular function in B which, according to the given proof, will be a solution

of the equation (1). If, however, B is a multiply-connected domain, then w will not, in general, be a single-valued function in B.

If w_1 and w_2 are two solutions of equation (1) then we have the following formula [II, 24]:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{w_2}{w_1}\right) = \frac{C}{w_1^2}\,\mathrm{e}^{-\int_{z_0}^{z} p(z)\,\mathrm{d}z}, \tag{17}$$

where C is a constant. If C is not zero, then the left-hand side will never be zero during the analytic continuation, i.e. the analytic continuation of linearly-independent solutions always gives linearly-independent solutions, and formula (15) therefore gives the analytic continuation of any solution in terms of the analytic continuation of two linearly-independent solutions.

If, for example, the coefficients $p(z)$ and $q(z)$ are rational functions then any solution of the equation can be analytically continued in the plane in any direction except through the poles of $p(z)$ or $q(z)$.

97. The neighbourhood of a singularity. Let us now investigate the behaviour of the solution in the neighbourhood of a singularity of the coefficients $p(z)$ and $q(z)$. Let us suppose that the point $z = z_0$ is a pole or an essential singularity of the coefficients $p(z)$ and $q(z)$, so that these coefficients can be represented by a Laurent series in an annulus K, centre z_0 and zero inner radius:

$$\left.\begin{aligned} p(z) &= \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k, \\ q(z) &= \sum_{k=-\infty}^{+\infty} b_k (z - z_0)^k \qquad (0 < |z - z_0| < R). \end{aligned}\right\} \tag{18}$$

Any solution of equation (1) can be analytically continued along any path in the annulus K, but if the path encircles the point $z = z_0$ the solution w may take new values, i.e. the point $z = z_0$, will, in general, be a branch-point. Let us explain in greater detail the character of this branch-point. Take any two linearly-independent solutions w_1 and w_2. If we cut the annulus from the centre along any radius, then in the connected domain so obtained, our solutions w_1 and w_2 will be regular single-valued functions, but will have different values on opposite edges of the cut. In other words, after describing the point $z = z_0$ the functions w_1 and w_2 will become new functions w_1^+ and w_2^+. They must also be solutions of the equation, and can therefore be

expressed as a linear combination of w_1 and w_2. Hence the formula, given below, must hold:

$$\left.\begin{aligned} w_1^+ &= a_{11} w_1 + a_{12} w_2 , \\ w_2^+ &= a_{21} w_1 + a_{22} w_2 , \end{aligned}\right\} \tag{19}$$

where a_{lk} are constants. In other words, by describing a singularity, linearly-independent solutions undergo a linear transformation. It can readily be seen that

$$a_{11} a_{22} - a_{12} a_{21} \neq 0 . \tag{20}$$

In fact, if we have $a_{11} a_{22} - a_{12} a_{21} = 0$ then the solutions w_1^+ and w_2^+ differ only in constant terms and are linearly-dependent; however this cannot be so for we saw earlier that the analytic continuation of linearly-independent solutions produces linearly-independent solutions. The form of the linear transformation (19) does, of course, depend on the choice of the solutions w_1 and w_2.

We shall try to construct a solution which by describing a singularity acquires a constant term, i.e. a solution which undergoes the simplest possible linear transformation

$$w^+ = \lambda w . \tag{21}$$

This solution, if it exists, should be a linear combination of the solutions w_1 and w_2:

$$w = b_1 w_1 + b_2 w_2 ,$$

where the coefficients b_1 and b_2 must be found. We have, from (21):

$$b_1 w_1^+ + b_2 w_2^+ = \lambda(b_1 w_1 + b_2 w_2) ,$$

and this, from (19), gives:

$$b_1(a_{11} w_1 + a_{12} w_2) + b_2(a_{21} w_1 + a_{22} w_2) = \lambda(b_1 w_1 + b_2 w_2) .$$

Comparing the coefficients of the linearly-independent solutions we obtain a system of homogeneous equations for b_1 and b_2:

$$\begin{aligned} (a_{11} - \lambda) b_1 + a_{21} b_2 &= 0 , \\ a_{12} b_1 + (a_{22} - \lambda) b_2 &= 0 . \end{aligned} \tag{22}$$

To obtain values for b_1 and b_2 which are not zero we must equate the determinant of the above system to zero

$$\begin{vmatrix} a_{11} - \lambda, & a_{21} \\ a_{12}, & a_{22} - \lambda \end{vmatrix} = 0 . \tag{23}$$

This is a quadratic equation in λ. Taking a zero $\lambda = \lambda_1$ of this equation and substituting this for the coefficients of the system (22) we obtain a solution for b_1 and b_2 which is not zero. Hence the zeros of equation (23) give the possible values of λ in formula (21), i.e. these zeros are equal to the constants by which the existing solutions of equation (1) must be multiplied when describing the singularity z_0 in the positive direction. If we take at the start different linearly-independent solutions, then the linear transformation (19) will be different but the zeros of equation (23) should remain the same, for they are fully defined and independent of the choice of the fundamental solutions.

Consider, first of all, the case when the quadratic equation has two different zeros

$$\lambda = \lambda_1 \text{ and } \lambda = \lambda_2 .$$

We then have two solutions which satisfy the conditions

$$w_1^+ = \lambda_1 w_1; \quad w_2^+ = \lambda_2 w_2 . \tag{24}$$

These two solutions must be linearly-independent. Otherwise w_2/w_1 would be a constant and would not alter by describing the singularity; but, according to (24), this fraction acquires the factor λ_2/λ_1 in doing so. Notice also that, from (20), λ_1 and λ_2 must not be zero.

Introduce the two constants

$$\varrho_1 = \frac{1}{2\pi i} \log \lambda_1; \quad \varrho_2 = \frac{1}{2\pi i} \log \lambda_2 , \tag{25}$$

where the logarithms are arbitrary. Construct two functions:

$$(z - z_0)^{\varrho_1} = e^{\varrho_1 \log (z - z_0)}; \quad (z - z_0)^{\varrho_2} = e^{\varrho_2 \log (z - z_0)} .$$

By describing the singularity they acquire the factors

$$e^{\varrho_1 2\pi i} = e^{\log \lambda_1} = \lambda_1; \quad e^{\varrho_2 2\pi i} = e^{\log \lambda_2} = \lambda_2 .$$

Hence the relationships

$$\frac{w_1}{(z - z_0)^{\varrho_1}} \quad \text{and} \quad \frac{w_2}{(z - z_0)^{\varrho_2}}$$

remain single-valued on describing the singularity, i.e. they are regular single-valued functions in the neighbourhood of the point $z = z_0$, and they can therefore be represented by a Laurent series in that neighbourhood. Hence the constructed solutions can be represented in the

neighbourhood of the singularity as follows:

$$\left.\begin{aligned} w_1 &= (z - z_0)^{\varrho_1} \sum_{k=-\infty}^{+\infty} c'_k (z - z_0)^k , \\ w_2 &= (z - z_0)^{\varrho_2} \sum_{k=-\infty}^{+\infty} c''_k (z - z_0)^k . \end{aligned}\right\} \tag{26}$$

Notice that $\log \lambda$ is accurate to the term $2m\pi i$, where m is any integer. Hence, from (25), ϱ_1 and ϱ_2 accurate to integers. This is in full agreement with the formulae (26), for by multiplying a Laurent series by $(z - z_0)^m$, where m is any integer, we obtain another Laurent series and therefore ϱ_1 and ϱ_2 in the formulae (26) are accurate to integers.

Consider now the case when the zeros of the equation (23) are the same, i.e. $\lambda_1 = \lambda_2$. As before, we can construct one solution of the equation which satisfies the condition

$$w_1^+ = \lambda_1 w_1 . \tag{27}$$

Take an arbitrary second solution w_2 which is linearly-independent of w_1. By describing the singularity it is subjected to the following linear transformation:

$$w_2^+ = a_{21} w_1 + a_{22} w_2 . \tag{28}$$

The quadratic equation (23) will have the following form for these constructed solutions

$$\begin{vmatrix} \lambda_1 - \lambda, & a_{21} \\ 0, & a_{22} - \lambda \end{vmatrix} = 0 .$$

It is given that this equation has a double zero $\lambda = \lambda_1$ and it follows that $a_{22} = \lambda_1$, i.e. formula (28) must have the form

$$w_2^+ = \lambda_1 w_2 + a_{21} w_1 . \tag{29}$$

It follows from (27) and (29) that the relationship w_2/w_1 only acquires a constant term by describing the singularity

$$\left(\frac{w_2}{w_1}\right)^+ = \frac{w_2}{w_1} + \frac{a_{21}}{\lambda_1}$$

and consequently the difference

$$\frac{w_2}{w_1} - \frac{a_{21}}{2\pi i \lambda_1} \log(z - z_0) = \frac{w_2}{w_1} - a \log(z - z_0)$$

is single-valued and can be represented by a Laurent series. Hence, bearing in mind that w_1 has the form (26) and that the product of a Laurent series and w_1 has the same form as w_1, we can see that, in this

case, our solutions can be expressed as follows in the neighbourhood of the singularity:

$$\left.\begin{aligned} w_1 &= (z-z_0)^{\varrho_1} \sum_{k=-\infty}^{+\infty} c'_k (z-z_0)^k , \\ w_2 &= (z-z_0)^{\varrho_1} \sum_{k=-\infty}^{+\infty} c''_k (z-z_0)^k + a w_1 \log (z-z_0) . \end{aligned}\right\} \tag{30}$$

We thus arrive at the following theorem.

THEOREM III. *If $z = z_0$ is a pole or an essential singularity of the coefficients $p(z)$ and $q(z)$, then two linearly-independent solutions exist which can be expressed in the form* (26) *or* (30) *in the neighbourhood of that point.* Notice that in the second case, when the zeros of equation (23) are the same, it may happen that the constant a_{21} and the dependent constant

$$a = \frac{a_{21}}{2\pi i \lambda_1}$$

are both zero and we apply formula (26) in the neighbourhood of this point.

All that was said above refers to the case when z_0 lies at a finite distance. When it lies at infinity we must replace z by a new variable t, according to the formula

$$z = \frac{1}{t}; \quad t = \frac{1}{z} .$$

Differentiating with respect to t, instead of z, we have:

$$\frac{\mathrm{d}}{\mathrm{d}z} = -t^2 \frac{\mathrm{d}}{\mathrm{d}t}; \quad \frac{\mathrm{d}^2}{\mathrm{d}z^2} = t^4 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2t^3 \frac{\mathrm{d}}{\mathrm{d}t} ,$$

and equation (1) with the new independent variable takes the form

$$t^4 \frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + \left[2t^3 - t^2 p\left(\frac{1}{t}\right)\right] \frac{\mathrm{d}w}{\mathrm{d}t} + q\left(\frac{1}{t}\right) w = 0 . \tag{31}$$

For this new equation the former point at infinity will become $t = 0$ and we shall now investigate the neighbourhood of this point.

Notice that all the arguments used above were purely theoretical. They do not give a practical method for constructing equation (23) or for finding the coefficients of the expansions (26) and (30). We shall now consider a practical method for finding these. We can do this only in one case, viz. when the expansions of these formulae contain a finite number of terms with negative powers.

The singularity $z = z_0$ is then known as a *regular singularity, i.e. a pole or an essential singularity of the coefficients in equation* (1) *is*

known as a regular singularity of that equation if Laurent's expansion (26) *or* (30) *contains only a finite number of terms with negative powers.* Replacing ϱ_1 and ϱ_2 by an integer we can, as with a regular singularity, always achieve the fact that the power series in the formulae (26) or (30) should contain no terms with negative powers and should begin with a constant term, i.e. for example, instead of (26) we can write

$$\left.\begin{aligned} w_1 &= (z-z_0)^{\varrho_1}\sum_{k=0}^{\infty} c_k'(z-z_0)^k\,, \\ w_2 &= (z-z_0)^{\varrho_2}\sum_{k=0}^{\infty} c_k''(z-z_0)^k\,. \end{aligned}\right\} \qquad (c_0' \text{ and } c_0'' \neq 0) \qquad (26_1)$$

Otherwise, if at least one expansion in the formulae (26) or (30) contains an infinite number of terms with negative powers, the singularity is known as *irregular*. We must, first of all, indicate criteria by which it is possible to determine from the coefficients of the equation whether the singularity is regular or irregular.

98. Regular singularity. Let us suppose that w_1 and w_2 are two linearly independent analytic functions. It is not difficult to construct a linear equation for which these functions are solutions. In fact, we should have:

$$w_1'' + p(z)\,w_1' + q(z)\,w_1 = 0;$$

$$w_2'' + p(z)\,w_2' + q(z)\,w_2 = 0,$$

whence the coefficients of the equation can readily be determined [II, 24]:

$$p(z) = -\frac{w_2''w_1 - w_1''w_2}{w_2'w_1 - w_1'w_2} \qquad (32)$$

and

$$q(z) = -\frac{w_1''}{w_1} - p(z)\frac{w_1'}{w_1}\,. \qquad (33)$$

Let us suppose that the point $z = z_0$ is a regular singularity and only consider the case when $\varrho_1 \neq \varrho_2$, since the formulae (30) can be investigated similarly. In future we shall denote by $P_k(z-z_0)$ any series in positive integral powers of $(z - z_0)$, the constant term of which is not zero. In the case under consideration it is given that the singularity $z = z_0$ is regular and the solution is of the form:

$$w_1 = (z-z_0)^{\varrho_1}P_1(z-z_0); \quad w_2 = (z-z_0)^{\varrho_2}P_2(z-z_0)\,.$$

Hence

$$\frac{w_2}{w_1} = (z - z_0)^{\varrho_2 - \varrho_1} P_3(z)\,,$$

since the quotient obtained by dividing two power series with constant terms is also a power series with a constant term. We have further:

$$\Delta(z) = w_2' w_1 - w_1' w_2 = w_1^2 \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{w_2}{w_1}\right) =$$

$$= (z - z_0)^{2\varrho_1} P_4(z - z_0)\,[(z - z_0)^{\varrho_2 - \varrho_1} P_3(z)]'\,,$$

or, differentiating the product and taking $(z - z_0)^{\varrho_2 - \varrho_1 - 1}$ outside the brackets:

$$\Delta(z) = (z - z_0)^{\varrho_1 + \varrho_2 - 1} P_5(z - z_0)$$

and differentiating with respect to z we obtain:

$$\Delta'(z) = (\varrho_1 + \varrho_2 - 1)\,(z - z_0)^{\varrho_1 + \varrho_2 - 2} P_5(z - z_0) +$$

$$+ (z - z_0)^{\varrho_1 + \varrho_2 - 1} P_5'(z - z_0)\,.$$

Hence:

$$p(z) = -\frac{\Delta'(z)}{\Delta(z)} = \frac{1 - \varrho_1 - \varrho_2}{z - z_0} + \frac{P_5'(z - z_0)}{P_5(z - z_0)}\,,$$

i.e. $p(z)$ can have a pole at the point $z = z_0$ of an order not higher than one.

By differentiating the expression for w_1 we find that w_1'/w_1 can have a pole of an order not higher than one at the point z_0, and w_1''/w_1 a pole of an order not higher than two. Formula (33) shows that $q(z)$ can have a pole of an order not higher than two at the point z_0.

We thus arrive at the following theorem.

THEOREM I. *The necessary condition for the point z_0 to be a regular singularity is that the coefficient $p(z)$ should have at the point z_0 a pole of an order not higher than one and the coefficient $q(z)$ should have at the point z_0 a pole of an order not higher than two, i.e. equation* (1) *should be of the following form*

$$w'' + \frac{p_1(z)}{z - z_0} w' + \frac{q_1(z)}{(z - z_0)^2} w = 0\,, \tag{34}$$

where $p_1(z)$ and $q_1(z)$ are regular functions at the point z_0.

We will now show that this condition is not only necessary but is also sufficient for the singularity to be regular. Let us recall that equations of the form (34) are similar to equations which we considered in [II, 47] and for which we constructed a formal solution in the form of a generalized power series. However, before we did not investigate the convergence of the constructed series. We shall now

investigate this problem fully and show that the formally constructed series will be convergent and will give a solution of the equation. To simplify our notation we put $z_0 = 0$.

Multiply equation (34) by z^2 and rewrite it in the form:

$$z^2 w'' + z(a_0 + a_1 z + \dots) w' + (b_0 + b_1 z + \dots) w = 0 . \qquad (35)$$

We shall seek a solution of this equation in the form:

$$w = z^\varrho \sum_{k=0}^{\infty} c_k z^k . \qquad (36)$$

Substituting this in the left-hand side of equation (35) and equating the coefficients of like powers of z to zero we obtain equations for the determination of the coefficients c_k. These equations have the form:

$$\left.\begin{aligned} c_0 f_0(\varrho) &= 0 \\ c_1 f_0(\varrho+1) + c_0 f_1(\varrho) &= 0 \\ c_2 f_0(\varrho+2) + c_1 f_1(\varrho+1) + c_0 f_2(\varrho) &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ c_n f_0(\varrho+n) + c_{n-1} f_1(\varrho+n-1) + \dots + c_0 f_n(\varrho) &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \end{aligned}\right\} \qquad (37)$$

where, for the sake of simplicity, we introduced the following notation:

$$\left.\begin{aligned} f_0(\lambda) &= \lambda(\lambda-1) + \lambda a_0 + b_0 \\ f_k(\lambda) &= \lambda a_k + b_k \end{aligned}\right\} \quad (k = 1, 2, \dots) . \qquad (38)$$

We have said already that we can assume that $c_0 \neq 0$; the first of the equations (37) gives a quadratic equation for the determination of ϱ:

$$f_0(\varrho) = \varrho(\varrho-1) + \varrho a_0 + b_0 = 0 . \qquad (39)$$

This equation is usually known as the determining equation for the singularity under consideration. Suppose that ϱ_1 is a zero of this equation which is such that for every whole positive n we have the condition:

$$f_0(\varrho_1 + n) \neq 0 \qquad (n = 1, 2, \dots) . \qquad (40)$$

In this case the equations (37), starting with the second equation, enable us to determine successively the coefficients $c_1, c_2, \dots$ The first coefficient c_0 remains arbitrary and will play the part of an arbitrary constant so that we can take, for example, $c_0 = 1$. We must also show that the constructed series which forms part of formula (36) will be a convergent series in the neighbourhood of the point $z = 0$.

Let R be the circle of convergence of the series which are the coefficients of equation (35). If R_1 is a positive number, smaller than R, then we obtain the following result for the coefficients a_k and b_k in these series [14]:

$$|a_k| < \frac{m_1}{R_1^k}; \quad |b_k| < \frac{m_2}{R_1^k},$$

where m_1 and m_2 are constants. Hence

$$|a_k| + |b_k| < \frac{m_1 + m_2}{R_1^k},$$

and, consequently, taking M sufficiently large, we obtain the result in the form:

$$|a_k| + |b_k| < \frac{M}{R_1^k}. \tag{41}$$

The relationship

$$\frac{|\varrho| + n}{f_0(\varrho + n)} = \frac{|\varrho| + n}{(\varrho + n)(\varrho + n - 1) + (\varrho + n) a_0 + b_0}$$

tends to zero as the integer n increases indefinitely, for the numerator is a polynomial of the first degree in n and the denominator is a polynomial of the second degree. Hence a positive integer N exists such that

$$|f_0(\varrho + n)| > |\varrho| + n \quad \text{when} \quad n \geqslant N. \tag{42}$$

We have from the formulae (37):

$$c_n = -\frac{f_1(\varrho + n - 1)}{f_0(\varrho + n)} c_{n-1} - \frac{f_2(\varrho + n - 2)}{f_0(\varrho + n)} c_{n-2} - \ldots - \frac{f_n(\varrho)}{f_0(\varrho + n)} c_0,$$

whence

$$|c_n| \leqslant \frac{|f_1(\varrho + n - 1)|}{|f_0(\varrho + n)|} |c_{n-1}| + \frac{|f_2(\varrho + n - 2)|}{|f_0(\varrho + n)|} |c_{n-2}| + \\ + \ldots + \frac{|f_n(\varrho)|}{|f_0(\varrho + n)|} |c_0|. \tag{43}$$

We have further:

$$f_k(\varrho + n - k) = b_k + (\varrho + n - k) a_k$$
$$|f_k(\varrho + n - k)| < |b_k| + (|\varrho| + n) |a_k| \qquad (k = 1, 2, \ldots, n).$$

and for this reason also:

$$|f_k(\varrho + n - k)| < (|\varrho| + n)(|a_k| + |b_k|). \tag{44}$$

We can always select a sufficiently large positive number P, so as to obtain the following result for the first N coefficients:

$$|c_k| \leqslant \frac{P^k}{R_1^k} \qquad (k = 0, 1, \ldots, N - 1). \tag{45}$$

We now recall that we took $c_0 = 1$. We also assumed that P is chosen so that

$$P > 1 + M. \tag{46}$$

For the remaining coefficients, starting with c_N, we can use the inequality (42). We will show by using it that if the result (45) holds for all c_k, from c_0 to c_n exclusively, then it will also hold for c_n. In fact, from (42), (43) and (44) we have:

$$|c_n| < (|a_1| + |b_1|)\,|c_{n-1}| + (|a_2| + |b_2|)\,|c_{n-2}| + \ldots + \\ + (|a_n| + |b_n|)\,|c_0|,$$

or from (41)

$$|c_n| < \frac{M}{R_1}|c_{n-1}| + \frac{M}{R_1^2}|c_{n-2}| + \ldots + \frac{M}{R_1^n}|c_0|,$$

or, assuming that for $c_0, c_1, \ldots, c_n - 1$ we have the result (45):

$$|c_n| < \frac{M}{R_1^n}(P^{n-1} + P^{n-2} + \ldots + 1) = \frac{M(P^n - 1)}{P - 1}\,\frac{1}{R_1^n}. \tag{47}$$

We will now show that

$$\frac{M(P^n - 1)}{P - 1} < P^n. \tag{48}$$

In fact, this inequality is equivalent to the following:

$$P^{n+1} - (1 + M)\,P^n + M > 0$$

or

$$P^n[P - (1 + M)] + M > 0.$$

This last inequality follows from (46). The inequalities (47) and (48) give

$$|c_n| \leqslant \frac{P^n}{R_1^n},$$

and our proposition is proved.

The result (45) is therefore valid up to $k = N - 1$ inclusively as a result of the choice of P. The inequality (42) holds for later symbols; by using this inequality we have shown that if the result (45) holds up to a certain suffix k it will also hold for the succeeding suffix $k + 1$ We have thus proved that the result (45) holds for every suffix, i.e. for every n we have:

$$|c_n| \leqslant \frac{P^n}{R_1^n}.$$

But the series

$$\sum_{n=0}^{\infty} \frac{P^n}{R_1^n} z^n$$

will converge absolutely in the circle $| z | < R_1/P$. Hence in this circle, the series in formula (36), the moduli of the terms of which are not greater than the terms of the preceeding series, will also converge absolutely and this series can be differentiated term by term, like any other convergent power series.

We have thus shown that formula (36) gives, in fact, a solution of our equation in the neighbourhood of the point $z = 0$. We will now show that the series (36) converges in the whole circle $| z | < R$, where the series which are the coefficients of the equation (35) converge. Otherwise the function which is determined in the neighbourhood of $z = 0$ by the power series in formula (36), would have a singularity in the circle $| z | < R$ in the course of the analytic continuation [18] (other than the point $z = 0$). But this is impossible since the coefficients of equation (35) are regular functions in the whole circle $| z | < R$, except at the point $z = 0$ and, from [97], the solution cannot have any singularities during analytic continuation at this point.

If the difference of the zeros of the quadratic equation (39) is not an integer, then the condition (40) is satisfied for each zero; two linearly-independent solutions of the form (36) ($\varrho_1 \neq \varrho_2$) can therefore be constructed.

We shall investigate the case when the quadratic equation (39) has zeros whose difference is either an integer or zero.

In the second case, using the above method with repeated zero of the equation, we can construct one solution of the form (36) and a second solution will have to be found. Consider the first case. Let ϱ_1 and ϱ_2 be the zeros of equation (39) where $\varrho_1 = \varrho_2 + m$ and m is a positive integer, i.e. ϱ_1 is that zero of the equation, the real part of which is greater than that of the second zero. Condition (40) is evidently satisfied by the zero ϱ_1 and, by using this zero, a solution can be constructed in the way described above. When attempting to use the second zero ϱ_2 for the construction of a solution we find the following obstacle: $\varrho_2 + m$ is a zero of the equation (39) and, consequently, if we take the $(m + 1)^{\text{th}}$ equation of the system (37)

$$c_m f_0(\varrho_2 + m) + c_{m-1} f_1(\varrho_2 + m - 1) + \ldots + c_0 f_m(\varrho_2) = 0\,,$$

then in this equation the coefficient $f_0(\varrho_2 + m)$ of the unknown c_m will be zero. The sum of the remaining terms will, in general not

be zero and this contradicts the equation. Hence even in this case we have to find the second solution in another way. Notice that if it should by chance happen that the above sum in the latter equation is zero, then we could take any value for c_m and continue to calculate the successive coefficients $c_{m+1}, \ldots$ Our previous results show that the series obtained will converge and we shall therefore also obtain a second solution in the form (36) for this particular case.

Let us now establish the form of the second solution by assuming generally that

$$\varrho_1 = \varrho_2 + m\,, \tag{49}$$

where m is a positive integer or zero. Let us recall that for a linear equation

$$w'' + p(z)\,w' + q(z)\,w = 0$$

we have a formula which gives a second solution w_2 of the equation when one solution w_1 is known [II, 24]:

$$w_2 = Cw_1 \int \mathrm{e}^{-\int p(z)\,\mathrm{d}z} \frac{\mathrm{d}z}{w_1^2}\,, \tag{50}$$

where C is an arbitrary constant. In this case

$$p(z) = \frac{a_0}{z} + a_1 + a_2\,z + \ldots$$

and

$$\int p(z)\,\mathrm{d}z = \log z^{a_0} + C_1 + a_1\,z + \frac{1}{2}a_2\,z^2 + \ldots,$$

whence

$$\mathrm{e}^{-\int p(z)\,\mathrm{d}z} = z^{-a_0}\,P_1(z)\,,$$

where, as before, $P_1(z)$ is Taylor's series in powers of z, the constant term of which is not zero. The constructed solution has the form

$$w_1 = z^{\varrho_1}\,P_2(z)\,, \tag{51}$$

whence

$$w_1^2 = z^{2\varrho_1}\,P_3(z)\,,$$

where $P_s(z)$ is a Taylor's series, the constant term of which is not zero. The integrand in formula (50) will therefore be:

$$\mathrm{e}^{-\int p(z)\,\mathrm{d}z} \frac{1}{w_1^2} = z^{-a_0-2\varrho_1}\,P_4(z)\,.$$

The numbers ϱ_1 and ϱ_2 are the zeros of the quadratic equation (39) and therefore:

$$\varrho_1 + \varrho_2 = 1 - a_0\,.$$

Hence, from (49):

$$-a_0 - 2\varrho_1 = \varrho_2 - \varrho_1 - 1 = -(1+m),$$

i.e. the integrand in formula (50) can be expressed as:

$$e^{-\int p(z)\,dz}\frac{1}{w_1^2} = z^{-(1+m)}\,P_4(z) = \frac{\gamma_{-(1+m)}}{z^{1+m}} + \ldots + \frac{\gamma_{-1}}{z} + \gamma_0 + \gamma_1 z + \ldots$$

$$(\gamma_{-(1+m)} \neq 0).$$

Integrating this expression we obtain one logarithmic term $\gamma_{-1}\log z$ and a series which begins with z^{-m}. Multiplying this by the expression for w_1 given by formula (51), we finally obtain the following:

$$w_2 = z^{-m}\,P_5(z)\cdot z^{\varrho_1}\,P_2(z) + \gamma_{-1}w_1\,\log\,z,$$

and from (49):

$$w_2 = z^{\varrho_2}\,P_6(z) + \gamma_{-1}w_1\,\log\,z, \tag{52}$$

where $P_6(z)$ is a Taylor's series with a constant term. The expression (52) has the same form as the second of the expressions (30) and in formula (52) the Laurent series has no terms with negative powers. Notice that the constant γ_{-1} is, in general, not zero but in isolated cases it can be zero. This will be so in the case described above. Hence in that case we also obtain a characteristic second solution for a regular singularity, i.e. we have the following theorem.

THEOREM II. *In order that the point $z = z_0$ should be a regular singularity it is sufficient that the coefficient $p(z)$ in equation* (1) *should have a pole at the point z_0 of an order not higher than one and the coefficient $q(z)$ a pole of an order not higher than two.*

The necessity of this condition has been explained above.

Notice that it may sometimes happen that at a regular singularity neither solution has any peculiarities. This will be so when ϱ_1 and ϱ_2 are positive integers and when the second solution does not contain a logarithmic term. Thus, for example, the equation

$$w'' - \frac{2}{z}w' + \frac{2}{z^2}w = 0$$

has the following two linearly-independent solutions:

$$w_1 = z; \qquad w_2 = z^2.$$

Notice also that, when $\varrho_1 = \varrho_2$, the constant γ_{-1} in formula (52) will not be zero; this follows from the above calculations when $m = 0$.

99. Equations of Fuchs's class. The first systematic investigation of regular singularities was undertaken in the middle of the nineteenth century by the German mathematician Fuchs. We shall now investigate equations, all the singularities of which are regular singularities. Such equations are usually known as *equations of Fuchs's class*. Let us write our equation in the form:

$$w'' + p(z)\,w' + q(z)\,w = 0\,. \tag{53}$$

On putting the independent variable

$$z = \frac{1}{t}\,,$$

we obtain, as we saw above, the equation

$$t^4 \frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + \left[2t^3 - t^2 p\left(\frac{1}{t}\right)\right]\frac{\mathrm{d}w}{\mathrm{d}t} + q\left(\frac{1}{t}\right)w = 0\,. \tag{53_1}$$

It is given that the point $t = 0$ is an essential singularity of this equation. Bearing in mind that, after dividing by t^4, the coefficient of $\mathrm{d}w/\mathrm{d}t$ cannot have a pole of an order higher than one at the point $t = 0$, we have for $p(1/t)$ the following expansion:

$$p\left(\frac{1}{t}\right) = d_1 t + d_2 t^2 + \ldots\,,$$

i.e. near $z = \infty$ the function $p(z)$ can be expanded as follows:

$$p(z) = d_1 \frac{1}{z} + d_2 \frac{1}{z^2} + \ldots \tag{54}$$

Similarly, bearing in mind that the coefficient $(1/t^4)q(1/t)$ cannot have a pole of order higher than two at the point $t = 0$, we obtain:

$$q\left(\frac{1}{t}\right) = d_2' t^2 + d_3' t^3 + \ldots\,,$$

and, consequently, we have for $q(z)$ the following expansion near $z = \infty$

$$q(z) = d_2' \frac{1}{z^2} + d_3' \frac{1}{z^3} + \ldots\,, \tag{55}$$

i.e. if the point at infinity $z = \infty$ is to be a regular singularity of the equation (53) it is necessary and sufficient that $p(z)$ should have a zero at the point $z = \infty$ and $q(z)$ a zero of order not less than two. Notice that if in the expansion (54) $d_1 = 2$ and in the expansion (55) $d_2' = d_3' = 0$, then the point $t = 0$ is not a singularity of the equation (53_1). In this case the equation has the following solution in

the neighbourhood of $z = \infty$:

$$w = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots + \frac{c_n}{z^n} + \ldots,$$

where the coefficients c_0 and c_1 are constant.

Let $a_1, \ldots, a_n$ be the singularities of our equation within a finite distance. The function $p(z)$ can have poles of the first order at these points and, according to (54), it must vanish at infinity, i.e. it will be a rational fraction of the form

$$p(z) = \frac{p_1(z)}{(z-a_1)\ldots(z-a_n)},$$

where the order of the numerator is at least by one unit less than the order of the denominator. Similarly, from (55), we can see that $q(z)$ must have the form

$$q(z) = \frac{q_1(z)}{(z-a_1)^2\ldots(z-a_n)^2},$$

where the order of the numerator is at least two units less than the order of the denominator. Converting the rational fraction into partial fractions we obtain the following general expressions for the coefficients of the equations of Fuchs's class:

$$\left.\begin{aligned} p(z) &= \sum_{k=1}^{n} \frac{A_k}{z-a_k}, \\ q(z) &= \sum_{k=1}^{n} \left[\frac{B_k}{(z-a_k)^2} + \frac{C_k}{z-a_k}\right]. \end{aligned}\right\} \tag{56}$$

We have from (55):

$$zq(z) \to 0 \quad \text{as} \quad z \to \infty,$$

and the second of the expressions (56) shows that the constants C_k satisfy the condition

$$C_1 + C_2 + \ldots + C_n = 0. \tag{57}$$

The expressions (56) *and the condition* (57) *are the necessary and sufficient conditions for the equation* (53) *to be an equation of Fuchs's class.*

Let us now construct the determining equation for the singularities $z = a_k$ and for the point $z = \infty$. For the point a_k the coefficient of $(z - a_k)^{-1}$ in the expression for $p(z)$ is equal to A_k and the coefficients of $(z - a_k)^{-2}$ in the expression for $q(z)$ is equal to B_k, so that the determining equation at that point will be

$$\varrho(\varrho - 1) + A_k \varrho + B_k = 0 \qquad (k = 1, 2, \ldots, n). \tag{58}$$

Let us now consider the point $z = \infty$, i.e. the point $t = 0$ for the equation (53_1). The coefficient of t^{-1} in the expression

$$\frac{1}{t^4}\left[2t^3 - t^2 p\left(\frac{1}{t}\right)\right]$$

must be determined as follows:

$$\lim_{t\to 0} \frac{1}{t^3}\left[2t^3 - t^2 p\left(\frac{1}{t}\right)\right]$$

or

$$\lim_{z\to\infty} z^3\left[\frac{2}{z^3} - \frac{1}{z^2} p(z)\right] = 2 - \lim_{z\to\infty} zp(z) .$$

Bearing in mind the first of the equations (56) we obtain the following expression for the coefficient:

$$2 - \sum_{k=1}^{n} A_k .$$

Similarly the coefficient of t^{-2} in the expression

$$\frac{1}{t^4} q\left(\frac{1}{t}\right)$$

will be

$$\lim_{t\to 0} \frac{1}{t^2} q\left(\frac{1}{t}\right) = \lim_{z\to\infty} z^2 q(z) .$$

But from (56) and (57):

$$q(z) = \sum_{k=1}^{n} \frac{B_k}{(z-a_k)^2} + \sum_{k=1}^{n} \frac{1}{z} \frac{C_k}{1 - \frac{a_k}{z}} =$$

$$= \sum_{k=1}^{n} \frac{B_k}{(z-a_k)^2} + \sum_{k=1}^{n} \left(\frac{a_k C_k}{z^2} + \frac{a_k^2 C_k}{z^3} + \cdots\right);$$

hence

$$\lim_{z\to\infty} z^2 q(z) = \sum_{k=1}^{n} (B_k + a_k C_k) .$$

Finally, the determining equation will have the following form when $z = \infty$:

$$\varrho(\varrho - 1) + \varrho\left(2 - \sum_{k=1}^{n} A_k\right) + \sum_{k=1}^{n} (B_k + a_k C_k) = 0 . \qquad (59)$$

Bearing in mind (58) and (59) it can readily be shown that the sum of the zeros of the determining equations at all singularities will be

equal to

$$n - \sum_{k=1}^{n} A_k + \sum_{k=1}^{n} A_k - 1 = n - 1 ,$$

i.e. *this sum will be equal to one less than the number of singularities within a finite distance.*

If we should wish to construct an equation of Fuchs's class with one singularity then we can always assume that this point lies at infinity so that there will be no singularities at all within a finite distance. In the formulae (56) we would have to assume that all coefficients A_k, B_k and C_k are equal to zero, when we obtain the uninteresting equation $w'' = 0$.

Let us now consider an equation of Fuchs's class with two singularities, one of which can always be assumed to lie at infinity. In that case the sums in the formulae (56) should consist of single terms and, according to the condition (57), we obtain:

$$w'' + \frac{A}{z-a} w' + \frac{B}{(z-a)^2} w = 0 ,$$

where a is the only singularity at a finite distance.

This equation is Euler's linear equation [II, 42] and we know that by a simple substitution

$$\tau = \log (z - a)$$

it can be converted into an equation with constant coefficients.

In the following section we shall investigate in detail equations of Fuchs's class with three singularities.

Let us recall the Bessel equation [II, 48]

$$z^2 w'' + z w' + (z^2 - p^2) w = 0 .$$

This equation has a regular singularity at the origin $z = 0$. The coefficient of w in the neighbourhood of infinity does not satisfy the condition (55) and therefore the point $z = \infty$ will be an irregular singularity of the Bessel equation, i.e. the Bessel equation has two singularities: the regular singularity $z = 0$ and the irregular singularity $z = \infty$.

100. The Gauss equation. Consider now an equation of Fuchs's class with three singularities. Using a bilinear transformation of the plane of the independent variable we can, without loss of generality, assume that these singularities lie at the points

$$z = 0; \qquad z = 1 \quad \text{and} \quad z = \infty .$$

We denote the zeros of the determining equation at these points as follows:

$$\alpha_1,\ \alpha_2;\quad \beta_1,\ \beta_2;\quad \gamma_1,\ \gamma_2\,.$$

We obtain the following expression for the coefficients of the equation:

$$p(z) = \frac{A_1}{z} + \frac{A_2}{z-1}\,,$$

$$q(z) = \frac{B_1}{z^2} + \frac{B_2}{(z-1)^2} + \frac{C_1}{z} + \frac{C_2}{z-1}\,,$$

where

$$C_1 + C_2 = 0\,. \tag{60}$$

It is given that the equation

$$\varrho(\varrho-1) + A_1\varrho + B_1 = 0$$

has the zeros α_1 and α_2, whence it follows that

$$A_1 = 1 - (\alpha_1 + \alpha_2); \qquad B_1 = \alpha_1\alpha_2\,.$$

Similarly, from the determining equation at the point $z = 1$ we obtain

$$A_2 = 1 - (\beta_1 + \beta_2); \qquad B_2 = \beta_1\beta_2\,.$$

The determining equation at the point $z = \infty$ has the form:

$$\varrho(\varrho-1) + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\,\varrho + \alpha_1\alpha_2 + \beta_1\beta_2 + C_2 = 0\,.$$

Substituting one of its zeros $\varrho = \gamma_1$ we find an expression for C_2:

$$C_2 = -\gamma_1(\gamma_1 - 1) - (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\,\gamma_1 - (\alpha_1\alpha_2 + \beta_1\beta_2)\,,$$

and the condition (60) finally gives $C_1 = -C_2$. It thus appears that in the case of three singularities the coefficients of the equation are fully determined by the zeros of the determining equations at the singularities. It follows from the above calculations that the zeros at the points $z = 0$ and $z = 1$ can be chosen arbitrarily but at the point $z = \infty$ only one zero can be chosen arbitrarily. The second zero is fully determined from the condition that the sum of all six zeros is equal to unity (one less than the number of singularities at a finite distance).

The solutions of the constructed equation are sometimes denoted by the following symbol:

$$P\begin{pmatrix} 0, & 1, & \infty \\ \alpha_1, & \beta_1, & \gamma_1; & z \\ \alpha_2, & \beta_2, & \gamma_2 & \end{pmatrix} \tag{61}$$

which was introduced by Riemann.

We shall now introduce an elementary transformation of the function w, so as to simplify the form of this equation. Notice that if we replace w by a new unknown function u, according to the formula:

$$w = z^p(z-1)^q u; \quad u = z^{-p}(z-1)^{-q} w,$$

we obtain another equation for the new function with three singularities: at $z = 0$, $z = 1$ and $z = \infty$, but instead of the zeros a_1 and a_2 of the determining equation the presence of the factor $z^{-p}(z-1)^{-q}$ will give at the point $z = 0$ new zeros $a_1 - \varrho$ and $a_2 - \varrho$. Similarly at the point $z = 1$ the new zeros of the determining equation will be $\beta_1 - q$ and $\beta_2 - q$. By choosing $p = a_1$ and $q = \beta_1$ we can require one of the zeros of the determining equation to be zero at the points $z = 0$ and $z = 1$; we shall assume this to be so in future.

We shall now introduce some new symbols. We denote by α and β the zeros of the determining equation at the point $z = \infty$. We have one zero at the point $z = 0$ which is equal to zero and the second zero we denote by $1 - \gamma$. Finally, at the point $z = 1$ we have one zero equal to zero and the other zero is determined from the condition that the sum of all six zeros is unity; hence it will be equal to $\gamma - \alpha - \beta$. Thus instead of the general symbol (61) we can investigate the particular case

$$P\begin{pmatrix} 0, & 1, & \infty \\ 0, & 0, & \alpha;\ z \\ 1-\gamma, & \gamma-\alpha-\beta, & \beta \end{pmatrix} \tag{61$_1$}$$

The coefficients of the equation are determined from the above calculations, if we assume that

$$a_1 = 0;\ a_2 = 1-\gamma;\ \beta_1 = 0;\ \beta_2 = \gamma - \alpha - \beta;\ \gamma_1 = \alpha;\ \gamma_2 = \beta.$$

We thus obtain the following equation:

$$w'' + \frac{-\gamma + (1+\alpha+\beta)z}{z(z-1)} w' + \frac{\alpha\beta}{z(z-1)} w = 0. \tag{62}$$

This equation is known as the *hypergeometric differential equation* or the *Gauss equation*. We shall now construct its solutions in the neighbourhood of the singularities.

101. The Hypergeometric series. Let us construct, to start with, the solutions of the equation (62) in the neighbourhood of the singularity $z = 0$. These solutions have the form:

$$P_1(z); \quad z^{1-\gamma} P_2(z), \tag{63}$$

where $P_1(z)$ and $P_2(z)$ are McLaurin's series with constant terms. Let us find, first of all, the first of the above solutions. To do so rewrite the equation (62) as follows:

$$z(z-1)w'' + [-\gamma + (1+\alpha+\beta)z]w' + \alpha\beta w = 0,$$

and in its left-hand side put

$$w_1 = 1 + c_1 z + c_2 z^2 + \dots$$

Using the usual method of undetermined coefficients we finally obtain a solution in the following form:

$$w_1 = F(\alpha, \beta, \gamma;\ z) = 1 + \frac{\alpha\beta}{1!\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} z^2 + \dots +$$

$$+ \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\gamma(\gamma+1)\dots(\gamma+n-1)} z^n + \dots, \qquad (64)$$

where $F(\alpha, \beta, \gamma; z)$ denotes the infinite series on the right-hand side. Owing to the fact that the singularity nearest the origin is the point $z = 1$ we can maintain that the above series must converge in the circle $|z| < 1$. This series is usually known as the hypergeometric series. When $\alpha = \beta = \gamma = 1$ the terms are in a geometric progression. We investigated this series earlier in [I, 141]

To find the second solution of (63) we shall use the elementary transformation of the function described in the previous section. We replace w by a new unknown function, according to the formula

$$w = z^{1-\gamma}u; \quad u = z^{\gamma-1}w = \frac{1}{z^{1-\gamma}}w. \qquad (65)$$

For this new unknown function the zeros 0 and $(1-\gamma)$ of the determining equation at the point $z = 0$ become $(\gamma - 1)$ and zero. The zeros 0 and $(\gamma - \alpha - \beta)$ at the point $z = 1$ remain unchanged and, finally, from the second of the formulae (65), the zeros α and β at the point $z = \infty$ become $(\alpha + 1 - \gamma)$ and $(\beta + 1 - \gamma)$ respectively.

In fact, before the transformation, the expansions in the neighbourhood of $z = \infty$ were of the form:

$$w_1 = \left(\frac{1}{z}\right)^{\alpha} P_1\left(\frac{1}{z}\right) \quad \text{and} \quad w_2 = \left(\frac{1}{z}\right)^{\beta} P_2\left(\frac{1}{z}\right).$$

After the transformation these expansions became

$$u_1 = \left(\frac{1}{z}\right)^{\alpha+1-\gamma} P_1\left(\frac{1}{z}\right) \quad \text{and} \quad u_2 = \left(\frac{1}{z}\right)^{\beta+1-\gamma} P_2\left(\frac{1}{z}\right).$$

Hence the new unknown function will be described by the following symbol:

$$P\begin{pmatrix} 0, & 1, & \infty \\ 0, & 0, & \alpha+1-\gamma;\ z \\ \gamma-1, & \gamma-\alpha-\beta, & \beta+1-\gamma \end{pmatrix}.$$

Comparing this Riemann symbol with the symbol (61_1) and denoting the parameters (α, β, γ) which correspond to this new Riemann symbol, by α_1, β_1 and γ_1 we obtain:

$$1-\gamma_1=\gamma-1;\quad \alpha_1=\alpha+1-\gamma;\quad \beta_1=\beta+1-\gamma,$$

i.e.

$$\alpha_1=\alpha+1-\gamma;\quad \beta_1=\beta+1-\gamma;\quad \gamma_1=2-\gamma.$$

The solution of the new equation, which is regular at the origin $z=0$, will, therefore, be as follows:

$$u=F(\alpha_1,\beta_1,\gamma_1;\ z)=F(\alpha+1-\gamma,\quad \beta+1-\gamma,\quad 2-\gamma;\ z),$$

hence, from (65), we obtain:

$$w_2=z^{1-\gamma}F(\alpha+1-\gamma,\quad \beta+1-\gamma,\quad 2-\gamma;\ z).$$

This will, in fact, be the second of the solutions (63).

Let us now try to find the solutions of the equation (62) in the neighbourhood of the singularity $z=1$. To do so let us introduce a new independent variable, according to the formula

$$z'=1-z.$$

The point $z=0$ becomes $z'=1$ and the point $z=1$ becomes $z'=0$; finally $z=\infty$ becomes $z'=\infty$. Thus this new independent variable also satisfies a Gauss equation and the function w will be described by the following symbol:

$$P\begin{pmatrix} 0, & 1, & \infty \\ 0, & 0, & \alpha;\ z' \\ \gamma-\alpha-\beta, & 1-\gamma, & \beta \end{pmatrix},$$

whence we obtain the following values for the parameters (α, β, γ):

$$\alpha_1=\alpha;\quad \beta_1=\beta;\quad \gamma_1=1+\alpha+\beta-\gamma.$$

In the neighbourhood of $z'=0$ we have two solutions:

$$F(\alpha,\beta,1+\alpha+\beta-\gamma;\ z');$$

$$z'^{\gamma-\alpha-\beta}F(\gamma-\beta,\quad \gamma-\alpha,\quad 1+\gamma-\alpha-\beta;\ z'),$$

or returning to the former independent variable, we obtain the following two solutions in the neighbourhood of $z = 1$:

$$\left.\begin{aligned} w_3 &= F(\alpha, \beta, \quad 1 + \alpha + \beta - \gamma; \quad 1 - z); \\ w_4 &= (1 - z)^{\gamma - \alpha - \beta} F(\gamma - \beta, \quad \gamma - \alpha, \quad 1 + \gamma - \alpha - \beta; \quad 1 - z). \end{aligned}\right\} \quad (64_2)$$

To construct the solutions in the neighbourhood of $z = \infty$ we perform the transformation of the independent variable:

$$z' = \frac{1}{z}; \quad z = \frac{1}{z'},$$

after which the point $z = 1$ remains in its former place and the positions of the points $z = 0$ and $z = \infty$ are interchanged. With this new variable the function w is described by the following symbol:

$$P\begin{pmatrix} 0, & 1, & \infty \\ \alpha, & 0, & 0; \ z' \\ \beta, & \gamma - \alpha - \beta, & 1 - \gamma \end{pmatrix}.$$

Continuing further the transformation:

$$w = z'^{\alpha} u; \quad u = \frac{1}{z'^{\alpha}} w, \quad (65_1)$$

we obtain the Riemann symbol for the function u, which corresponds to a Gauss equation:

$$P\begin{pmatrix} 0, & 1, & \infty \\ 0, & 0, & \alpha; \ z \\ \beta - \alpha, & \gamma - \alpha - \beta, & 1 + \alpha - \gamma \end{pmatrix}.$$

The parameters of the Gauss equation will be:

$$\alpha_1 = \alpha; \quad \beta_1 = 1 + \alpha - \gamma; \quad \gamma_1 = 1 + \alpha - \beta,$$

and we have the following two solutions of the function u in the neighbourhood of $z' = 0$:

$$\begin{aligned} u_1 &= F(\alpha, \quad 1 + \alpha - \gamma, \quad 1 + \alpha - \beta; \quad z'); \\ u_2 &= z'^{\beta - \alpha} F(\beta, \quad 1 + \beta - \gamma, \quad 1 + \beta - \alpha; \ z'), \end{aligned}$$

whence, bearing in mind (65_1) and the fact that $z' = 1/z$, we obtain two solutions of the equation (62) in the neighbourhood of $z = \infty$:

$$\left.\begin{aligned} w_5 &= \left(\frac{1}{z}\right)^{\alpha} F\left(\alpha, \quad 1 + \alpha - \gamma, \quad 1 + \alpha - \beta; \ \frac{1}{z}\right), \\ w_6 &= \left(\frac{1}{z}\right)^{\beta} F\left(\beta, \quad 1 + \beta - \gamma, \quad 1 + \beta - \alpha; \ \frac{1}{z}\right). \end{aligned}\right\} \quad (64_3)$$

We thus see that all six solutions obtained in the neighbourhood of each singularity are expressed by hypergeometric series. We assumed in all above calculations that the difference of the zeros of the determining equation is not an integer. Bearing in mind the expansions in the neighbourhood of the singularities we can maintain that the formulae (64_2) holds when $|z-1|<1$ and the formulae (64_3) when $|z|>1$. Notice that the solution (64) also has a meaning when γ is an integer.

In [I, 141] we investigated the convergence of a hypergeometric series when $x=1$ and showed that it will converge provided the following condition is satisfied:

$$\gamma-\alpha-\beta>0\,, \tag{66}$$

where α, β and γ are real. In this case, according to the second of Abel's theorems [I, 149] we have $F(\alpha,\beta,\gamma;x)\to F(\alpha,\beta,\gamma;1)$ as $x\to 1-0$ and

$$F(\alpha,\beta,\gamma;1)=1+\frac{\alpha\beta}{1!\,\gamma}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}+\dots$$

We shall prove the formula

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,. \tag{67}$$

Comparing the coefficients of x^n it is easy to prove the relationship:

$$\gamma[\gamma-1-(2\gamma-\alpha-\beta-1)\,x]\,F(\alpha,\beta,\gamma;x)+(\gamma-\alpha)\,(\gamma-\beta)\,xF(\alpha,\beta,\gamma+1;x)=$$
$$=\gamma(\gamma-1)\,(1-x)\,F(\alpha,\beta,\gamma-1;x)\qquad(|x|<1)$$

or

$$\gamma[\gamma-1-(2\gamma-\alpha-\beta-1)\,x]\,F(\alpha,\beta,\gamma;x)+(\gamma-\alpha)\,(\gamma-\beta)\,xF(\alpha,\beta,\gamma+1;x)=$$
$$=\gamma(\gamma-1)\left[1+\sum_{n=1}^{\infty}(v_n-v_{n-1})\,x^n\right], \tag{68}$$

where v_n is the coefficient of x^n in the expansion of $F(\alpha,\beta,\gamma-1;x)$. From the condition (66) we will show that it follows that $v_n\to 0$ as $n\to\infty$.

We have [I, 141]:

$$\frac{|v_n|}{|v_{n+1}|}=1+\frac{\gamma-\alpha-\beta}{n}+\frac{\omega_n}{n^2}\,,$$

where the modulus of ω_n is bounded as $n\to\infty$. Let p be a positive integer which is such that $p(\gamma-\alpha-\beta)>1$. We can then write:

$$\frac{|v_n|^p}{|v_{n+1}|^p}=1+\frac{p(\gamma-\alpha-\beta)}{n}+\frac{\omega_n'}{n^2}\,,$$

where the modulus of ω_n' is bounded. From this equation and from the inequality $p(\gamma-\alpha-\beta)>1$ it follows that the series, which is the sum of the terms $|v_n|^p$ converges [I, 141] and therefore $v_n\to 0$ as $n\to\infty$. If in the

formula (68) x tends to unity, then using the second of Abel's theorems we obtain:

$$\gamma(\alpha+\beta-\gamma)\,F(\alpha,\beta,\gamma;\,1)+(\gamma-\alpha)\,(\gamma-\beta)\,F(\alpha,\beta,\gamma+1;\,1)=0\,,$$

i.e.

$$F(\alpha,\beta,\gamma;\,1)=\frac{(\gamma-\alpha)\,(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)}\,F(\alpha,\beta,\gamma+1;\,1)\,.$$

Using this relationship several times we can write:

$$F(\alpha,\beta,\gamma;\,1)=\left[\prod_{k=0}^{m-1}\frac{(\gamma-\alpha-k)\,(\gamma-\beta+k)}{(\gamma+k)\,(\gamma-\alpha-\beta+k)}\right]F(\alpha,\beta,\gamma+m;\,1)\,. \tag{69}$$

As $m\to\infty$ the product in the square brackets has the limit [73]:

$$\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,.$$

We will now show that $F(\alpha,\beta,\gamma+m;\,1)\to 1$ as $m\to\infty$. Denoting by $u_n(\alpha,\beta,\gamma)$ the coefficient of x^n in the expansion of $F(\alpha,\beta,\gamma;x)$ we can write:

$$|\,F(\alpha,\beta,\gamma+m;\,1)-1\,|\leqslant\sum_{n=1}^{\infty}|\,u_n(\alpha,\beta,\gamma+m)\,|\,.$$

In the numerator of the expression $u_n(\alpha,\beta,\gamma+m)$ we replace α and β by $|\,\alpha\,|$ and $|\,\beta\,|$ and the sum $\gamma+m$ by the difference $m-|\,\gamma\,|$, where $m>|\,\gamma\,|$ to obtain:

$$|\,F(\alpha,\beta,\gamma+m;\,1)-1\,|\leqslant\sum_{n=1}^{\infty}u_n(|\,\alpha\,|,|\,\beta\,|,m-|\,\gamma\,|)\,,$$

where the series on the right-hand side is composed of positive terms. Taking $|\,\alpha\,|\,|\,\beta\,|\,(m-|\,\gamma\,|)$ outside the brackets and replacing $m!$ by $(m-1)!$ we have:

$$|\,F(\alpha,\beta,\gamma+m;\,1)-1\,|<\frac{|\,\alpha\,|\,|\,\beta\,|}{m-|\,\gamma\,|}\cdot\sum_{n=0}^{\infty}u_n(|\,\alpha\,|+1,\,|\,\beta\,|+1,\,m-|\,\gamma\,|+1)\,.$$

When m is sufficiently large the arguments $\alpha_1=|\,\alpha\,|+1$, $\beta_1=|\,\beta\,|+1$ and $\gamma_1=m-|\,\gamma\,|+1$ satisfy the condition (66) and the series on the right-hand side of the above inequality converges while its terms, and therefore the sum on the whole, decrease as m increases. The first factor $|\,\alpha\,|\,|\,\beta\,|\,(m-|\gamma|)\to 0$ as $m\to\infty$ and, therefore, $F(\alpha,\beta,\gamma+m;\,1)\to 1$, as $m\to\infty$. Formula (69) finally brings us back to (67).

Using the formula (67) we can express the solution w_1 in terms of the linearly-independent solutions w_3 and w_4. All these solutions hold in that domain of the plane common to both the circles, centres $z=0$ and $z=1$ and unit radii. We have:

$$F(\alpha,\beta,\gamma;x)=C_1\,F(\alpha,\beta,1+\alpha+\beta-\gamma;\,1-x)+$$
$$+C_2(1-x)^{\gamma-\alpha-\beta}\,F(\gamma-\alpha,\,\gamma-\beta,\,1+\gamma-\alpha-\beta;\,1-x)\,.$$

Assuming that α, β and γ satisfy the inequalities: $1 > \gamma > \alpha + \beta$ we can also assume in this equation that $x = 1$ and $x = 0$ and thus determine C_1 and C_2. Using the formula (67), the equation (122) from [71] and the formula given below which can easily be proved

$$\sin \pi\alpha \sin \pi\beta = \sin \pi(\gamma - \alpha) \sin \pi(\gamma - \beta) - \sin \pi\gamma \sin \pi(\gamma - \alpha - \beta),$$

we arrive at the following equation:

$$\begin{aligned} \Gamma(\gamma - \alpha)\,\Gamma(\gamma - \beta)\,\Gamma(\alpha)\,\Gamma(\beta)\,F(\alpha, \beta, \gamma; x) = \\ = \Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\gamma - \alpha - \beta)\,F(\alpha, \beta, 1 + \alpha + \beta - \gamma; 1 - x) + \\ + \Gamma(\gamma)\,\Gamma(\gamma - \beta)\,\Gamma(\alpha + \beta - \gamma)\,(1 - x)^{\gamma - \alpha - \beta} \times \\ \times F(\gamma - \alpha; \gamma - \beta, 1 + \gamma - \alpha - \beta; 1 - x). \end{aligned} \tag{70}$$

We have proved this formula for $1 > \gamma > \alpha + \beta$. It can be shown that this formula will hold in all cases when $(\gamma - \alpha - \beta)$ is not an integer.

102. The Legendre polynomials. Let us now consider a very important form of the hypergeometric series. To start with let us establish a general transformation for linear equations of the second order. Let us consider the following equation of the second order:

$$a(z)\,w'' + b(z)\,w' + c(z)\,w = 0. \tag{71}$$

We can find a factor $f(z)$ which is such that when the first two terms on the left-hand side of (71) are multiplied by it they become derivatives of a product, i.e.

$$a(z)\,f(z)\,w'' + b(z)\,f(z)\,w' = \frac{\mathrm{d}}{\mathrm{d}z}\left[a(z)\,f(z)\,w'\right].$$

Where we must have:

$$b(z)\,f(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left[a(z)\,f(z)\right],$$

whence

$$a(z)\,f'(z) + \left[a'(z) - b(z)\right] f(z) = 0$$

or

$$\frac{f'(z)}{f(z)} + \frac{a'(z)}{a(z)} - \frac{b(z)}{a(z)} = 0,$$

i.e. we can take

$$f(z) = \frac{1}{a(z)}\,\mathrm{e}^{\int \frac{b(z)}{a(z)}\,\mathrm{d}z}, \tag{72}$$

from which we have:

$$p_1(z) = a(z)\,f(z) = \mathrm{e}^{\int \frac{b(z)}{a(z)}\,\mathrm{d}z};\quad q_1(z) = c(z)\,f(z) = \frac{c(z)}{a(z)}\,\mathrm{e}^{\int \frac{b(z)}{a(z)}\,\mathrm{d}z}, \tag{73}$$

and the equation (71) takes the form:

$$\frac{d}{dz}[p_1(z)\,w'] + q_1(z)\,w = 0\,. \tag{74}$$

Performing the above operations with the Gauss equation (62) we obtain it in the form:

$$\frac{d}{dz}[z^{\gamma}(z-1)^{\alpha+\beta+1-\gamma}\,w'] + \alpha\beta z^{\gamma-1}(z-1)^{\alpha+\beta-\gamma}\,w = 0\,. \tag{75}$$

We shall now try to establish a general formula for the hypergeometric series. Differentiating the series (64) n times we obtain:

$$w_1^{(n)} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)} \times \\ \times \left[1 + \frac{(\alpha+n)\,(\beta+n)}{1!(\gamma+n)}\,z + \ldots\right]$$

or

$$w_1^{(n)} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)} \times \\ \times F(\alpha+n, \beta+n, \gamma+n;\ z) \tag{76}$$

i.e. the nth derivative of the hypergeometric series (64) is equal to a factor multiplied by the hypergeometric series with the parameters $\alpha+n$, $\beta+n$ and $\gamma+n$. Hence the function $w_1^{(n)}$ will satisfy the equation (75) if we replace α, β and γ in it by $\alpha+n$, $\beta+n$ and $\gamma+n$, i.e.

$$\frac{d}{dz}\left[z^{\gamma+n}(z-1)^{\alpha+\beta+1-\gamma+n}\frac{dw_1^{(n)}}{dz}\right] + \\ + (\alpha+n)\,(\beta+n)\,z^{\gamma-1+n}(z-1)^{\alpha+\beta-\gamma+n}\,w_1^{(n)} = 0\,.$$

Differentiating this identity n times we obtain the new identity:

$$\frac{d^{n+1}}{dz^{n+1}}\left[z^{\gamma+n}(z-1)^{\alpha+\beta+1-\gamma+n}\frac{dw_1^{(n)}}{dz}\right] = \\ = -(\alpha+n)\,(\beta+n)\,\frac{d^n}{dz^n}[z^{\gamma-1+n}(z-1)^{\alpha+\beta-\gamma+n}\,w_1^{(n)}]\,.$$

We shall write out this identity for the values $n = 0, 1, 2, \ldots, k-1$ and multiply term by term the identities so obtained. The left and right-hand sides of this identity will contain similar factors and after

simplification we arrive at the required identity:

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left[z^{\gamma+k-1}(z-1)^{\alpha+\beta-\gamma+k}\,w_1^{(k)}\right] =$$

$$= (-1)^k\,\alpha(\alpha+1)\ldots(\alpha+k-1)\times$$

$$\times\,\beta(\beta+1)\ldots(\beta+k-1)\,z^{\gamma-1}(z-1)^{\alpha+\beta-\gamma}\,w_1 \qquad (77)$$

$$(k=1,\ 2,\ 3,\ldots).$$

Let us recall that in this identity the symbol w_1 denotes the hypergeometric series (64).

Notice that a hypergeometric series generally ends abruptly and becomes a polynomial when either α or β, which are symmetrical in the hypergeometric series, is a negative integer. We shall now consider one particular case when this is so, viz. we shall take the hypergeometric series

$$F(k+1,\ -k,\ 1;\ z) \qquad (\alpha=k+1,\ \beta=-k,\ \gamma=1), \qquad (78)$$

where k is a positive integer or zero. The function (78) will simply be a polynomial of degree k and the coefficient of the last term z^k of this polynomial will be

$$\frac{(k+1)\,(k+2)\ldots 2k(-k)\,(-k+1)\ldots(-1)}{k!\ 1\cdot 2\ldots k} = (-1)^k\,\frac{2k!}{(k!)^2}.$$

On supposing in the formula (77) that $w_1 = F(k+1,\ -k,\ 1;\ z)$, i.e. $\alpha = k+1,\ \beta = -k$ and $\gamma = 1$ we obtain $\omega_1^{(k)} = (-1)^k\ 2k\,!/k!$ and performing the obvious simplifications we have:

$$F(k+1,\ -k,\ 1;\ z) = \frac{(-1)^k}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left[z^k(z-1)^k\right]. \qquad (79)$$

Let us now replace z by a new independent variable x according to the formula

$$z = \frac{1-x}{2}. \qquad (80)$$

The points $z = 0$ and $z = 1$ become $x = 1$ and $x = -1$. Put

$$P_k(x) = F\left(k+1,\ -k,\ 1;\ \frac{1-x}{2}\right). \qquad (81)$$

Substituting (80) in (79) we obtain the following expression for the polynomial $P_k(x)$:

$$P_k(x) = \frac{1}{k!\,2^k}\,\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[(x^2-1)^k\right]. \qquad (82)$$

These polynomials $P_k(x)$ are usually known as the *Legendre polynomials*. We shall use them in future in connection with spherical functions.

We shall now explain some of their fundamental properties. The function (79) satisfies an equation which can be obtained from the equation (75) by writing

$$\alpha = k + 1; \qquad \beta = -k; \qquad \gamma = 1, \tag{83}$$

i.e. the function (79) satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}z}[z(z-1)w'] - k(k+1)w = 0.$$

Replacing the independent variable, according to the formula (80) we can see that *Legendre's polynomials* $P_k(x)$ *are the solutions of the equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_k(x)}{\mathrm{d}x}\right] + k(k+1)P_k(x) = 0. \tag{84}$$

Let us now consider a more general equation in the form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \lambda y = 0, \tag{85}$$

where λ is a parameter. Both zeros of this equation are equal to zero at the singularities $x = \pm 1$. This can easily be seen from the form of the equation and it also follows from the condition (83). We have

$$\gamma - 1 = 0; \qquad \gamma - \alpha - \beta = 0.$$

Hence at the point $x = \pm 1$ we have one regular solution and another solution containing a logarithm; this latter solution will have the following form at the point $x = 1$:

$$P_1(x-1) + P_2(x-1)\ \log\ (x-1),$$

where $P_1(x-1)$ and $P_2(x-1)$ are Taylor's series with constant terms. It follows from this that the solution containing the logarithm will, in any case, become infinite at the corresponding point. Notice that when both zeros of the determining equation are the same, the coefficient γ_{-1}, mentioned in [98], cannot vanish, i.e. when the zeros of the determining equation are the same there must be a solution containing a logarithm.

Let us now return to the equation (85) and take one of its solutions y_1, which is regular at the point $x = -1$. In the process of analytic continuation of this solution along the line $-1 \leqslant x \leqslant +1$

we obtain a solution which will, in general, be logarithmic at the point $x = 1$ where it becomes infinite. For certain exceptional values of the parameter λ in the equation (85), the solution, which is regular at the point $x = -1$, will also be regular at the point $x = +1$, i.e. we obtain a solution of the equation (85) which is finite in the whole interval $(-1, +1)$, including its ends. Such exceptional values will be the values given below

$$\lambda_k = k(k+1), \tag{86}$$

for which the equation (85) has a solution in the form $P_k(x)$. It can be shown, but we shall not do so here, that *the values* (86) *include all values of the parameter* λ, *for which the equation* (85) *has a finite solution in the interval* $(-1, +1)$, *including its ends.*

We shall now explain some other properties of the Legendre polynomials. Let us consider the equations of two different Legendre polynomials:

$$\frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)\,P_m'(x)] + \lambda_m P_m(x) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)\,P_n'(x)] + \lambda_n P_n(x) = 0\,. \qquad (n \neq m)$$

Multiplying the first of these equations by $P_n(x)$ and the second by $P_m(x)$, subtracting and integrating over the interval $(-1, +1)$ we obtain:

$$(\lambda_m - \lambda_n) \int_{-1}^{1} P_m(x)\,P_n(x)\,\mathrm{d}x =$$

$$= \int_{-1}^{1} \left\{ P_m(x) \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)P_n'(x)] - P_n(x) \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)\,P_m'(x)] \right\} \mathrm{d}x\,.$$

Integrating the first term on the right-hand side by parts we obtain:

$$\int_{-1}^{1} P_m(x) \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)\,P_n'(x)]\,\mathrm{d}x =$$

$$= (1-x^2)\,P_m(x)\,P_n'(x)\Big|_{x=-1}^{x=1} - \int_{-1}^{1} (1-x^2)\,P_m'(x)\,P_n'(x)\,\mathrm{d}x\,,$$

or

$$\int_{-1}^{1} P_m(x) \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)\,P_n'(x)]\,\mathrm{d}x = -\int_{-1}^{1} (1-x^2)\,P_m'(x)\,P_n'(x)\,\mathrm{d}x\,.$$

Similarly

$$\int_{-1}^{1} P_n(x) \frac{\mathrm{d}}{\mathrm{d}x} [(1 - x^2) P'_m(x)] \, \mathrm{d}x = - \int_{-1}^{1} (1 - x^2) P'_m(x) P'_n(x) \, \mathrm{d}x .$$

We thus obtain:

$$(\lambda_m - \lambda_n) \int_{-1}^{1} P_m(x) P_n(x) \, \mathrm{d}x = 0$$

or

$$\int_{-1}^{1} P_m(x) P_n(x) \, \mathrm{d}x = 0 \qquad (m \neq n) , \tag{87}$$

i.e. *Legendre's polynomials are orthogonal in the interval* $(-1, +1)$. If we were to attempt the integration of the square of the Legendre polynomial

$$I_n = \int_{-1}^{1} P_n^2(x) \, \mathrm{d}x , \tag{88}$$

we would find that the integral is not unity. Legendre's polynomials form an orthogonal but not normal system of functions. Bearing in mind (82) and using Leibnitz's formula we can write:

$$P_k(x) = \frac{1}{k! \, 2^k} \left\{ (x + 1)^k \frac{\mathrm{d}^k (x - 1)^k}{\mathrm{d}x^k} + \frac{k}{1} \frac{\mathrm{d}(x + 1)^k}{\mathrm{d}x} \cdot \frac{\mathrm{d}^{k-1}(x - 1)^k}{\mathrm{d}x^{k-1}} + \ldots \right\}.$$

We therefore have:

$$\frac{\mathrm{d}^k (x - 1)^k}{\mathrm{d}x^k} = k! \quad \text{and} \quad \frac{\mathrm{d}^{k-s}(x - 1)^k}{\mathrm{d}x^{k-s}} \bigg|_{x=1} = 0 \qquad (s = 1, 2, \ldots, k) ,$$

from which follow the equations

$$P_k(1) = 1 . \tag{89}$$

Let us evaluate the integral I_n. Using the formula (82) we can write

$$I_n = \frac{1}{(n!)^2 \, 2^{2n}} \int_{-1}^{1} \frac{\mathrm{d}^n (x^2 - 1)^n}{\mathrm{d}x^n} \cdot \frac{\mathrm{d}^n (x^2 - 1)^n}{\mathrm{d}x^n} \, \mathrm{d}x .$$

Integrating by parts we obtain:

$$I_n = \frac{1}{(n!)^2 \, 2^{2n}} \left[\frac{\mathrm{d}^{n-1}(x^2 - 1)^n}{\mathrm{d}x^{n-1}} \cdot \frac{\mathrm{d}^n (x^2 - 1)^n}{\mathrm{d}x^n} \bigg|_{x=-1}^{x=1} - \right.$$

$$\left. - \int_{-1}^{1} \frac{\mathrm{d}^{n-1}(x^2 - 1)^n}{\mathrm{d}x^{n-1}} \cdot \frac{\mathrm{d}^{n+1}(x^2 - 1)^n}{\mathrm{d}x^{n+1}} \, \mathrm{d}x \right].$$

The polynomial $(x^2-1)^n$ has zeros $x=\pm 1$ of order n. Differentiating it $(n-1)$ times we obtain a polynomial which also has zeros $x=\pm 1$ (of order one). Hence in the above formula the term outside the integral vanishes. Continuing the integration by parts we obtain in every case a zero term outside the integral and we thus arrive at the formula:

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{(-1)^n}{(n!)^2\,2^{2n}} \int_{-1}^{1} (x^2-1)^n \frac{d^{2n}(x^2-1)^n}{dx^{2n}}\,dx .$$

We have further

$$\frac{d^{2n}(x^2-1)^n}{dx^{2n}} = \frac{d^{2n}}{dx^{2n}}(x^{2n}+\ldots) = 1\cdot 2\ldots 2n = (2n)!$$

and consequently:

$$\int_{-1}^{1} P_n^2(x)\,dx = (-1)^n \frac{(n+1)(n+2)\ldots 2n}{n!\,2^{2n}} \int_{-1}^{1} (x^2-1)^n\,dx .$$

Putting $x=\cos\varphi$ we obtain:

$$\int_{-1}^{1} (x^2-1)^n\,dx = (-1)^n \int_0^{\pi} \sin^{2n+1}\varphi\,d\varphi = (-1)^n\,2 \int_0^{\frac{\pi}{2}} \sin^{2n+1}\varphi\,d\varphi ,$$

i.e. [I, 100]:

$$\int_{-1}^{1} (x^2-1)^n\,dx = (-1)^n\,2\,\frac{2\cdot 4\ldots 2n}{3\cdot 5\ldots(2n+1)} ,$$

and the above formula finally gives:

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} . \tag{90}$$

Using the expression (82) and applying Rolle's theorem it can easily be shown that all zeros of the polynomial $P_n(x)$ are different and lie in the interval $-1 \leqslant x \leqslant +1$. In fact, the polynomial $d(x^2-1)^n/dx$ of the $(2n-1)$th degree has zeros $x=\pm 1$ of order $(n-1)$; according to Rolle's theorem there is still another zero $x=a$ in the interval $(-1, +1)$. There are no more other zeros. The polynomial $d^2(x^2-1)^n/dx^2$ of the $(2n-2)$th degree has the zeros $x=\pm 1$ of order $(n-2)$; also, according to Rolle's theorem, it has two more real

zeros, one in the interval $(-1, \alpha)$ and the other in the interval $(\alpha, +1)$. Continuing in this way we can see that $P_n(x)$ *has n different zeros in the interval* $(-1, +1)$.

103. Jacobian polynomials. Legendre's polynomials are only one particular example of polynomials obtained when a hypergeometric series ends abruptly and becomes a polynomial. We shall now investigate the general case. Let us introduce the following notation:

$$\gamma - 1 = p; \qquad \alpha + \beta - \gamma = q \tag{91}$$

and suppose that p and q are fixed numbers, greater than (-1); we shall always assume that the parameters α, β and γ are real numbers. If a hypergeometric series is to end abruptly and become a polynomial of degree k we must assume that either α or β is equal to $(-k)$. Without loss of generality we can, for example, assume that $\beta = -k$ and determine α and γ subsequently from the equation (91). The polynomial so obtained we denote as follows:

$$Q_k^{(p,q)}(z) = C_k F(p + q + k + 1, \quad -k, \quad p + 1; \; z), \tag{92}$$

where C_k is an arbitrary constant.

Applying to the above hypergeometric series the formula (64) we can see that the coefficient of z^k in the polynomial (92) is equal to

$$(-1)^k \frac{(p+q+k+1)(p+q+k+2)\ldots(p+q+2k)}{(p+1)(p+2)\ldots(p+k)} C_k.$$

Applying the formula (77) to the polynomial (92) we have:

$$\begin{aligned} k!(p+q+k+1)(p+q+k+2)\ldots \\ \ldots(p+q+2k)\, z^p(z-1)^q Q_k^{(p,q)}(z) = \\ = (-1)^k \frac{(p+q+k+1)(p+q+k+2)\ldots(p+q+2k)\,k!}{(p+1)(p+2)\ldots(p+k)} \times \\ \times C_k \frac{d^k}{dz^k}[z^{p+k}(z-1)^{q+k}]. \end{aligned}$$

The constant C_k is determined from the formula

$$C_k = \frac{(p+1)(p+2)\ldots(p+k)}{k!}.$$

In this case we obtain the following formula for the constructed polynomial:

$$z^p(z-1)^q Q_k^{(p,q)}(z) = \frac{(-1)^k}{k!} \frac{d^k}{dz^k}[z^{p+k}(z-1)^{q+k}].$$

If we replace z by x, according to the formula (80), we obtain polynomials in x which are known as *Jacobian polynomials:*

$$(1-x)^p (1+x)^q P_k^{(p,q)}(x) = \frac{(-1)^k}{k!\, 2^k} \frac{\mathrm{d}^k}{\mathrm{d}x^k} [(1-x)^{p+k} (1+x)^{q+k}]. \quad (93)$$

When $p = q = 0$ these polynomials are the same as Legendre's polynomials. When $k = 0$, $P_0^{(p,q)}(x) = 1$.

It follows directly from the determination of C_k that the coefficient of z^k in the polynomial (92) is equal to:

$$(-1)^k \frac{(p+q+k+1)(p+q+k+2)\ldots(p+q+2k)}{k!},$$

and in the polynomial $P_k^{(p,q)}(x)$ the coefficient of x^k is equal to:

$$a_k = \frac{(p+q+k+1)(p+q+k+2)\ldots(p+q+2k)}{k!\, 2^k}.$$

In the case under consideration we have:

$$\alpha = p+q+k+1, \quad \beta = -k, \quad \gamma = p+1,$$

and the function (92) is a solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}z} [z^{p+1}(z-1)^{q+1} w'] - k(p+q+k+1) z^p (z-1)^q w = 0.$$

Replacing the independent variable according to (80) we obtain the following equation for Jacobian polynomials:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[(1-x)^{p+1} (1+x)^{q+1} \frac{\mathrm{d}P_k^{(p,q)}(x)}{\mathrm{d}x}\right] + \\ + k(p+q+k+1)(1-x)^p (1+x)^q P_k^{(p,q)}(x) = 0. \quad (94)$$

In this case the Jacobian polynomials (93), when p and $q \geqslant 0$, are a solution of the following limit problem: what is the value of the parameter λ for which the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[(1-x)^{p+1} (1+x)^{q+1} \frac{\mathrm{d}y}{\mathrm{d}x}\right] + \lambda (1-x)^p (1+x)^q y = 0 \quad (95)$$

has a finite solution in the interval $(-1, +1)$, including its ends. The values of this parameter are

$$\lambda_k = k(p+q+k+1), \quad (96)$$

and the corresponding solution is given by Jacobian polynomials.

Using the equation (94) for Jacobian polynomials it can readily be shown that, as for Legendre polynomials, the following equations hold:

$$\int_{-1}^{1} (1-x)^p (1+x)^q P_m^{(p,q)}(x) P_n^{(p,q)}(x)\,dx = 0 \quad (m \neq n). \tag{97}$$

The properties (97) can be expressed as follows: *Jacobian polynomials are orthogonal in the interval* $(-1, +1)$ *with the weight*

$$r(x) = (1-x)^p (1+x)^q. \tag{98}$$

As with Legendre's polynomials, we can deduce from the formula (93), that

$$P_k^{(p,q)}(1) = \frac{(p+k)(p+k-1)\ldots(p+1)}{k!}. \tag{99}$$

Let us now evaluate the integral:

$$I_k = \int_{-1}^{1} (1-x)^p (1+x)^q [P_k^{(p,q)}(x)]^2\,dx.$$

Using the formula (93) we can write:

$$I_k = \frac{(-1)^k}{k!\,2^k} \int_{-1}^{1} P_k^{(p,q)}(x) \frac{d^k}{dx^k} [(1-x)^{p+k}(1+x)^{q+k}]\,dx.$$

Integrating by parts, as in [102], we obtain:

$$I_k = \frac{1}{k!\,2^k} \int_{-1}^{1} (1-x)^{p+k}(1+x)^{q+k} \frac{d^k P_k^{(p,q)}(x)}{dx^k}\,dx,$$

since the terms other than the integral vanish when $p > -1$ and $q > -1$. Denoting as before the coefficient of x^k in the polynomial $P_k^{(p,q)}(x)$ by a_k we obtain:

$$I_k = \frac{a_k}{2^k} \int_{-1}^{1} (1-x)^{p+k}(1+x)^{q+k}\,dx.$$

Introducing a new variable of integration $t = (1-x)/2$:

$$I_k = a_k\, 2^{p+q+k+1} \int_0^1 t^{p+k}(1-t)^{q+k}\,dt,$$

i.e. [72]:

$$I_k = a_k\, 2^{p+q+k+1} B(p+k+1,\ q+k+1) =$$

$$= a_k\, 2^{p+q+k+1} \frac{\Gamma(p+k+1)\,\Gamma(q+k+1)}{\Gamma(p+q+2k+2)},$$

or, substituting the above expression for a_k and using the formula (120) from [71]:

$$I_k = \frac{2^{p+q+1}}{p+q+2k+2} \cdot \frac{\Gamma(p+k+1)\,\Gamma(q+k+1)}{k!\,\Gamma(p+q+k+1)},$$

i.e. the formula holds:

$$\int_{-1}^{1} (1-x)^p (1+x)^q [P_n^{(p,q)}(x)]^2 \,dx =$$

$$= \frac{2^{p+q+1}}{2n+p+q+1} \cdot \frac{\Gamma(n+p+1)\,\Gamma(n+q+1)}{n!\,\Gamma(n+p+q+1)} \quad (n = 1, 2, \ldots). \quad (100)$$

When $n = 0$, owing to the fact that $\Gamma(x+1) = x\,\Gamma(x)$, the latter expression has the form:

$$2^{p+q+1} \frac{\Gamma(p+1)\,\Gamma(q+1)}{\Gamma(p+q+2)}.$$

Notice another particular case, viz. when $p = q = -1/2$. We shall introduce special symbols for these polynomials.

$$T_k(x) = C_k P_k^{\left(-\frac{1}{2}, -\frac{1}{2}\right)}(x), \quad (101)$$

where C_k is a constant.

They are determined, from (93), by the following relationship:

$$(1-x^2)^{-\frac{1}{2}} T_k(x) = \frac{(-1)^k C_k}{k!\,2^k} \frac{d^k}{dx^k}\left[(1-x^2)^{-\frac{1}{2}+k}\right]. \quad (102)$$

We shall now introduce another expression for these polynomials i.e. a differential equation which they must satisfy. These equations can be obtained from the equation (94) by assuming that $p = q = -1/2$. We have for $T_k(x)$ the following equation:

$$\sqrt{1-x^2}\,\frac{d}{dx}\left[\sqrt{1-x^2}\,\frac{dT_k(x)}{dx}\right] + k^2 T_k(x) = 0. \quad (103)$$

The zeros of the determining equation at the singularity $x = 1$ will be z and $1/2$. The first zero corresponds to a solution in the form of a polynomial, but the second solution will not be a polynomial. To find a polynomial in a convenient form which would satisfy the equation (103) we replace x by a new independent variable φ according to the formula

$$x = \cos\varphi. \quad (104)$$

Differentiating with respect to φ instead of x we have, according to the differentiation law for complicated functions:

$$\sqrt{1-x^2}\,\frac{\mathrm{d}}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}\varphi}\,.$$

Substituting this in the equation (103) we obtain

$$\frac{\mathrm{d}^2 T_k(\cos\varphi)}{\mathrm{d}\varphi^2} + k^2 T_k(\cos\varphi) = 0\,.$$

The latter equation has the following solutions

$$\cos k\varphi \quad \text{and} \quad \sin k\varphi\,,$$

and for the equation (103) we obtain solutions in the form

$$\cos(k \operatorname{arc}\cos x) \quad \text{and} \quad \sin(k \operatorname{arc}\cos x)\,.$$

Using the formula from [I, 174]

$$\cos k\varphi = \cos^k\varphi - \binom{k}{2}\cos^{k-2}\varphi\sin^2\varphi + \ldots,$$

we can see that the first of these solutions is a polynomial in x and, consequently, except for the arbitrary term, the polynomial which gives the solution of the equation will be

$$T_k(x) = \cos(k \operatorname{arc}\cos x)\,, \tag{105}$$

and this polynomial is known as *Tschebyshev's polynomial.* When $\varphi = 0$ we have $x = 1$ and, consequently, $T_k(1) = 1$; on the other hand, from the formula (99):

$$P_k^{\left(-\frac{1}{2},\,-\frac{1}{2}\right)}(1) = \frac{1\cdot 3\ldots(2k-1)}{2\cdot 4\ldots 2k}$$

whence the constant term in the formula (101) can easily be determined.

$$T_k(x) = \frac{2\cdot 4\ldots 2k}{1\cdot 3\ldots(2k-1)}\,P_k^{\left(-\frac{1}{2},\,-\frac{1}{2}\right)}(x)\,. \tag{106}$$

104. Conformal transformation and the formula of Gauss. We shall now try to explain the connection between the equation of Gauss and a certain problem of conformal transformation; we shall again assume, as in the previous paragraph, that the coefficients α, β and γ are real. We will prove, first of all, that the solution of the equation of Gauss (62) cannot have multiple zeros in the plane of the complex

variable, except at its singularities. For suppose we have at a point $z = z_0$ a zero of order higher than one, i.e.

$$w(z_0) = w'(z_0) = 0\,,$$

then it follows from the equation (62) that $w''(z_0) = 0$. Differentiating the equation (62) and putting $z = z_0$ we obtain $w''(z_0) = 0$, etc. However, it is well known that if all the derivatives of an analytic function vanish at a certain point then the function is identically zero; but we assumed that w gives a solution which does not vanish. The above proof is also valid for any linear equation of the second degree with analytic coefficients. The result obtained can also be deduced directly from the existence and uniqueness theorems [95].

Let us now consider the quotient of two solutions of the equation of Gauss:

$$\eta(z) = \frac{w_2(z)}{w_1(z)}\,. \tag{107}$$

When this function is analytically continued it can only have singularities at the points $z = 0$, 1 and ∞, as well as at points at which the values of z are zeros of the solution $w_1(z)$. These values of z will be simple poles of the function (107). If $w_1(z_0) = 0$, we can say that $w_2(z_0) \neq 0$. In fact, if we had

$$w_2(z_0) = 0\,,$$

then our two solutions would be determined from the initial conditions

$$\left.\begin{aligned} w_1(z_0) &= 0; \quad w_1'(z_0) = \alpha \\ w_2(z_0) &= 0; \quad w_2'(z_0) = \beta \end{aligned}\right\} \qquad \alpha \text{ and } \beta \neq 0\,,$$

and, according to the existence and uniqueness theorems, we would have had

$$w_2(z) = \frac{\beta}{\alpha} w_1(z)\,,$$

i.e. the solutions $w_1(z)$ and $w_2(z)$ would be linearly dependent; we are given, however, that the numerator and the denominator in the formula (107) are linearly independent solutions.

Consider the upper half-plane of the complex variable z. In this connected domain B the analytic functions $w_1(z)$ and $w_2(z)$ have no singularities in the course of analytical continuation and, consequently, are single-valued regular functions of z. The function (107) will also be single-valued in the upper half-plane, where its only singularities are simple poles. We will now show that the derivative

of the function (107) cannot vanish except at the poles. We have, from [II, 24], the following expression for this derivative:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{w_2(z)}{w_1(z)}\right) = \frac{C}{w_1^2(z)}\,\mathrm{e}^{-\int p(z)\,\mathrm{d}z}, \tag{108}$$

where C is a constant and $p(z)$ is the coefficient of w' in the equation (62).

Our hypothesis follows directly from the formula (108). Bearing in mind that conformity is not impaired at a simple pole, we can say that the function (107) gives a conformal transformation of the domain B into a new domain B_1, in which there are no branch-points. We shall now determine the contour of the new domain B_1.

When the point z, in the upper half-plane approaches a point z_0 on the real axis, which is not one of the singularities 0, 1 or ∞, the function (107) tends to a definite limit and, moreover, it will remain regular at the point z_0; it will also be regular on each of the following three lines:

$$(-\infty,\ 0);\ (0,\ 1);\ (1,\ \infty) \tag{109}$$

on the real axis. We will now show that the function (107) will also tend to a definite limit even though z tends to one of the singularities. Consider, for example, the point $z = 0$.

To start with let us explain one circumstance which will be important in what follows. Suppose that instead of $w_1(z)$ and $w_2(z)$ we had taken two other independent solutions of the equation, viz. $w_1^*(z)$ and $w_2^*(z)$. These can be expressed linearly in terms of the former solutions:

$$w_1^*(z) = a_{11}\,w_1(z) + a_{12}\,w_2(z)\,,$$

$$w_2^*(z) = a_{21}\,w_1(z) + a_{22}\,w_2(z)\,,$$

where

$$a_{11}\,a_{22} - a_{12}\,a_{21} \neq 0\,.$$

Let us construct a new function $\eta^*(z)$ using these new solutions:

$$\eta^*(z) = \frac{w_2^*(z)}{w_1^*(z)} = \frac{a_{21}\,w_1(z) + a_{22}\,w_2(z)}{a_{11}\,w_1(z) + a_{12}\,w_2(z)}$$

or

$$\eta^*(z) = \frac{a_{21} + a_{22}\,\eta(z)}{a_{11} + a_{12}\,\eta(z)}\,,$$

i.e. when choosing different independent solutions in the formula (107) the corresponding functions $\eta(z)$ will be simply connected with each

other by a bilinear transformation, the determinant of which is not zero.

We shall now investigate the function (107) in the neighbourhood of the point $z = 0$. We select the independent solutions as follows:

$$\left.\begin{aligned} w_1(z) &= F(\alpha, \beta, \gamma; z); \\ w_2(z) &= z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma; z), \end{aligned}\right\} \tag{110}$$

Also

$$\eta(z) = z^{1-\gamma} \frac{F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma; z)}{F(\alpha, \beta, \gamma; z)}. \tag{111}$$

This latter formula must be interpreted as follows: in the neighbourhood of $z = 0$ the function $\eta(z)$ is determined by the formula (111); in the half-plane B this single-valued function can be determined by analytic continuation. It follows from the formula (111) that, for example,

$$\eta(z) \to 0, \quad \text{if} \quad z \to 0 \quad \text{and} \quad \gamma < 1.$$

If we choose any other solutions, the new $\eta(z)$ will be expressed by (111) as a bilinear transformation, and, consequently, it will also have a definite limit as $z \to 0$.

We will now show that the function (107) transforms the lines (109) into circular arcs. For example, let us consider, the line (0, 1) and take a point z_0 on this line. We shall determine the solutions $w_1(z)$ and $w_2(z)$ at the point z_0 from the initial conditions; we select these conditions so that $w(z_0)$ and $w'(z_0)$ are real. Bearing in mind the fact that the coefficients in the equation of Gauss are also real we obtain for $w_1(z)$ and $w_2(z)$ Taylor's series with real coefficients in the neighbourhood of the point z_0. The analytic continuation of these solutions along the line (0, 1) will evidently also lead to Taylor's series, in other words, for this choice of solutions the function $\eta(z)$ will have real values on the line (0, 1), i.e. it will transform this line into another line on the real axis. For any other choice of solutions the new function $\eta(z)$ will be obtained from the former function by the bilinear transformation which transforms the line on the real axis into the arc of a circle. Therefore the function (107) will, in fact, transform each of the lines (109) into the arc of a circle.

Let us again consider the case when the fundamental solutions $w_1(z)$ and $w_2(z)$ are real on the line (0, 1). Applying the formula (108) to this line we can see that the derivative of the function $\eta(z)$ does not change its sign on this line, i.e. the function $\eta(z)$ is a monotonic

function of the variable z on that line. In other words, if the point z moves along the line (0, 1) in a definite direction, the point $\eta(z)$ always moves along the corresponding line in the same direction.

Notice that the point $\eta(z)$ can also pass through infinity, so that the line, described by the point $\eta(z)$, can be infinite. In certain cases this line can overlap itself. In the general case, when the independent soutions in the formula (107) are chosen arbitrarily, the point z moves along the line (0, 1) in a definite direction and the point $\eta(z)$ moves along the arc of a circle always in one direction; in certain cases it is not an arc of a circle but a full circle overlaping itself which corresponds to the line (0, 1).

From what was said above we obtain the following result: *the function* (107), *i.e. the quotient of two independent solutions of the equation of Gauss, conformally transforms the upper half-plane into a domain bounded by three circular arcs or, in other words, into a circular triangle which contains no branch-points.* Let us now determine the angles of this circular triangle. Take the apex A of the triangle which corresponds to the point $z = 0$. Select the fundamental solutions by using formula (110) and assume that $\gamma < 1$. Turn to the formula (111). In the neighbourhood of the point $z = 0$ we have $\eta(z) > 0$, when $z > 0$ and we assume that $\arg z = 0$. Describing the point $z = 0$ from the upper half-plane we obtain $\arg z$ and, consequently, $\arg z^{1-\gamma} = \pi(1 - \gamma)$; the fraction in formula (111), when z is close to zero, will be real and close to unity. Hence assuming that $\gamma < 1$, we obtain two straight lines in the plane $\eta(z)$; one of these goes from the origin in the direction of the positive part of the real axis and the other makes an angle of $\pi(1 - \gamma)$ with this direction. If $\gamma > 1$, then instead of the relationship (111) we must take the inverse relationship. Thus for the given choice of fundamental solutions we have in our circular triangle an angle $\pi \mid 1 - \gamma \mid$ at the apex corresponding to the point $z = 0$. For any other choice of fundamental solutions we obtain another triangle which can be obtained from the former by a bilinear transformation. We know that this transformation conserves the angles and therefore we obtain, in general, the angle $\pi \mid 1 - \gamma \mid$ at the apex A. The procedure is similar when we determine the other two angles of the circular triangle, which correspond to the points $z = 1$ and $z = \infty$. We obtain then two angles equal to $\pi \mid \gamma - \alpha - \beta \mid$ and $\pi \mid \beta - \alpha \mid$ respectively. The direction in which these angles are measured is determined, as always in conformal transformation, from the fact that when the point moves along the real axis in the positive direction the corresponding

point moves along the contour of the circular triangle so that this triangle lies to the left of the moving observer.

The above result can be formulated as follows: *an angle of the triangle in the $\eta(z)$ plane is equal to the product of π and the modulus of the difference of the zeros of the determining equation at the corresponding singularity of the equation* (62). We notice, without going into the proof, that this will still be so when the difference is equal to zero (the arcs of the circles touch), or to an integer.

It can be shown conversely, that any circular triangle, even one with several sheets but without branch-points either inside or on the sides, can be obtained from the upper half-plane as a result of the conformal transformation by the quotient of two solutions of the equation of Gauss, when the parameters α, β and γ are suitably chosen. In particular the usual triangle with straight sides can be taken: this is a particular case of a triangle bounded by circular arcs. In this case we can express the function which performs the conformal transformation more simply, viz. by means of Christoffel's formula.

105. Irregular singularities. We shall now deal with the problem of obtaining a solution in the neighbourhood of an irregular singularity. By performing a bilinear transformation of the independent variable we can always achieve the fact that this singularity should lie at infinity and we shall assume in future that this is so. Consider the equation

$$w'' + p(z)\, w' + q(z)\, w = 0\,.$$

If $p(z)$ and $q(z)$ have the following expansions near infinity:

$$p(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k}\,; \quad q(z) = \sum_{k=2}^{\infty} \frac{b_k}{z^k}\,, \tag{112}$$

then, as we know from [99], this point will be a regular singularity. We now assume that the coefficients cannot be expanded in accordance with (112), but that the expansions of $p(z)$ and $q(z)$ near infinity do not contain positive powers of z, i.e. we consider an equation of the type:

$$w'' + \left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\right) w' + \left(b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots\right) w = 0, \tag{113}$$

where at least one of the coefficients a_0, b_0 and b_1 is not zero. This equation must be formally satisfied by an expression of the form:

$$w = z^\varrho \left(c_0 + c_1 \frac{1}{z} + \dots\right), \tag{114}$$

where $c_0 \neq 0$; substituting this in the left-hand side of the equation we obtain a single term which contains z^ϱ, viz. a term of the form $b_0 c_0 z^\varrho$. Hence it will not be possible to satisfy our equation formally by an expression of the form (114) if $b_0 \neq 0$. To eliminate the coefficient b_0 we replace w by a new function u, according to the formula

$$w = \mathrm{e}^{\alpha z} u .$$

Hence

$$w' = \mathrm{e}^{\alpha z} u' + \alpha \mathrm{e}^{\alpha z} u; \quad w'' = \mathrm{e}^{\alpha z} u'' + 2\alpha \mathrm{e}^{\alpha z} u' + \alpha^2 \mathrm{e}^{\alpha z} u ,$$

and, substituting in the equation we obtain a new equation

$$u'' + \left(2\alpha + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right) u' + \\ + \left(\alpha^2 + \alpha a_0 + b_0 + \frac{\alpha a_1 + b_1}{z} + \frac{\alpha a_2 + b_2}{z^2} + \dots\right) u = 0 .$$

The constant α now remains to be chosen from the condition

$$\alpha^2 + \alpha a_0 + b_0 = 0 . \tag{115}$$

As a result we obtain an equation of the form:

$$u'' + \left(2\alpha + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right) u' + \left(\frac{b_1'}{z} + \frac{b_2'}{z^2} + \dots\right) u = 0 \tag{116}$$

$$(b_k' = \alpha a_k + b_k) ,$$

where α is a root of the equation (115). This equation can be formally satisfied by an expression of the form (114). Let us suppose that

$$u = z^\varrho v ,$$

hence

$$u' = z^\varrho v' + \varrho z^{\varrho-1} v; \quad u'' = z^\varrho v'' + 2\varrho z^{\varrho-1} v' + \varrho(\varrho - 1) z^{\varrho-2} v .$$

Substituting in (116) we have for the new function v the following equation

$$v'' + p_1(z) v' + q_1(z) v = 0 . \tag{117}$$

where

$$\left.\begin{aligned} p_1(z) &= 2\alpha + a_0 + \frac{2\varrho + a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \dots, \\ q_1(z) &= \frac{(2\alpha + a_0)\,\varrho + b_1'}{z} + \frac{\varrho(\varrho - 1) + a_1\varrho + b_2'}{z^2} + \\ &\quad + \frac{a_2\varrho + b_3'}{z^3} + \frac{a_3\varrho + b_4'}{z^4} + \dots \end{aligned}\right\} \tag{118}$$

We will now determine ϱ from the condition that the coefficient $q_1(z)$ must not contain a term in z^{-1}, i.e.

$$(2\alpha + a_0)\,\varrho + b_1' = 0; \quad \varrho = -\frac{\alpha a_1 + b_1}{2\alpha + a_0}. \tag{119}$$

If we assume that the equation (115) has different zeros, it follows that $2\alpha + a_0 \neq 0$.

The new equation for v will be:

$$v'' + \left(2\alpha + a_0 + \frac{2\varrho + a_1}{z} + \dots\right)v' + \\ + \left(\frac{\varrho^2 + (a_1 - 1)\,\varrho + b_2'}{z^2} + \frac{a_2\varrho + b_3'}{z^3} + \dots\right)v = 0. \tag{120}$$

This equation can be formally satisfied by the series

$$v = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \tag{121}$$

Differentiating, substituting in the left-hand side of the equation and equating coefficients of like powers of x to zero we obtain a system of equations from which $c_1, c_2, \dots$, can be determined successively and where c_0 will be an arbitrary factor. Let us write the first of these equations

$$-(2\alpha + a_0)\,c_1 + [\varrho^2 + (a_1 - 1)\,\varrho + b_2']\,c_0 = 0,$$

Hence

$$c_1 = \frac{\varrho^2 + (a_1 - 1)\,\varrho + \alpha a_2 + b_2}{2\alpha + a_0}\,c_0. \tag{122}$$

We finally obtain for equation (113) a solution of the form:

$$w = e^{\alpha z} z^{\varrho}\left(c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots\right). \tag{123}$$

If the quadratic equation (115) has different zeros then by using every zero we can construct by the above method two expressions of the form (123). It appears, however, that the infinite series in the expression (123) will, in general, be a divergent series for every value of z.

We will show this to be true in one particular case. Consider the equation

$$w'' + \left(a_0 + \frac{a_1}{z}\right) w' + \frac{b_2}{z^2} w = 0 . \tag{124}$$

In this case we can assume that $\alpha = \varrho = 0$ and, substituting in the left-hand side of the equation (124) a series of the form (121), we obtain the following formulae for the determination of the coefficients:

$$[n(n+1) - na_1 + b_2] c_n - (n+1) a_0 c_{n+1} = 0 .$$

Consider the relationship of two successive terms of the series (121). Using the above formula we obtain the following formula for this relationship:

$$\frac{c_{n+1}}{z^{n+1}} : \frac{c_n}{z^n} = \frac{n(n+1) - na_1 + b_2}{(n+1) a_0} \frac{1}{z} ,$$

from which it follows that for any given z the above relationship tends to infinity together with n and, consequently, the constructed infinite series cannot converge for any value of z.

The divergence of the series in the expression (123) at first sight appears to make the above relationship devoid of meaning. It becomes evident, however, that this expression can be used to give the solution of the equation (113). To explain this circumstance we must introduce a new concept, viz. the concept of the *asymptotic expansion of a function.*

We shall lastly consider the case when the equation (115) has a double zero. In this case $2\alpha + a_0 = 0$ and the equation (116) becomes:

$$u'' + \left(\frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right) u' + \left(\frac{b_1'}{z} + \frac{b_2'}{z^2} + \dots\right) u = 0 .$$

Replacing z by a new independent variable $t = \sqrt{z}$ we obtain the equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \left(\frac{2a_1 - 1}{t} + \frac{2a_2}{t^3} + \frac{2a_3}{t^5} + \dots\right) \frac{\mathrm{d}u}{\mathrm{d}t} + \\ + \left(4b_1' + \frac{4b_2'}{t^2} + \frac{4b_3'}{t^4} + \dots\right) u = 0 . \tag{125}$$

For this differential equation the quadratic equation (116) becomes $\alpha^2 + 4b_1' = 0$ and, provided $b_1' \neq 0$, we have for the equation (125) the case of the different zeros which we considered above. If, however, $b_1' = 0$ then $t = \infty$ is a regular singularity of the equation (125).

106. Asymptotic expansion. Let us suppose that we are given an infinite series

$$c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \tag{126}$$

and denote by $S_n(z)$ the sum of its first n terms:

$$S_n(z) = c_0 + \frac{c_1}{z} + \dots + \frac{c_{n-1}}{z^{n-1}}.$$

The convergence of this series is equivalent to the existence of a limit $S_n(z)$, as n increases indefinitely. We shall now consider it differently viz.: we suppose that n is fixed and z tends to infinity along a definite straight line L. In future we shall assume that this straight line is the positive part of the real axis, i.e. $z > 0$.

We now suppose that a function $f(z)$ is determined on L, so that for any fixed n the difference

$$f(z) - S_n(z)$$

as $z \to \infty$, represents an infinitely small quantity of an order greater than $(1/z)^{n-1}$, i.e. the difference $f(z) - S_n(z)$ is an infinitely small quantity of an order higher than the last of the terms in the expression $S_n(z)$. The above conditions can be written as follows

$$\lim_{z\to\infty} [f(z) - S_n(z)]\, z^{n-1} = 0 \qquad (\text{on } L). \tag{127}$$

We usually say that *the series* (126) *gives the asymptotic expansion of the function* $f(z)$ on L and it can be written as follows:

$$f(z) \sim c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \qquad (\text{on } L). \tag{128}$$

Bearing in mind that $(c_n/z^n)z^{n-1} \to 0$ as $z \to \infty$ we can, instead of the condition (127), write the following equivalent condition:

$$\lim_{z\to\infty} \left[f(z) - \left(c_0 + \frac{c_1}{z} + \dots + \frac{c_n}{z^n} \right) \right] z^{n-1} = 0. \tag{129}$$

Consider, for example, the function which is determined when $x > 0$, by the integral

$$f(x) = \int_x^\infty t^{-1} e^{x-t}\, dt. \tag{130}$$

Performing successive integrations by parts we can write:

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \dots + \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^n n! \int_x^\infty \frac{e^{x-t}}{t^{n+1}}\, dt.$$

Construct the series

$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots \tag{131}$$

On considering the relationship of two successive terms it can readily be shown that this series will diverge for every value of x. We will show that it gives the asymptotic expression of the function (130). We have in this case:

$$f(x) - S_{n+1}(x) = (-1)^n n! \int_x^\infty \frac{e^{x-t}}{t^{n+1}} dt ,$$

whence, since $t \geqslant x$, the factor e^{x-t} lies between zero and unity, and we obtain:

$$| f(x) - S_{n+1}(x) | < n! \int_x^\infty \frac{dt}{t^{n+1}} = (n-1)! \frac{1}{x^n} ,$$

from which the condition (129) follows directly and, consequently:

$$\int_x^\infty t^{-1} e^{x-t} dt \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots \tag{132}$$

Suppose that we are given the asymptotic expansion (128). When $n = 1$, the condition (127) gives

$$\lim_{z \to \infty} [f(z) - c_0] = 0 ,$$

i.e.

$$c_0 = \lim_{z \to \infty} f(z) .$$

Furthermore the same condition, when $n = 2$, gives

$$\lim_{z \to \infty} \left[f(z) - c_0 - \frac{c_1}{z} \right] z = 0 ,$$

whence

$$c_1 = \lim_{z \to \infty} [f(z) - c_0] z ,$$

and in general we have:

$$c_n = \lim_{z \to \infty} \left[f(z) - \left(c_0 + \frac{c_1}{z} + \dots + \frac{c_{n-1}}{z^{n-1}} \right) \right] z^n . \tag{133}$$

These formulae determine in a unique way the coefficients of the asymptotic expansion when this expansion exists. It follows that *a given function can only have one asymptotic expansion.*

Consider the function e^{-x} on the straight line $x > 0$. We know that for every n we have:

$$\lim_{x\to\infty} e^{-x} x^n = 0 ,$$

i.e. the asymptotic representation of the function e^{-x} on the straight line $x > 0$ will be $e^{-x} \to 0$. Hence, for example, if a function $f(z)$ can have an asymptotic expansion along the straight line $x > 0$, then the function $f(x) + e^{-x}$ will have the same asymptotic expansion. We can see in this case that the addition to the function $f(x)$ of the term e^{-x} which decreases more rapidly than any whole negative power of x, does not change the asymptotic expansion of the function.

From the definition of the asymptotic expansion the laws of term by term multiplication and term by term integration of asymptotic expansions can be proved, viz. when:

$$f(z) \sim \sum_{k=0}^{\infty} \frac{c_k}{z^k} \quad \text{and} \quad \varphi(z) \sim \sum_{k=0}^{\infty} \frac{d_k}{z^k} ,$$

then

$$f(z)\,\varphi(z) \sim \sum_{k=0}^{\infty} \frac{c_k d_0 + c_{k-1} d_1 + \ldots + c_0 d_k}{z^k} .$$

Similarly, when

$$f(z) \sim \sum_{k=2}^{\infty} \frac{c_k}{z^k} ,$$

then

$$\int_z^{\infty} f(z)\,\mathrm{d}z \sim \sum_{k=2}^{\infty} \frac{c_k}{(k-1)\,z^{k-1}} .$$

We shall not give the proofs of these laws, which follow directly from the definition of the asymptotic expansion.

It can be shown that the infinite series in the expression (123) gives the asymptotic expansion of a certain function, viz. a solution of the equation (113) exists for which the following asymptotic expansion holds on the straight line $z > 0$:

$$w(z)\, e^{-\alpha z} z^{-\varrho} \sim c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots$$

We shall prove this for a particular case of the equation (113) viz. for the case when both a_k and b_k vanish for $k \geqslant 2$. We shall use in this proof a special method for integrating the equation (113),

viz. we shall give the solution of this equation in the form of a contour integral. To start with let us try to solve the problem of integrating the equation by means of a contour integral.

Above we found that the condition (127) is satisfied when z lies at infinity with reference to a straight line L. If this condition is satisfied when z moves to infinity along a certain sector then it is said that the asymptotic representation (128) takes place in this sector.

107. The Laplace transformation. Consider the equation

$$w'' + \left(a_0 + \frac{a_1}{z}\right) w' + \left(b_0 + \frac{b_1}{z}\right) w = 0$$

or, multiplying by z

$$zw'' + (a_0 z + a_1) w' + (b_0 z + b_1) w = 0 . \tag{134}$$

We shall try to find the solution of this equation in the form:

$$w(z) = \int_l v(z') \, e^{zz'} \, dz' , \tag{135}$$

where $v(z')$ is the required function z' and l is the required path of integration which is independent of z. Differentiating with respect to z we have:

$$w'(z) = \int_l v(z') \, z' \, e^{zz'} \, dz'; \quad w''(z) = \int_l v(z') \, z'^2 \, e^{zz'} \, dz' . \tag{136}$$

Multiplying by z and integrating by parts we obtain:

$$zw(z) = \int_l v(z') \, de^{zz'} = [v(z') \, e^{zz'}]_l - \int_l \frac{dv(z')}{dz'} e^{zz'} \, dz' ,$$

where the symbol

$$[\varphi(z')]_l$$

denotes the increment in the function $\varphi(z')$, when z' has described the contour l. Similarly we have

$$zw'(z) = [v(z') \, z' \, e^{zz'}]_l - \int_l \frac{d[v(z') \, z']}{dz'} e^{zz'} \, dz'$$

and

$$zw''(z) = [v(z') \, z'^2 \, e^{zz'}]_l - \int_l \frac{d[v(z') \, z'^2]}{dz'} e^{zz'} \, dz' .$$

We must, first of all, make the condition that

$$[v(z') \, (z'^2 + a_0 z' + b_0) \, e^{zz'}]_l = 0 . \tag{137}$$

Substituting the above expressions in the left-hand side of the equation (134), the terms outside the integral will vanish, as a result of (137), and we can write this equation in the form:

$$\int_l \left\{ \frac{\mathrm{d}[v(z')\,z'^2]}{\mathrm{d}z'} + a_0 \frac{\mathrm{d}[v(z')\,z']}{\mathrm{d}z'} + b_0 \frac{\mathrm{d}v(z')}{\mathrm{d}z'} - \right.$$

$$\left. - a_1 z'\, v(z') - b_1\, v(z') \right\} \mathrm{e}^{zz'}\, \mathrm{d}z' = 0\,.$$

This equation will certainly be satisfied if we determine the function $v(z')$ from the equation

$$\frac{\mathrm{d}[v(z')\,z'^2]}{\mathrm{d}z'} + a_0 \frac{\mathrm{d}[v(z')\,z']}{\mathrm{d}z'} + b_0 \frac{\mathrm{d}v(z')}{\mathrm{d}z'} - a_1 z'\, v(z') - b_1\, v(z') = 0\,. \quad (138)$$

Consider the quadratic equation

$$z'^2 + a_0 z' + b_0 = 0\,, \quad (139)$$

which is the same as the equation (115), and assume that it has two different zeros α_1 and α_2. The equation (138) gives:

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}z'} = \frac{(a_1 - 2)\,z' + (b_1 - a_0)}{(z' - \alpha_1)\,(z' - \alpha_2)}$$

or, converting the fraction into partial fractions:

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}z'} = \frac{p-1}{z' - \alpha_1} + \frac{q-1}{z' - \alpha_2}\,, \quad (140)$$

where

$$p = \frac{(a_1 - 2)\,\alpha_1 + (b_1 - a_0) + (\alpha_1 - \alpha_2)}{\alpha_1 - \alpha_2}\,;$$

$$q = \frac{(a_1 - 2)\,\alpha_2 + (b_1 - a_0) + (\alpha_2 - \alpha_1)}{\alpha_2 - \alpha_1}\,.$$

On the other hand, we obtain from the quadratic equation (139):

$$\alpha_1 + \alpha_2 = -\,a_0\,,$$

and the above expressions for p and q can be transformed as follows:

$$p = \frac{a_1\,\alpha_1 + b_1}{2\alpha_1 + a_0}\,; \quad q = \frac{a_1\,\alpha_2 + b_1}{2\alpha_2 + a_0}\,. \quad (141)$$

Comparing this with the formula (119) we can see that

$$p = -\,\varrho_1; \quad q = -\,\varrho_2\,, \quad (142)$$

where ϱ_1 and ϱ_2 are two different values of ϱ which correspond to the two different zeros α_1 and α_2 of the equation (115).

On solving the equation (140) we have

$$v(z') = C(z' - a_1)^{p-1}(z' - a_2)^{q-1}, \tag{143}$$

and, consequently, *the solution of the equation* (134) *will be*

$$w(z) = C\int_l (z' - a_1)^{p-1}(z' - a_2)^{q-1} e^{zz'} dz', \tag{144}$$

where C is an arbitrary constant and the contour l, as a result of (137) *and* (143), *must satisfy the condition*

$$[(z' - a_1)^p (z' - a_2)^q e^{zz'}]_l = 0. \tag{145}$$

108. The choice of solutions. On choosing the contour l, which satisfies the equation (145) in different ways we can obtain different solutions for the equation (134). This equation, like the Bessel equation, has a regular singularity at $z = 0$ and an irregular singularity at $z = \infty$. The determining equation at the singularity $z = 0$ will be

$$\varrho(\varrho - 1) + a_1 \varrho = 0,$$

and its zeros will be $\varrho_1 = 0$ and $\varrho_2 = 1 - a_1$; we assume, for simplicity, that $1 - a_1$ is not a positive integer. Hence one of the solutions of the equation (134) will be a regular function at the point $z = 0$ and this solution can be represented in the z plane by the series:

$$1 + c_1 z + c_2 z^2 + \dots \tag{146}$$

We shall first of all, indicate a choice of the path of integration l for which the formula (144) gives this solution which is regular at the origin.

The integrand in the expression (144) has singularities at the points $z' = a_1$ and $z' = a_2$, which will, in general, be branch-points, since p and q will, on the whole, not be integers. When circumscribing the point $z' = a_1$ in the positive direction the above integrand will gain the factor $e^{(p-1)2\pi i} = e^{p2\pi i}$ and when circumscribing the point $z' = a_2$ it will gain the factor $e^{(q-1)2\pi i} = e^{q2\pi i}$. In future we shall assume that the numbers p and q are fractions.

Consider a finite point z_0 other than a_1 or a_2 in the plane and denote by l_1 and l_2 closed contours which originate at z_0 and encircle the points a_1 and a_2.

We denote the symbol by (l_1, l_2) the contour which consists of the following successive circuits: the circuit along l_1 in the positive direction, the circuit along l_2 in the positive direction, the circuit along l_1 in the negative direction and the circuit along l_2 in the negative direction. On completing the first circuit the function (145) gains

the factor $e^{p2\pi i}$. After the second circuit is completed it gains the factor $e^{q2\pi i}$; after the third circuit the factor $e^{-p2\pi i}$ and, finally, after the fourth circuit the factor $2^{-q2\pi i}$. Thus when we finally return to the point z_0 we have on the left-hand side of (145) the same branch as the one we took by starting out from z_0. Thus by taking for l the contour (l_1, l_2) we satisfy the condition (145) and formula (44) gives the solution of the equation. Notice that had we taken for the contour l a closed contour which did not encircle the singularities a_1 and a_2 of the integrand then this function would, of course, also have returned to the initial value but, according to Cauchy's theorem, the integral round this closed contour would not have been equal to zero and we would have obtained the solution of the equation (134). In this case we have chosen the contour (l_1, l_2) so that by describing the singularities we have, nevertheless, returned to the initial branch of the function.

We therefore have the solution

$$w(z) = C \int_{(l_1, l_2)} (z' - a_1)^{p-1} (z' - a_2)^{q-1} e^{zz'} dz' . \qquad (147)$$

The variable z' lies on the contour consisting entirely of finite points and we can therefore write the expansion of the series

$$e^{zz'} = \sum_{k=0}^{\infty} \frac{z^k}{k!} z'^k ,$$

which is uniformly convergent on the contour of integration. Substituting this series and integrating term by term we can write our solution in the form:

$$w_0(z) = C \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_{(l_1, l_2)} z'^k (z' - a_1)^{p-1} (z' - a_2)^{q-1} dz' , \qquad (148)$$

where C is an arbitrary constant, i.e. *the constructed solution happens to be a solution which is regular (at the origin)*. It must, however, be remembered that this solution is not identically zero; this can occur only in exceptional cases, when p and q are positive integers.

It can readily be seen that the value of the integral in formula (148) is independent of the choice of the initial point z_0. This can be shown, for example, by applying Cauchy's theorem but it must be remembered that the complete contour (l_1, l_2) must be closed, for when describing it fully we return to the starting branch of the function and, therefore, the use of Cauchy's theorem is permissible.

We shall now consider the particular case when the real part of the numbers p and q is greater than zero. We shall assume that the point z_0 lies on the straight line $a_1 a_2$ near the point a_1, that l_1 is a small circle, centre a_1, and that l_2 consists of a straight line $z_0 z_1$ and a small circle, centre a_2, where the above straight line must be described twice. We will show that when the radii of the above circles become indefinitely small the integrals round these circles tend to zero. Consider, for example, the circle, centre a_1, and assume, for simplicity, that p is a real number which, from the given conditions, must be greater than zero. Let ε be the radius of the circle. On this circumference we have the following inequality for the integrand:

$$|z'^k(z'-a_1)^{p-1}(z'-a_2)^{q-1}| = |z'-a_1|^{p-1}|z'^k(z'-a_2)^{q-1}| < \varepsilon^{p-1}M,$$

where M is a positive constant. For the whole integral over the above circumference we have the inequality:

$$|\int z'^k(z'-a_1)^{p-1}(z'-a_2)^{q-1}\,dz'| < \varepsilon^{p-1}M2\pi\varepsilon = \varepsilon^p 2\pi M,$$

from which it follows that the integral tends to zero together with ε. For the complex power $p = p_1 + ip_2$, where $p_1 > 0$, we have:

$$|(z'-a_1)^{p-1}| = |e^{[(p_1-1)+ip_2]\log(z'-a_1)}| = e^{(p_1-1)\log|z'-a_1| - p_2\arg(z'-a_1)}$$

or

$$|(z'-a_1)^{p-1} = \varepsilon^{p_1-1}\cdot e^{-p_2\arg(z'-a_1)},$$

and the result will be the same.

Hence using the above method of integration we can, in the limit, neglect the integration round the circle; we then have the path of integration l_2 and we must integrate along the straight line $a_1 a_2$ encircling the point a_2 and returning to the point a_1 along that same line.

Bearing in mind the factors which the integrand gains by describing the points a_1 and a_2 we obtain the following formula for the solution (147):

$$w_0(z) = Ce^{p2\pi i}(1-e^{q2\pi i})\int_{a_1}^{a_2}(z'-a_1)^{p-1}(z'-a_2)^{q-1}e^{zz'}\,dz' +$$

$$+C(e^{q2\pi i}-1)\int_{a_1}^{a_2}(z'-a_1)^{p-1}(z'-a_2)^{q-1}e^{zz'}\,dz'$$

or

$$w_0(z) = -C(e^{p2\pi i}-1)(e^{q2\pi i}-1)\int_{a_1}^{a_2}(z'-a_1)^{p-1}(z'-a_2)^{q-1}e^{zz'}\,dz'.$$

If we suppose that p and q are not integers and reject the constant factor we can, in this case, write the solution of the equation (134) which is regular at the origin simply in the form of an integral along the line $a_1 a_2$:

$$w_0(z) = C\int_{a_1}^{a_2}(z'-a_1)^{p-1}(z'-a_2)^{q-1}e^{zz'}\,dz'. \qquad (149)$$

This last result can also be obtained directly. If the real parts of p and q are greater than zero, the expression (145) must vanish when $z' = a_1$ and $z' = a_2$, so that we can simply take the line $a_1 a_2$ as the path of integration l. In this case we are not using the fact that p and q are not integers.

Let us now return to the general case. Notice that if we take for the path of integration only one contour l_1 or l_2 then the value of the integral will, in general, depend on the initial point z_0 and we shall not obtain a solution of our equation. However, the point z_0 can be so chosen that we can, in fact, obtain a solution of the equation.

We shall assume in future that z is a positive number and notice that the expression

$$(z' - a_1)^p (z' - a_2)^q e^{zz'} \quad (150)$$

tends to zero as z' tends to infinity, so that the real part of z' tends to $(-\infty)$ and its imaginary part remains bounded. We shall say that z' tends to $(-\infty)$. If we chose for the contour l_1' a contour the ends of which are at the point $(-\infty)$ and which circumscribes a_1, then at the ends of this contour the expression (150) will vanish and the condition (145) will be satisfied; consequently the integral round this contour will give a solution of the equation (134). We obtain a second solution similarly by taking for the path of integration l_2' a contour originating at $(-\infty)$, which circumscribes the point a_2 in the positive direction. We thus obtain two solutions for the equation (134):

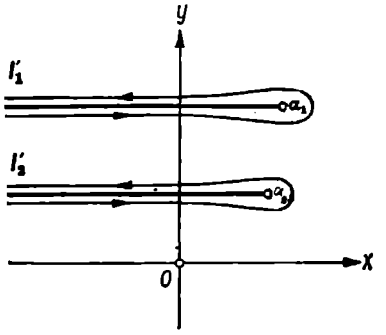

FIG. 68

$$\left.\begin{aligned} w_1(z) &= \int_{l_1'} (z' - a_1)^{p-1} (z' - a_2)^{q-1} e^{zz'} dz', \\ w_2(z) &= \int_{l_2'} (z' - a_1)^{p-1} (z' - a_2)^{q-1} e^{zz'} dz'. \end{aligned}\right\} \quad (151)$$

The integrand has branch points at $z' = a_1$ and $z' = a_2$. To make this function single-valued we must cut the plane from these points to $(-\infty)$ and, assuming that the imaginary parts of a_1 and a_2 are different, we cut the plane along straight lines parallel to the real axis (Fig. 68). In the cut plane we select that branch of the integrand for which $\arg(z' - a_1) = 0$, when $z' - a_1 > 0$, i.e. the

continuation of the first cut, and $\arg(z' - a_2) = 0$, when $z' - a_2 > 0$. The contours l_1 and l_2 are situated as shown in Fig. 68. The above conditions fully determine the solution (151) when $z > 0$.

Notice that the exponential function $e^{zz'}$ tends to zero as $z' \to -\infty$ not only for positive values of z but for any value of z, the amplitude of which lies within the limits

$$-\frac{\pi}{2} < \arg z < \frac{\pi}{2} . \tag{152}$$

In fact, assuming that $z = x + yi$ we have, at the same time, $x > 0$. Also $z' = x' + y'i$, where $x' \to -\infty$ and $|y'|$ remains bounded. Thus the real part of the product zz', which is equal to $xx' - yy'$, will also, in this case, tend to $(-\infty)$ and the function (150) will vanish at the ends of l_1' and l_2'. Hence formula (151) gives the unique solution for all z's in the sector (152).

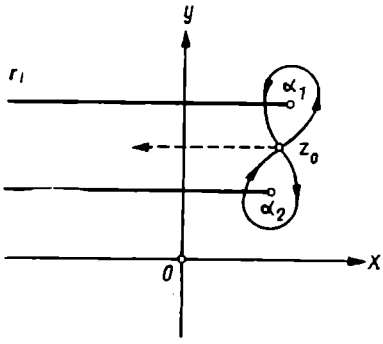

FIG. 69

We shall now establish the connection between the solution (151) and the solution (134) which is regular at the origin. Bearing in mind further applications of the Bessel equation we shall restrict ourselves to the case when $p = q$. To obtain a solution regular at the origin we can take as the contour of integration not the contour (l_1, l_2) which we mentioned above, but a simpler contour, viz. a contour from a point z_0, which describes the point a_1 in the positive direction and the point a_2 in the negative direction. On completing the first circuit the integrand gains the factor $e^{p2\pi i}$ and on completing the second circuit the factor $e^{-q2\pi i} = e^{-p2\pi i}$, so that we return to the initial value and the condition (145) is satisfied. As before the constructed solution is independent of the choice of the point z_0. Let us remove this point, without affecting either of the points a_1 or a_2, to $(-\infty)$ for example, along the lower edge of the cut r_1 which goes to the point a_1 (Fig. 69). The completion of a circuit around the point a_1 will then give the solution w_1. On completing this circuit we shall find ourselves on the upper edge of the above cut and we shall then have to complete a circuit round the point a_2 in the negative direction. Had we completed this circuit from the lower edge of the

cut r_1, we would have obtained the solution $(-w_2)$. However, by coming onto the upper edge of the cut, from where the circuit round the point a_2 is being described, the integrand acquired the factor $e^{2p\pi i}$, and therefore the completion of a circuit round a_2 in the negative direction gives $(-e^{p2\pi i} w_2)$. We finally obtain the following rule: *when $p = q$ the regular integral obtained by integrating round the contour shown in Fig. 69 can be expressed in terms of the solution* (151) *in the form:*

$$w_1(z) - e^{p2\pi i} w_2(z). \tag{153}$$

109. The asymptotic representation of solutions. We shall now deduce the asymptotic expansions for the solutions (151) for large positive values of z. Let us recall that we have determined these solutions for values of z in the sector (152). We begin with the first solution. Replace z' by a new variable of integration t, according to the formula

$$z' - a_1 = t, \tag{154}$$

and assume, to simplify the notation, that $\beta = a_1 - a_2$. The first solution will then have the form:

$$w_1(z) = \int_{l_0} t^{p-1} (t+\beta)^{q-1} e^{z(a_1+t)} \, dt, \tag{155}$$

where l_0 is a contour which goes from $t = -\infty$ and surrounds the origin. The points $t = 0$ and $t = -\beta$ are branch-points of the integrand function. Instead of the cuts shown in Fig. 68 we cut the t-plane in two places, viz. from $(-\infty)$ to the points $t = 0$ and $t = -\beta = a_2 - a_1$, respectively; here $\arg t = 0$ when $t > 0$, while $\arg(t+\beta) = 0$ when $t + \beta > 0$, i.e. on the continuations of these cuts.

Using Newton's binomial expansion we have, when $|t| < |\beta|$

$$(t+\beta)^{q-1} = \beta^{q-1}\left(1 + \frac{t}{\beta}\right)^{q-1} = \sum_{k=0}^{\infty} d_k t^k, \tag{156}$$

where

$$d_k = \beta^{q-1}\beta^{-k} \frac{(q-1)(q-2)\ldots(q-k)}{k!} \quad (d_0 = \beta^{q-1}). \tag{157}$$

From the above condition for the amplitude of $(t+\beta)$ in the t-plane with the cut from $(-\infty)$ to $(-\beta)$, we can assume that in the expression β^{q-1} which is equal to the function (156) when $t = 0$, the amplitude of β lies within the limits

$$-\pi < \arg\beta < \pi, \tag{158}$$

and we assume that β is not a real negative number.

When $|t| \geqslant |\beta|$ we cannot use the formula (156) and in this case we can simply write:

$$(t+\beta)^{q-1} = d_0 + d_1 t + \ldots + d_n t^n + R_n(t),$$

where

$$R_n(t) = (t+\beta)^{q-1} - (d_0 + d_1 t + \ldots + d_n t^n). \tag{159}$$

Using the above formulae we can write:

$$w_1(z) = e^{a_1 z} \sum_{k=0}^{n} d_k \int_{l_0} e^{zt} t^{p+k-1}\, dt + e^{a_1 z} \int_{l_0} e^{zt} t^{p-1} R_n(t)\, dt. \tag{160}$$

Consider the sum on the right-hand side. Replacing t by a new variable of integration τ, according to the formula

$$zt = -\tau = e^{-\pi i}\tau,$$

we can obtain the integral in the form:

$$\int_{l_0} e^{zt} t^{p+k-1}\, dt = e^{-\pi p i} (-1)^k z^{-p-k} \int_{\lambda} e^{-\tau} \tau^{p+k-1}\, d\tau.$$

where the path of integration λ is along a cut from $\tau = +\infty$ and around $\tau = 0$ in the positive direction. Owing to the fact that $zt = e^{-\pi i \tau}$ we have $\tau = z e^{\pi i} t$, where we assume that $z > 0$ and $\arg z = 0$, i.e. the τ-plane is obtained from the t-plane by rotation about the origin by an angle π, so that the lower edge of the cut l_0 in the t plane, where $\arg t = -\pi$, becomes the upper edge of the cut λ in the τ plane; as a result of the above formula we must assume on this upper edge that $\arg \tau = 0$.

Remembering the connection between the above contour integral and the function $\Gamma(z)$ we have [74]:

$$\int_{l_0} e^{zt} t^{p+k-1}\, dt = e^{-\pi p i} (-1)^k z^{-p-k} (e^{(p+k)2\pi i} - 1)\, \Gamma(p+k),$$

and formula (160) gives:

$$w_1(z) = e^{a_1 z} z^{-p} (e^{2\pi p i} - 1)\, e^{-\pi p i} \sum_{k=0}^{n} (-1)^k d_k \Gamma(p+k)\, z^{-k} + \\ + e^{a_1 z} \int_{l_0} e^{zt} t^{p-1} R_n(t)\, dt \tag{161}$$

or

$$e^{-a_1 z} z^p w_1(z) = e^{-\pi p i} (e^{2\pi p i} - 1) \sum_{k=0}^{n} (-1)^k d_k \Gamma(p+k)\, z^{-k} + \\ + z^p \int_{l_0} e^{zt} t^{p-1} R_n(t)\, dt. \tag{162}$$

We will show that the infinite series

$$e^{-\pi pi}(e^{2\pi p} - 1)\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(p+k)\,d_k}{z^k} \tag{163}$$

represents the asymptotic expansion of the function $e^{-a_1 z} z^p w_1(z)$ when $z > 0$.

To prove this we must show that the product of z^n and the last term of the formula (162) tends to zero, i.e.

$$\lim_{z\to\infty} z^{n+p}\int_{l_0} e^{zt}\, t^{p-1} R_n(t)\,dt = 0.$$

Consider the following path of integration: the line $(-\infty, -r)$ of the real axis, a circle, centre $t = 0$ and radius r and the line $(-r, -\infty)$ of the real axis, where r is a positive number.

We will show, to start with, that the expression

$$z^{n+p}\int_{-\infty}^{-r} e^{zt}\, t^{p-1} R_n(t)\,dt \tag{164}$$

tends to zero as $z \to +\infty$. The same thing must also apply to the integral along the line $(-r, -\infty)$, since only the factor $e^{(p-1)2\pi i}$ is added on describing the point $t = 0$.

Returning to the formula (159) we can see that a sufficiently large positive number N can be found such that

$$\left|\frac{R_n(t)}{t^N}\right| \to 0 \quad \text{as} \quad t \to -\infty,$$

and therefore the modulus of the quotient $R_n(t)/t^N$ will be bounded along the whole path of integration, and we can write the inequality

$$|R_n(t)| < m\,|t|^N \qquad (-\infty < t \leqslant -r), \tag{165}$$

where m is a fixed positive number. Let ε be a small positive number. Remembering that an exponential function increases more rapidly than any power function and bearing in mind (165) we can write

$$\frac{t^{p-1} R_n(t)}{e^{-\varepsilon t}} \to 0 \quad \text{as} \quad t \to -\infty$$

$$\text{or} \quad |t^{p-1} R_n(t)| < m_1 e^{-\varepsilon t} \qquad (-\infty < t \leqslant -r),$$

where m_1 is a positive constant.

We therefore have the following inequality for the expression (164):

$$\left| z^{n+p} \int_{-\infty}^{-r} e^{zt} t^{p-1} R_n(t)\, dt \right| < |z^{n+p}| \int_{-\infty}^{-r} m_1 e^{(z-\varepsilon)t}\, dt \qquad (z > 0),$$

i.e., integrating

$$\left| z^{n+p} \int_{-\infty}^{-r} e^{zt} t^{p-1} R_n(t)\, dt \right| < \frac{|z^{n+p}|}{z-\varepsilon} m_1 e^{-(z-\varepsilon)r}.$$

It follows that the expression (164) does, in fact, tend to zero as $z \to +\infty$. This will be so for every fixed r. It now remains to be shown that the expression given below also tends to zero

$$z^{n+p} \int_C e^{zt} t^{p-1} R_n(t)\, dt,$$

where the path of integration C is the circle, centre the origin and radius r. We assume that r is very small, for example, $r < (1/2)\,|\beta|$. We can then use Newton's binomial expansion (156) on the circumference $|t| = r$.

According to Cauchy's inequality we have the following inequality for the coefficients of the expansion d_k

$$|d_k| < \frac{m_2}{(|\beta| - \varepsilon)^k},$$

where m_2 is a positive number. We now assume that $|\beta| - \varepsilon$ is equal to, say, $\varrho = (1/2)\,|\beta|$. We have further:

$$R_n(t) = d_{n+1} t^{n+1} + d_{n+2} t^{n+2} + \dots$$

The above inequalities give

$$|d_k| < m_2 \left(\frac{1}{2}|\beta|\right)^{-k}; \qquad |t| = r < \frac{1}{2}|\beta|$$

and, consequently,

$$|R_n(t)| \leqslant |d_{n+1}|\,|t|^{n+1} + |d_{n+2}|\,|t|^{n+2} + \dots < \frac{m_2 |t|^{n+1}}{\varrho^{n+1}(1-\theta)}, \quad (166)$$

where

$$\theta = \frac{r}{\varrho} < 1.$$

This inequality for $R_n(t)$ is also valid when $|t| < r$, i.e. when t lies in C. We again replace t by a new variable of integration τ, according to the formula $zt = -\tau$, and we obtain the following expression:

$$z^{n+p} \int_C e^{zt} t^{p-1} R_n(t)\, dt = (-1)^p z^n \int_{C'} e^{-\tau} \tau^{p-1} R_n\left(-\frac{\tau}{z}\right) d\tau, \quad (167)$$

where the path of integration C' is a circle, centre the origin and radius rz. According to Cauchy's theorem we can deform this contour and take for the contour of integration any closed contour, which originates at the point rz of the real axis, encircles the origin and lies in C'. In this case the corresponding contour in the t plane will lie in C and the inequality (166) will be valid. We can, for example, take for the contou of integration C'' the following path: the segment of the real axis from rz to a point c which lies to the right of the origin, a circle, centre the origin and radius c and, again, the part (c, rz) of the real axis, where c is a fixed positive number which is independent of z.

We suppose at first that p is real. Using (166) to find an upper bound of the expression (167) we obtain [4]:

$$\left|(-1)^p z^n \int_{C''} e^{-\tau} \tau^{p-1} R_n\left(-\frac{\tau}{z}\right) d\tau\right| < \frac{1}{z} \int_{C''} \frac{m_2 |\tau|^{n+p}}{\varrho^{n+1}(1-\theta)} |e^{-\tau}| \, ds,$$

where ds is the differential of the arc of the contour. We will show that the coefficient of $1/z$ will remain bounded as z increases indefinitely. In fact, integration round the circle with radius c gives an expression which is independent of z. Let us now also consider the integral along the line (c, rz). This gives the following coefficient of z^{-1}:

$$\frac{m_2}{\varrho^{n+1}(1-\theta)} \int_c^{rz} e^{-\tau} \tau^{n+p} \, d\tau.$$

As z increases indefinitely this last integral tends to the finite limit:

$$\int_c^{\infty} e^{-\tau} \tau^{n+p} \, d\tau,$$

where the existence of the above integral is fully assured by the factor $e^{-\tau}$ in the integrand. Hence our proposition is proved for a real p. When p is complex, i.e. $p = p_1 + ip_2$, we only have to use the usual property of a complex power, viz.

$$\tau^p = e^{(p_1+ip_2)\log\tau} = e^{(p_1+ip_2)(\log|\tau|+i\arg\tau)},$$

whence

$$|\tau^p| = |\tau|^{p_1} \cdot e^{-p_2 \arg\tau}.$$

We can therefore say that *the series* (163) *gives the asymptotic representation of the function* $e^{-a_1 z} z^p w_1(z)$ *when* $z > 0$:

$$e^{-a_1 z} z^p w_1(z) \sim e^{-\pi p i} (e^{2\pi p i} - 1) \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(p+k)\, d_k}{z^k}, \tag{168}$$

where

$$d_k = (a_1 - a_2)^{q-1-k} \frac{(q-1)(q-2)\ldots(q-k)}{k!} \tag{169}$$

$$(d_0 = (a_1 - a_2)^{q-1}; \quad -\pi < \arg(a_1 - a_2) < \pi).$$

We will not consider the case when in the formula (168) p is an integer.

Similarly, *for the second of the solutions* (151) *we obtain the asymptotic representation:*

$$e^{-a_2 z} z^q w_2(z) \sim e^{-\pi q i} (e^{2\pi q i} - 1) \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(q+k)\, d'_k}{z^k}, \tag{170}$$

where

$$d'_k = (a_2 - a_1)^{p-1-k} \frac{(p-1)(p-2)\ldots(p-k)}{k!} \tag{171}$$

$$(d'_0 = (a_2 - a_1)^{p-1}; \quad -\pi < \arg(a_2 - a_1) < \pi).$$

For the powers z^p and z^q it must be assumed that $\arg z = 0$ when $z > 0$.

110. Comparison of results. We now return to the calculations in [105]. In that section we constructed the following solution of the equation (113):

$$e^{az} z^\varrho \left(c_0 + c_1 \frac{1}{z} + \ldots\right), \tag{172}$$

which formally satisfied this equation. Let us compare the expression (172) which we obtained at the time with the expression determined by the asymptotic formula (168), i.e. with the expression:

$$e^{a_1 z} z^{-p} e^{-\pi p i} (e^{2\pi p i} - 1) \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(p+k)\, d_k}{z^k}, \tag{173}$$

and let us show that these expressions are exactly alike except for the constant factor which enters formula (172) in the form of the arbitrary c_0.

Comparing the equation (134), for which we deduced the asymptotic formula (168), with the equation (113), we can see, first of all, that

we must assume that $a_k = b_k = 0$ when $k \geqslant 2$. The exponential and power factors coincide in the formulae (172) and (173) since the equation (139), from which we obtained a_1, coincides with the equation (115); also, from (142), we have $p = -\varrho_1$, where ϱ_1 is the value of ϱ determined by the formula (119) when $a = a_1$. The equivalence of the power series in the formulae (172) and (173) remains to be shown; for this purpose it is sufficient to show that the coefficients of these series satisfy the same relationships from which they can be determined.

The series in the formula (172), obtained as the formal solution of the equation (120) satisfies the following differential equation for $a_k = b_k = 0$, when $k \geqslant 2$:

$$u'' + \left(2a_1 + a_0 + \frac{2\varrho_1 + a_1}{z}\right) u' + \frac{\varrho_1^2 + (a_1 - 1)\varrho_1}{z^2} u = 0. \quad (174)$$

Remembering the form of the equation (115) we can write:

$$a_1 + a_2 = -a_0; \quad 2a_1 + a_0 = a_1 - a_2; \quad 2a_2 + a_0 = a_2 - a_1,$$

whence

$$\varrho_1 = -\frac{a_1 a_1 + b_1}{2a_1 + a_0} = \frac{a_1 a_1 + b_1}{a_2 - a_1}; \quad \varrho_2 = -\frac{a_2 a_1 + b_1}{2a_2 - a_0} = \frac{a_2 a_1 + b_1}{a_1 - a_2},$$

and therefore

$$\varrho_1 + a_1 = -\varrho_2; \quad \varrho_1^2 + (a_1 - 1)\varrho_1 = -\varrho_1 \varrho_2 - \varrho_1. \quad (175)$$

The equation (174) has the same form as (124) and we obtain the following relationship for the coefficients c_n:

$$[n(n+1) - n(2\varrho_1 + a_1) + \varrho_1^2 + (a_1 - 1)\varrho_1]\, c_n = \\ = (n+1)(2a_1 + a_0) c_{n+1}$$

or, from the equation $a_1 + a_2 = -a_0$ and from (175):

$$[n(n+1) - n(\varrho_1 - \varrho_2) - \varrho_1 \varrho_2 - \varrho_1]\, c_n = (n+1)(a_1 - a_2)\, c_{n+1}. \quad (176)$$

If we denote by c_n the coefficients of the infinite series in the formula (173):

$$c_n = (-1)^n d_n\, p(p+1)\ldots(p+n-1)\, \Gamma(p) = (-1)^n d_n\, \Gamma(p+n),$$

whence

$$\frac{c_{n+1}}{c_n} = -\frac{d_{n+1}(p+n)}{d_n},$$

or from the formula (169):

$$\frac{c_{n+1}}{c_n} = -\frac{(q-n-1)(p+n)}{(n+1)(\alpha_1-\alpha_2)},$$

or

$$(n+1-q)(n+p)c_n = (n+1)(\alpha_1-\alpha_2)c_{n+1}.$$

Bearing in mind that $p = -\varrho_1$ and $q = -\varrho_2$ we can see that the above relationship is the same as (176). We can thus see that *formal solutions of the equation* (134) *which are constructed by the methods given in* [105] *give the asymptotic representation when* $z \to +\infty$, *of solutions determined by the formulae* (151) *except for the constant term.*

111. The Bessel equation. Let us apply the above theory to the Bessel equation [II, 48]:

$$z^2 w'' + zw' + (z^2 - n^2) w = 0. \tag{177}$$

Replacing w by a new unknown function u, defined by the formula

$$w = z^n u,$$

we can obtain the equation (177) in the form:

$$zu'' + (2n+1)u' + zu = 0, \tag{178}$$

and this happens to be an equation of exactly the same type as the one we have considered above. In this case we have:

$$a_0 = 0; \quad a_1 = 2n+1; \quad b_0 = 1; \quad b_1 = 0.$$

The quadratic equation (139) will be $z'^2 + 1 = 0$, so that we obtain:

$$\alpha_1 = i; \quad \alpha_2 = -i,$$

and, similarly, from the formulae (141) we have:

$$p = -\frac{2n+1}{2}; \qquad q = \frac{2n+1}{2}.$$

The final solution (151) will, in this case, have the following form:

$$u_1 = \int_{l_1'} (z'^2+1)^{-\frac{2n-1}{2}} e^{zz'} dz'; \quad u_2 = \int_{l_2'} (z'^2+1)^{-\frac{2n-1}{2}} e^{zz'} dz', \tag{179}$$

where the contours l_1 and l_2 originate at the point $-\infty$ and surround the points $z' = i$ and $z' = -i$

These solutions are given by the formulae (179) where $-\pi/2+\varepsilon<\arg z<\pi/2-\varepsilon$. From the condition in [108] $\arg(z'+i)=0$, when $z'+i>0$, and $\arg(z'-1)=0$, when $z'-i>0$. It follows that $\arg(z'^2+1)=\arg(z'+i)+\arg(z'-i)=0$ when z' is real.

For the first of the solutions (179) we have, from (168):

$$e^{-iz}z^{n+\frac{1}{2}}u_1 \sim e^{-\pi\left(n+\frac{1}{2}\right)i}(e^{\pi(2n+1)i}-1)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(n+\frac{1}{2}+k\right)d_k}{z^k}.$$

Remembering that

$$e^{\pi(2n+1)i}-1=-(1+e^{2\pi ni})$$

and the expression (157) for d_k

$$d_k=(2i)^{n-\frac{1}{2}-k}\binom{q-1}{k}=(2i)^{n-\frac{1}{2}-k}\binom{n-\frac{1}{2}}{k}$$

$$\left(-\pi<\arg 2i<\pi \quad \text{or} \quad \arg 2i=\frac{\pi}{2}\right),$$

i.e.

$$d_k=2^{n-\frac{1}{2}-k}e^{\frac{\pi}{2}\left(n-\frac{1}{2}\right)i}i^{-k}\binom{n-\frac{1}{2}}{k},$$

we obtain the following asymptotic representation:

$$e^{-iz}z^{n+\frac{1}{2}}u_2 \sim e^{-\frac{\pi}{2}\left(n-\frac{1}{2}\right)i}(1+e^{2\pi ni})2^{n-\frac{1}{2}}\times$$

$$\times\sum_{k=0}^{\infty}\binom{n-\frac{1}{2}}{k}\Gamma\left(n+\frac{1}{2}+k\right)\left(\frac{i}{2z}\right)^k. \quad (180)$$

Similarly, analogous calculations give the following formula for the second solution (179):

$$e^{iz}z^{n+\frac{1}{2}}u_2 \sim e^{-\frac{3\pi}{2}ni+\frac{3\pi}{4}i}(1+e^{2\pi ni})\,2^{n-\frac{1}{2}}\times$$

$$\times\sum_{k=0}^{\infty}\binom{n-\frac{1}{2}}{k}\Gamma\left(n+\frac{1}{2}+k\right)\left(-\frac{i}{2z}\right)^k, \quad (181)$$

where the only difference is in the coefficients d'_k which are expressed by the formula

$$d'_k=(-2i)^{n-\frac{1}{2}-k}\binom{n-\frac{1}{2}}{k}\left(\arg(-2i)=-\frac{\pi}{2}\right).$$

Let us recall that the symbol $\binom{a}{k}$ for an integer $k \geqslant 0$ denotes the following:

$$\binom{a}{k} = \frac{a(a-1)\dots(a-k+1)}{k!} \quad \text{and} \quad \binom{a}{0} = 1.$$

Remembering the expression (153) for a solution of the equation (178), which is regular at the origin, where the letter w replaces the letter u, we can replace the solution u_2 by a new solution u_2^*:

$$u_2^* = e^{p2\pi i} u_2 = e^{(2n+1)\pi i} u_2.$$

For this new solution we obtain the asymptotic representation

$$e^{iz} z^{n+\frac{1}{2}} u_2^* \sim e^{\frac{\pi}{2}\left(n-\frac{1}{2}\right)i} (1 + e^{2\pi n i})\, 2^{n-\frac{1}{2}} \times \\ \times \sum_{k=0}^{\infty} \binom{n-\frac{1}{2}}{k} \Gamma\left(n + \frac{1}{2} + k\right) \left(-\frac{i}{2z}\right)^k. \quad (182)$$

The corresponding solution of the equation (177) can be obtained from $w = z^n u$, by adding the factor z^n.

The solution (179) is sometimes written in a slightly different form, viz. z' is replaced by a new variable of integration τ, according to the formula: $z' = i\tau = e^{\pi i/2}\tau$; this corresponds to the rotation of the z' plane through an angle $(-\pi/2)$:

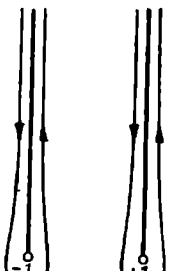

FIG. 70

$$\left.\begin{aligned} u_1 &= i \int_{\lambda_1} (1-\tau^2)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau, \\ u_2 &= i \int_{\lambda_2} (1-\tau^2)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau, \end{aligned}\right\} \quad (183)$$

where λ_1 and λ_2 are contours which originate at the point $\tau = +i\infty$ and surround the points $\tau = +1$ and $\tau = -1$ (Fig. 70); $\arg(1-\tau^2) = 0$ when τ is purely imaginary, which corresponds to a real z' or, which comes to the same thing, $\arg(1-\tau^2) = \pi$ when $\tau > 1$. Assuming that $1 - \tau^2 = e^{\pi i}(\tau^2 - 1)$, we obtain in place of (183):

$$\left.\begin{aligned} u_1 &= e^{\pi\left(n-\frac{1}{2}\right)i} i \int_{\lambda_1} (\tau^2-1)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau, \\ u_2 &= e^{\pi\left(n-\frac{1}{2}\right)i} i \int_{\lambda_2} (\tau^2-1)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau, \end{aligned}\right\} \quad (184)$$

where

$$\arg(\tau^2 - 1) = 0 \text{ for } \tau > 1. \tag{185}$$

The corresponding solutions of the equation (177) will be:

$$\left.\begin{aligned} w_1 &= e^{\pi\left(n-\frac{1}{2}\right)} iz^n \int_{\lambda_1} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau, \\ w_2 &= e^{\pi\left(n-\frac{1}{2}\right)i} iz^n \int_{\lambda_2} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau, \end{aligned}\right\} \tag{186}$$

and the asymptotic representation of these solutions for large positive values of z will be obtained from the above formulae by the addition of the factor z^n to the right-hand sides. Replace the second solution by $w_2^* = e^{(2n+1)\pi i} w_2$, so that

$$w_2^* = e^{\pi\left(3n+\frac{1}{2}\right)i} iz^n \int_{\lambda_2} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau. \tag{187}$$

The difference $u_1 - u_2^*$ gives the solution of the equation (178) which is regular at the origin $z = 0$ and, similarly, the difference $w_1 - w_2^*$ gives the solution of the Bessel equation which has the following form near the origin:

$$z^n \sum_{k=0}^{\infty} \beta_k z^k.$$

We know already that this solution of the Bessel equation is given by the following series [II, 48]:

$$Cz^n\left[1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2\cdot 4\cdot(2n+2)\cdot(2n+4)} - \cdots\right].$$

When n is a positive integer or zero then, as we said before, we select the constant C equal to $1/2^n\, n!$ where, as always, $0! = 1$; with this choice of the constant we obtain the Bessel function of the first kind:

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!}\left(\frac{z}{2}\right)^{n+2k},$$

or, using the function $\Gamma(z)$:

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1^k)}{\Gamma(k+1)\,\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k}.$$

When n is not an integer the constant C is equal to

$$C = \frac{1}{2^n \Gamma(n+1)}$$

and we arrive, similarly, at the solution

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)(n+k-1)\ldots(n+1)\Gamma(n+1)} \left(\frac{z}{2}\right)^{n+2k}$$

or, from the fundamental property of the function $\Gamma(z)$:

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(n+k+1)} \left(\frac{z}{2}\right)^{n+2k}. \tag{188}$$

The Bessel function of any variable is thus determined. The difference $w_1 - w_2^*$ will not give the exact value of the Bessel function; it will differ from it by a constant factor which we shall now attempt to find. We must, therefore, take in place of the solution (186) a new solution which differs from the one above only by a constant term; the latter is so chosen that the difference of the new solutions should give the exact value of the Bessel function $J_n(z)$. Introducing the minus sign in the second solution we shall look for a constant a from the condition, that half the sum of the solutions

$$\left.\begin{aligned} H_n^{(1)}(z) &= bw_1 = az^n \int_{\lambda_1} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau, \\ H_n^{(2)}(z) &= -bw_2^* = -ae^{(2n+1)\pi i} z^n \int_{\lambda_2} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau \\ &\qquad \left(a = be^{\pi\left(n-\frac{1}{2}\right)i}\right), \end{aligned}\right\} \tag{189}$$

gives the Bessel function.

In all the above calculations we have assumed that $(n - 1/2)$ is not equal to an integer $m \geqslant 0$. This latter case we shall investigate in the detailed treatment of the Bessel function.

112. The Hankel function. In the above method of choosing the constant a the formulae (189) gave two solutions for the equation (177) which are known as the *Hankel functions;* they are denoted in the same way as in the formulae (189). We saw earlier [108], that by adding the solutions (189) we obtain one integral along the path of integration C which will have the form of the figure eight, as shown

in Fig. 71. Let us recall that this figure can be obtained from Fig. 69, when $a_1 = i$ and $a_2 = -i$, by rotation about the origin through a right angle in the clockwise direction.

Bearing in mind the fact that half the sum of the functions (189) should give the Bessel function (188) we have:

$$\frac{1}{2} az^n \int_C (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau} d\tau = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\,\Gamma(n+k+1)} \left(\frac{z}{2}\right)^{n+2k}. \quad (190)$$

Simplifying both sides by dividing by z^n and assuming subsequently that $z = 0$ we arrive at an equation which gives a:

$$\frac{1}{2} a \int_C (\tau^2 - 1)^{n-\frac{1}{2}} d\tau = \frac{1}{2^n \Gamma(n+1)}, \quad (191)$$

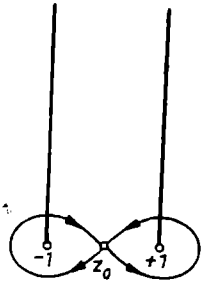

FIG. 71

and the integral on the left-hand side only remains to be evaluated. Assuming that n is real and that $(n - 1/2)$ is greater than (-1) we can, as in [108], transform the path of integration C so that it runs along the straight line $(-1, +1)$; we must integrate along the lower edge of the line from (-1) to $(+1)$ and along the upper edge of the line from $(+1)$ to (-1). We said above that $\arg(\tau^2 - 1) = 0$ when $\tau > 1$, from which follows that $\arg(\tau^2 - 1) = \pi$ on the upper edge of the line $(-1, +1)$ and $\arg(\tau^2 - 1) = -\pi$ on the lower edge of this line, i.e.

$$(\tau^2 - 1)^{n-\frac{1}{2}} = e^{i\pi\left(n-\frac{1}{2}\right)} (1 - \tau^2)^{n-\frac{1}{2}} \text{ (on the upper edge)},$$

$$(\tau^2 - 1)^{n-\frac{1}{2}} = e^{-i\pi\left(n-\frac{1}{2}\right)} (1 - \tau^2)^{n-\frac{1}{2}} \text{ (on the lower edge)};$$

and, finally, adding the integrals, we obtain:

$$\int_C (\tau^2 - 1)^{n-\frac{1}{2}} d\tau = -2i \sin\left(n - \frac{1}{2}\right)\pi \int_{-1}^{1} (1 - \tau^2)^{n-\frac{1}{2}} d\tau,$$

where

$$(1 - \tau^2)^{n-\frac{1}{2}} = e^{\left(n-\frac{1}{2}\right)\log(1-\tau^2)} \qquad (1 - \tau^2 > 0).$$

Bearing in mind the fact that the integrand is an even function we can write

$$\int_C (\tau^2 - 1)^{n-\frac{1}{2}} d\tau = -4i \sin\left(n - \frac{1}{2}\right)\pi \int_0^1 (1 - \tau^2)^{n-\frac{1}{2}} d\tau,$$

or, substituting τ by a new variable of integration x, according to the formula $\tau^2 = x$:

$$\int_C (\tau^2 - 1)^{n-\frac{1}{2}} d\tau = -2i \sin\left(n - \frac{1}{2}\right)\pi \int_0^1 x^{-\frac{1}{2}} (1 - x)^{n-\frac{1}{2}} dx.$$

But we saw earlier that

$$\int_0^1 x^{p-1} (1 - x)^{q-1} dx = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)},$$

so that the integral in the equation (191) will be

$$\int_C (\tau^2 - 1)^{n-\frac{1}{2}} d\tau = -2i \sin\left(n - \frac{1}{2}\right)\pi \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} =$$

$$= 2i \sin\left(n + \frac{1}{2}\right)\pi \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}.$$

But we had earlier:

$$\Gamma(z)\,\Gamma(1 - z) = \frac{\pi}{\sin \pi z} \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

whence

$$\Gamma\left(n + \frac{1}{2}\right) \sin\left(n + \frac{1}{2}\right)\pi = \frac{\pi}{\Gamma\left(\frac{1}{2} - n\right)}, \tag{192}$$

so that finally

$$\int_C (\tau^2 - 1)^{n-\frac{1}{2}} d\tau = -\frac{2\pi^{\frac{3}{2}} i}{\Gamma\left(\frac{1}{2} - n\right)\Gamma(n+1)}.$$

We deduced this formula on the assumption that n is real and that $n - 1/2 > -1$. Bearing in mind the fact that both sides are

analytic functions of n, we can say that this formula will hold for every value of n. Formula (191) thus gives us the following value for the constant a:

$$a = \frac{\Gamma\left(\frac{1}{2} - n\right)}{2^n \pi^{\frac{3}{2}} i}.$$

Substituting this value in the formula (189) we obtain an expression for the Hankel function:

$$\left.\begin{aligned} H_n^{(1)}(z) &= \frac{\Gamma\left(\frac{1}{2} - n\right)}{\pi^{\frac{3}{2}} i} \left(\frac{z}{2}\right)^n \int_{\lambda_1} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau} d\tau, \\ H_n^{(2)}(z) &= -\frac{\Gamma\left(\frac{1}{2} - n\right)}{\pi^{\frac{3}{2}} i} e^{(2n+1)\pi i} \left(\frac{z}{2}\right)^n \int_{\lambda_2} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau} d\tau. \end{aligned}\right\} \quad (193)$$

We assume in both integrals that $\arg(\tau^2 - 1) = 0$ when $\tau > 1$. If we assume in the second integral that $\arg(\tau^2 - 1) = 2\pi$ when $\tau > 1$, we can write:

$$\left.\begin{aligned} H_n^{(1)}(z) &= \frac{\Gamma\left(\frac{1}{2} - n\right)}{\pi^{\frac{3}{2}} i} \left(\frac{z}{2}\right)^n \int_{\lambda_1} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau} d\tau, \\ H_n^{(2)}(z) &= -\frac{\Gamma\left(\frac{1}{2} - n\right)}{\pi^{\frac{3}{2}} i} \left(\frac{z}{2}\right)^n \int_{\lambda_2} (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau} d\tau. \end{aligned}\right\} \quad (193_1)$$

When $\tau \to +i\infty$ the real part of $iz\tau$ tends to $(-\infty)$ provided the real part of z is greater than zero; formula (193) thus determines the Hankel function to the right of the imaginary axis. Let us remind you that we have assumed that $n - 1/2$ is not a positive integer.

The Hankel function $H_n^{(1)}(z)$, as given by the formula (193), differs from the first of the functions (186) by the factor

$$-\frac{e^{-\pi\left(n-\frac{1}{2}\right)i}\,\Gamma\left(\frac{1}{2} - n\right)}{2^n \pi^{\frac{3}{2}}}.$$

Taking the asymptotic expansion (180) and bearing in mind that $w = z^n u$, we obtain after performing elementary transformations:

$$e^{-iz} z^{\frac{1}{2}} H_n^{(1)}(z) \sim -\frac{2^{\frac{1}{2}} e^{-\frac{\pi}{2}ni+\frac{3\pi}{4}} \Gamma\left(\frac{1}{2}-n\right) \sin\left(n+\frac{1}{2}\right)\pi}{\pi^{\frac{3}{2}}} \times$$

$$\times \sum_{k=0}^{\infty} \binom{n-\frac{1}{2}}{k} \Gamma\left(n+\frac{1}{2}+k\right)\left(\frac{i}{2z}\right)^k$$

or, from (192):

$$e^{-iz} z^{\frac{1}{2}} H_n^{(1)}(z) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-i\left(\frac{\pi n}{2}+\frac{\pi}{4}\right)}}{\Gamma\left(n+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \binom{n-\frac{1}{2}}{k} \Gamma\left(n+\frac{1}{2}+k\right)\left(\frac{i}{2z}\right)^k. \tag{194}$$

We can write this as follows:

$$H_n^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{i\left(z-\frac{\pi n}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(n+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \binom{n-\frac{1}{2}}{k} \Gamma\left(n+\frac{1}{2}+k\right)\left(\frac{i}{2z}\right)^k. \tag{195}$$

Similarly, we obtain:

$$H_n^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{-i\left(z-\frac{\pi n}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(n+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \binom{n-\frac{1}{2}}{k} \Gamma\left(n+\frac{1}{2}+k\right)\left(-\frac{i}{2z}\right)^k. \tag{196}$$

The latter formulae can be written as follows:

$$H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{i\left(z-\frac{\pi n}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(n+\frac{1}{2}\right)} \times$$

$$\times \left[\sum_{k=0}^{p-1} \binom{n-\frac{1}{2}}{k} \Gamma\left(n+\frac{1}{2}+k\right)\left(\frac{i}{2z}\right)^k + O(|z|^{-p})\right], \tag{195$_1$}$$

$$H_n^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{-i\left(z-\frac{\pi n}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(n+\frac{1}{2}\right)} \times$$

$$\times \left[\sum_{k=0}^{p-1} \binom{n-\frac{1}{2}}{k} \Gamma\left(n+\frac{1}{2}+k\right)\left(-\frac{i}{2z}\right)^k + O(|z|^{-p})\right], \tag{196$_1$}$$

where $0(|z|^{-k})$ denotes a number which is such that the product $|z|^k 0(|z|^{-k})$ remains bounded as $|z|$ increases indefinitely. In the expression $(2/\pi z)^{1/2}$ it must be assumed that $\arg z = 0$, i.e. the positive radical must be taken.

We proved the above asymptotic formulae for the ray $z > 0$. It can be shown that these formulae will be valid in certain sectors, viz. formula (195_1) holds in the sector:

$$-\pi + \varepsilon < \arg z < 2\pi - \varepsilon,$$

and formula (196_1) in the sector:

$$-2\pi + \varepsilon < \arg z < \pi - \varepsilon,$$

where ε is any small positive number.

113. The Bessel function. Substituting the expression we found for a in the formula (190) we can represent the Bessel function $J_n(z)$ by the integral

$$J_n(z) = \frac{\Gamma\left(\frac{1}{2} - n\right)}{2\pi^{\frac{3}{2}} i} \left(\frac{z}{2}\right)^n \int_C (\tau^2 - 1)^{n-\frac{1}{2}} e^{iz\tau}\, d\tau. \tag{197}$$

This formula, like formula (193), holds for all values of n except for values of the form $m + 1/2$, where m is an integer $\geqslant 0$.

If the real part of n is greater than $(-1/2)$ then integration in the above formula can be reduced to integration along the double line $(-1, +1)$ and, using the same arguments as above, we arrive at the formula:

$$J_n(z) = \frac{1}{\sqrt{\pi}\,\Gamma\left(n + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \int_{-1}^{1} (1 - \tau^2)^{n-\frac{1}{2}} e^{iz\tau} d\tau \tag{198}$$

$$\left(\mathcal{R}[n] > -\frac{1}{2}\right).$$

If we put $\tau = \sin\varphi$ we obtain:

$$J_n(z) = \frac{1}{\sqrt{\pi}\,\Gamma\left(n + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n}\varphi \times$$

$$\times [\cos(z\sin\varphi) + i\sin(z\sin\varphi)]\, d\varphi,$$

or, bearing in mind that the coefficient of i is odd:

$$J_n(z) = \frac{1}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n}\varphi \cdot \cos(z\sin\varphi)\,d\varphi \qquad (199)$$

$$\left(\mathcal{R}[n] > -\frac{1}{2}\right),$$

which can also be written as follows:

$$J_n(z) = \frac{2}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^n \int_{0}^{\frac{\pi}{2}} \cos^{2n}\varphi \cos(z\sin\varphi)\,d\varphi \qquad (200)$$

$$\left(\mathcal{R}[n] > -\frac{1}{2}\right).$$

Taking half the sum of the asymptotic expansions (195) and (196) we obtain the asymptotic representation for the Bessel function.

For simplicity we shall only consider the ray $z > 0$. In this case the modulus of the factor $e^{\pm iz}$ is unity. Taking half the sum of the expressions (195_1) and (196_1) and bearing in mind that

$$\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{\pm i\left(z-\frac{\pi n}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(n+\frac{1}{2}\right)} O(z^{-p}) = O\left(z^{-p-\frac{1}{2}}\right) \qquad (z > 0),$$

we obtain:

$$J_n(z) = \frac{1}{2}\left[H_n^{(1)}(z) + H_n^{(2)}(z)\right] =$$

$$= \frac{1}{\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sum_{k=0}^{p-1} \binom{n-\frac{1}{2}}{k} \frac{\Gamma\left(n+\frac{1}{2}+k\right)}{(2z)^k} \times$$

$$\times \left\{ \begin{array}{l} (-1)^{\frac{k}{2}} \cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right) \\ (-1)^{\frac{k+1}{2}} \sin\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right) \end{array} \right\} + O\left(z^{-p-\frac{1}{2}}\right), \qquad (201)$$

where the first expression in the braces refers to the even k and the second to the odd k.

Considering only the first terms of the asymptotic representation we can write:

$$\left.\begin{aligned} H_n^{(1)}(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)}[1+O(|z|^{-1})], \\ H_n^{(2)}(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)}[1+O(|z|^{-1})] \end{aligned}\right\} \tag{202}$$

and for the Bessel function

$$J_n(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right) + O\left(z^{-\frac{3}{2}}\right) \quad (z>0). \tag{203}$$

The difference in the asymptotic expressions for the Hankel and Bessel functions plays an essential part in solving problems of mathematical physics in infinite domains which include the point at infinity; we shall deal with this later.

114. The Laplace transformation in more general cases. The Laplace transformation can be applied to equations of a more general type than the equation (134). Consider, for example, an equation the coefficients of which are polynomial expressions of the second degree:

$$(a_0 z^2 + a_1 z + a_2) w'' + (b_0 z^2 + b_1 z + b_2) w' + (c_0 z^2 + c_1 z + c_2) w = 0, \tag{204}$$

where we assume that $a_0 \neq 0$. If we divide the above equation by the coefficient of w'', then the coefficients of w' and w will lie in the neighbourhood of $z = \infty$, and they will have the same form as in the equation (113). We shall again try to find the solution of the equation (204) in the form:

$$w(z) = \int_l v(z')\, e^{zz'}\, dz'. \tag{205}$$

Using the same arguments as in [107] we obtain for $v(z')$ a differential equation of the second order in the form:

$$(a_0 z'^2 + b_0 z' + c_0)\frac{d^2 v}{dz'^2} + p(z')\frac{dv}{dz'} + q(z')\, v = 0, \tag{206}$$

where the coefficients which are omitted are polynomials, the degree of which is not greater than two. Construct the quadratic equation

$$a_0 a^2 + b_0 a + c_0 = 0 \tag{207}$$

and assume that it has different zeros $a = a_1$ and $a = a_2$. The equation (206) has regular singularities at the points $z' = a_1$ and $z' = a_2$. At each of these points the determining equation has one zero equal to zero. Denote by $(p-1)$ and $(q-1)$ the second zeros of the determining equation at the above points;

we assume that p and q are not integers. At each of the above singularities there is one regular solution and the second integral has the form:

$$v_1(z') = (z' - a_1)^{p-1}\varphi_1(z'); \quad v_2(z') = (z' - a_2)^{q-1}\varphi_2(z'), \tag{208}$$

where $\varphi_k(z')$ is regular at the point $z' = a_k$ $(k = 1, 2)$. The contour l in formula (205) should be so chosen that the increment of the term outside the integral vanishes on integration by parts, as in [107]), round l. This will be so if the following conditions are satisfied:

$$\left[\frac{d^n(vz'^m)}{dz'^n}\,e^{zz'}\right]_l = 0 \quad \begin{pmatrix} n = 0, 1 \\ m = 0, 1, 2 \end{pmatrix}. \tag{209}$$

Take for $v(z')$ the integral $v_k(z')$ and for l the contour l_k' shown in [108]. We thus obtain, as in [108], two linearly independent solutions of the equation (204):

$$w_k(z) = \int_{l_k'} v_k(z')\,e^{zz'}\,dz' \quad (z > 0)\,(k = 1, 2).$$

These solutions will have the following asymptotic representation as $z \to +\infty$:

$$w_1(z) = e^{a_1 z} z^{-p}\left(c_0 + \frac{c_1}{z} + \ldots\right),$$

$$w_2(z) = e^{a_2 z} z^{-q}\left(d_0 + \frac{d_1}{z} + \ldots\right),$$

which are the same as the expansion in [105] except for the constant terms. The Laplace transformation can be applied in cases when the coefficients of the equation are polynomials of degree m. We then obtain for $v(z')$ an equation of the mth order, the coefficients of which are polynomials of the second degree. As before this equation for $v(z')$ has at the singularities $z_1 = a_1$ and $z' = a_2$ (the zeros of the coefficient of $d^m v/dz^m$) unique solutions of the form (208). The remaining solutions are regular at the above singularities. Otherwise the arguments used are exactly the same as before.

115. The generalized Laguerre polynomials. Investigations of the condition of an electron in the Coulomb field and some other problems of modern physics lead us to a linear differential equation of the second order in the following form:

$$w'' + \frac{1}{z}w' + \left(2\varepsilon + \frac{2}{z} - \frac{s^2}{4z^2}\right)w = 0. \tag{210}$$

Here s is a given real positive number and ε is a real parameter. The problem involves the finding of those values of the parameter for which the equation (210) has a solution which remains bounded along the whole line $0 \leqslant z < +\infty$ on the real axis.

Let us consider, to start with, the region of negative values of the parameter ε and replace z by a new independent variable x, given by the formula:

$$x = z\sqrt{-8\varepsilon} \tag{211}$$

and replace the parameter ε by a new positive parameter λ given by the formula:

$$\lambda = \frac{1}{\sqrt{-2\varepsilon}}\,. \tag{212}$$

After these transformations the equation (210) will have the form:

$$x\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{\mathrm{d}w}{\mathrm{d}x} + \left(-\frac{x}{4} + \lambda - \frac{s^2}{4x}\right)w = 0\,. \tag{213}$$

This equation has a regular singularity at the point $x = 0$ and the determining equation at this point will be of the form:

$$\sigma(\sigma - 1) + \sigma - \frac{s^2}{4} = 0.$$

This latter equation has zeros $\sigma = \pm s/2$. Bearing in mind the condition that the solutions must be bounded at the origin we take the zero $\sigma = s/2$, i.e. we must isolate the factor $x^{s/2}$, and our solution will have the following form near the origin:

$$w = x^{\frac{s}{2}} \sum_{k=0}^{\infty} b_k x^k \qquad (b_0 \neq 0)\,. \tag{214}$$

At infinity we must, according to [105], try to satisfy formally the equation (213) by an expression

$$\mathrm{e}^{ax} x^{\varrho} \sum_{k=0}^{\infty} \frac{c_k}{x^k} \qquad (c_0 \neq 0).$$

The quadratic equation for a will be

$$a^2 - \frac{1}{4} = 0\,.$$

It gives the values $a = \pm 1/2$ and the corresponding values of the constant ϱ will be [from (119)]:

$$\varrho_1 = -\left(\lambda + \frac{1}{2}\right); \quad \varrho_2 = \lambda - \frac{1}{2}\,.$$

Bearing in mind the condition that the solution must be bounded, we must take that solution which has the following asymptotic representation at infinity:

$$e^{-\frac{x}{2}} x^{\lambda-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{c_k}{x^k}. \tag{215}$$

The problem thus involves the determination of those values of λ for which a solution in the form (214), after analytic continuation along the line $(0, +\infty)$, has the form (125) at infinity.

The above considerations naturally bring us to the conclusion that w should be replaced by a new function y, according to the formula

$$w = e^{-\frac{x}{2}} x^{\frac{s}{2}} y. \tag{216}$$

Substituting in the equation (213) we obtain for y the equation

$$x \frac{d^2y}{dx^2} + (s+1-x) \frac{dy}{dx} + \left(\lambda - \frac{s+1}{2}\right) y = 0. \tag{217}$$

This equation happens to have the same form as the equation (134) which we investigated before. Bearing in mind our former considerations we should be able to find a solution of the equation (217), which is regular at the origin and is of the order $x^{\lambda-(s+1)/2}$ at infinity.

If we suppose, for briefness, that

$$\frac{s+1}{2} - \lambda = p, \tag{218}$$

then the equation (217) can be written as follows:

$$x \frac{d^2y}{dx^2} + (s+1-x) \frac{dy}{dx} - py = 0. \tag{219}$$

We shall attempt to find a solution of this equation, which is regular at the origin, in the form of the usual power series

$$y = 1 + b_1 x + b_2 x^2 + \dots$$

Substituting in the equation (219) and using the usual method of undetermined coefficients we obtain the solution in the form of a series, very similar to the hypergeometric series; thus if we denote

$$F(\alpha, \gamma, x) = 1 + \frac{\alpha}{\gamma} \frac{x}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \dots, \tag{220}$$

then the solution of the equation (219), which is regular at the origin, will be

$$y = CF(p, s+1; x), \tag{221}$$

where C is an arbitrary constant. Notice that the series (220) converges for every value of x; this follows from the form of the equation (219) and can readily be proved by the d'Alembert test. It is evident that the series (200) ends abruptly when α is either zero or a negative integer and in this case our solution will satisfy the necessary condition at infinity. We thus obtain from (218) the following equation for the determination of the parameter λ:

$$\frac{s+1}{2} - \lambda_n = -n \quad (n = 0,\ 1,\ 2, \ldots),$$

whence

$$\lambda_n = \frac{s+1}{2} + n \quad (n = 0,\ 1,\ 2, \ldots). \tag{222}$$

For this value of the parameter the required solution of the equation (219) will be

$$Q_n(x) = C_n F(-n,\ s+1;\ x) = C_n\Big[1 - \frac{n}{1!}\frac{x}{s+1} + \\ + \frac{n(n-1)}{2!}\frac{x^2}{(s+1)(s+2)} + \ldots + (-1)^n \frac{x^n}{(s+1)(s+2)\ldots(s+n)}\Big].$$

To eliminate the letter s from the denominator we choose the constant C_n as follows:

$$C_n = (s+1)(s+2)\ldots(s+n) = \frac{\Gamma(s+n+1)}{\Gamma(s+1)},$$

which finally gives us the following solution for the equation (217) in the form of a polynomial in x and s:

$$Q_n^{(s)}(x) = \frac{\Gamma(s+n+1)}{\Gamma(s+1)} F(-n,\ s+1;\ x), \tag{223}$$

or

$$Q_n^{(s)}(x) = (-1)^n\Big[x^n - \frac{n}{1}(s+n)\,x^{n-1} + \\ + \frac{n(n-1)}{2!}(s+n)(s+n-1)\,x^{n-2} + \ldots + \\ + (-1)^n (s+n)(s+n-1)\ldots(s+1)\Big]. \tag{224}$$

These polynomials are known as *the generalized Laguerre polynomials*. We shall deal with them in greater detail later.

It can be shown that formula (222) gives all the values of the parameter for which our problem has a solution which satisfies the given conditions at the points $x = 0$ and $x = +\infty$.

Using the same arguments as in [102] we can deduce a simple expression for the generalized Laguerre polynomials. The series (220) is the solution of the equation:

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0. \tag{225}$$

If we differentiate this series m times we obtain the series:

$$\frac{\alpha(\alpha+1)\dots(\alpha+m-1)}{\gamma(\gamma+1)\dots(\gamma+m-1)} F(\alpha+m,\ \gamma+m;\ x),$$

and, writing $F(\alpha, \gamma; x) = y_1$ we can see that the derivative of the mth order:

$$y_1^{(m)} = F^{(m)}(\alpha, \gamma; x) \tag{226}$$

is the solution of the equation (225) in which α and γ are replaced by $(\alpha + m)$ and $(\gamma + m)$, i.e.

$$x\frac{d^2y_1^{(m)}}{dx^2} + (\gamma + m - x)\frac{dy_1^{(m)}}{dx} - (\alpha + m)\, y_1^{(m)} = 0.$$

Multiplying both sides of this equation by $x^{\gamma+m-1}e^{-x}$ we can rewrite it in the following form [102]:

$$\frac{d}{dx}\left[x^{\gamma+m}e^{-x}\frac{dy_1^{(m)}}{dx}\right] - (\alpha + m)\, x^{\gamma+m-1}e^{-x}y_1^{(m)} = 0.$$

Differentiating this identity m times we have:

$$\frac{d^{m+1}}{dx^{m+1}}\left[x^{\gamma+m}e^{-x}\frac{dy_1^{(m)}}{dx}\right] = (\alpha + m)\frac{d^m}{dx^m}\left[x^{\gamma+m-1}e^{-x}y_1^{(m)}\right].$$

On constructing similar equations for $m = 0, 1, \dots, k-1$, multiplying them term by term and simplifying the results we have:

$$\frac{d^k}{dx^k}\left(x^{\gamma+k-1}e^{-x}y_1^{(k)}\right) = \alpha(\alpha+1)\dots(\alpha+k-1)\, x^{\gamma-1}e^{-x}y_1. \tag{227}$$

Let us suppose that α is a negative integer $\alpha = -k$, so that the series (220) is a polynomial of the kth degree and $y_1^{(k)}$ is a constant, then

$$F^{(k)}(-k, \gamma; x) = \frac{-k(-k+1)(-k+2)\dots(-k+k-1)}{\gamma(\gamma+1)(\gamma+2)\dots(\gamma+k-1)} =$$

$$= (-1)^k \frac{k!}{\gamma(\gamma+1)\dots(\gamma+k-1)}.$$

Formula (227) gives

$$(-1)^k \frac{k!}{\gamma(\gamma+1)\dots(\gamma+k-1)} \frac{d^k}{dx^k}(x^{\gamma+k-1}e^{-x}) = \\ = (-1)^k k! x^{\gamma-1}e^{-x} F(-k, \gamma; x),$$

so that we finally obtain:

$$F(-k, \gamma; x) = \frac{x^{1-\gamma}e^x}{\gamma(\gamma+1)\dots(\gamma+k-1)} \frac{d^k}{dx^k}(x^{\gamma+k-1}e^{-x}). \quad (228)$$

We thus have, from (223), *the following expression for the generalized Laguerre polynomials* ($\gamma = s+1$; $k = n$):

$$Q_n^{(s)}(x) = x^{-s}e^x \frac{d^n}{dx^n}(x^{s+n}e^{-x}). \quad (229)$$

116. Positive values of the parameter. Let us now consider the equation (210) for positive values of the parameter ε. For this purpose we replace z by a new independent variable x_1 according to the formula

$$x_1 = z\sqrt{8\varepsilon}$$

and the parameter ε by a new parameter λ_1:

$$\lambda_1 = \frac{1}{\sqrt{2\varepsilon}}.$$

The equation (210) thus becomes:

$$x_1 \frac{d^2w}{dx_1^2} + \frac{dw}{dx_1} + \left(\frac{x_1}{4} + \lambda_1 - \frac{s^2}{4x_1}\right) w = 0. \quad (230)$$

This latter equation is obtained from equation (213) by the following change of independent variable and parameter:

$$x = ix_1; \quad \lambda = -i\lambda_1.$$

We thus replace w by a new function y_1, according to the formula

$$w = e^{-\frac{ix_1}{2}} x_1^{\frac{s}{2}} y_1, \quad (231)$$

and we obtain for y_1 the equation

$$x_1 \frac{d^2y_1}{dx_1^2} + (s+1-ix_1)\frac{dy_1}{dx_1} + \left[\lambda_1 - \frac{i}{2}(s+1)\right] y_1 = 0. \quad (232)$$

Comparing this with the equation (134) we can see that in this case

$$a_0 = -i; \quad a_1 = s+1; \quad b_0 = 0; \quad b_1 = \lambda_1 - \frac{i}{2}(s+1).$$

The quadratic equation for a will have the form:

$$a^2 - ia = 0,$$

this has two zeros

$$a_1 = 0; \quad a_2 = i,$$

and the corresponding values of p and q are [107]:

$$p = \frac{b_1}{a_0} = \frac{1}{2}(s+1) + i\lambda_1;$$

$$q = \frac{i(s+1) + \lambda_1 - \frac{i}{2}(s+1)}{2i - i} = \frac{1}{2}(s+1) - i\lambda_1.$$

In the case under consideration we therefore have the following two solutions for the equation (232):

$$\begin{aligned} y_1^{(1)} &= C_1 \int_{l_1'} z'^{\frac{1}{2}(s-1)+i\lambda_1} (z'+i)^{\frac{1}{2}(s-1)-i\lambda_1} e^{x_1 z'} \, dz', \\ y_1^{(2)} &= C_2 \int_{l_2'} z'^{\frac{1}{2}(s-1)+i\lambda_1} (z'-i)^{\frac{1}{2}(s-1)-i\lambda_1} e^{x_1 z'} \, dz', \end{aligned} \tag{233}$$

where l_1' and l_2' are contours, with ends at the point $z' = -\infty$, which surround the points $z' = 0$ and $z' = i$. According to the formulae in [109] these solutions will have the following asymptotic expressions for large positive values of x_1:

$$C_3 x_1^{-\frac{1}{2}(s+1)-i\lambda_1} \sum_{k=0}^{\infty} \frac{c_k}{x_1^k},$$

$$C_4 e^{ix_1} x_1^{-\frac{1}{2}(s+1)+i\lambda_1} \sum_{k=0}^{\infty} \frac{c_k'}{x_1^k},$$

where C_3 and C_4 are constants. Using formula (231) also we can see that the corresponding solutions, w_1 and w_2 of the equation (230) tend to zero as $x_1 \to +\infty$. Consequently the same can be said about every solution of the equation (230) and in particular about the solution which, near $x_1 = 0$, is represented in the form

$$w = x_1^{\frac{s}{2}} \sum_{k=0}^{\infty} b_k x_1^k \quad (b_0 \neq 0),$$

i.e. for every real λ_1 the solution of the equation (230) *vanishes at both ends of the interval* $(0, +\infty)$.

117. The degeneration of the equation of Gauss. Let us consider the general case of an equation, the coefficients of which are polynomials of the first degree:

$$(p_0 t + p_1)\frac{d^2u}{dt^2} + (q_0 t + q_1)\frac{du}{dt} + (r_0 t + r_1)u = 0, \qquad (234)$$

where we assume that $p_0 \neq 0$. We will show that this equation can be obtained in the form (225). Replacing t by a new independent variable $z = p_0 t + p_1$ we can obtain the equation (234) in the form (134):

$$z\frac{d^2u}{dz^2} + (a_0 z + a_1)\frac{du}{dz} + (b_0 z + b_1)u = 0. \qquad (235)$$

If we now assume that $u = e^{\alpha z} z^p y$ and replace z by a new variable $x = kz$, then with a suitable choice of the constants p and k we arrive at the equation (225). We will now show that this latter equation can be obtained by a limit transition from the equation of Gauss:

$$z(z-1)\frac{d^2y}{dz^2} + [-\gamma + (1 + \alpha + \beta)z]\frac{dy}{dz} + \alpha\beta y = 0.$$

Replacing z by a new variable $x = az$ we can rewrite this equation in the form:

$$x\left(\frac{x}{a} - 1\right)\frac{d^2y}{dx^2} + \left[-\gamma + x + \frac{(1+\beta)x}{a}\right]\frac{dy}{dx} + \beta y = 0.$$

If we assume in the above equation that a tends to infinity we obtain the equation (225) which, as we have shown, is connected with the equation (234) by a simple replacement of variables. As a result of the above limit transition, two of the regular singularities of the equation of Gauss coincide and form one irregular singularity at infinity while the other regular singularity remains.

Connected with the above is Whittaker's equation:

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right)w = 0. \qquad (236)$$

If we replace w by a new function u, where $w = z^{m+1/2}u$, we obtain an equation in the form (225):

$$z\frac{d^2u}{dz^2} + (1 + 2m)\frac{du}{dz} + \left(-\frac{1}{4}z + k\right)u = 0.$$

Constructing a solution in the form of (151) for this equation, we have:

$$w = Cz^{m+\frac{1}{2}} \int_l \left(z' - \frac{1}{2}\right)^{m-\frac{1}{2}+k} \left(z' + \frac{1}{2}\right)^{m-\frac{1}{2}-k} e^{zz'}\, dz',$$

where l is a contour which goes from $(-\infty)$ and surrounds the point $z' = -1/2$. Replacing the variable of integration as follows:

$$z' = -\frac{1}{2} - \frac{t}{z},$$

we obtain:

$$w = C_1 e^{-\frac{1}{2}z} z^k \int_{l_0} (-t)^{m-\frac{1}{2}-k} \left(1 + \frac{t}{z}\right)^{m-\frac{1}{2}+k} e^{-t}\, dt,$$

where l_0 is a contour which originates at $(+\infty)$ and surrounds the point $t = 0$ in the positive direction; we are assuming that the point $t = -z$ lies outside this contour. Choosing the constant C_1 in a deliberate manner we obtain the functions deduced by Whittaker:

$$w_{k,m}(z) = \tag{237}$$

$$= -\frac{1}{2\pi i} \Gamma\left(k + \frac{1}{2} + m\right) e^{-\frac{1}{2}z} z^k \int_{l_0} (-t)^{m-\frac{1}{2}-k} \left(1 + \frac{t}{z}\right)^{m-\frac{1}{2}+k} e^{-t}\, dt.$$

In this formula it is assumed that z is not a negative, that $\arg z$ has principal values, that $|\arg(-t)| \leqslant \pi$ and $\arg(1 + t/2)$ tend to zero as $t \to 0$ and when they lie inside the contour l_0. Formula (237) becomes devoid of meaning when $(k - 1/2 - m)$ is a negative integer.

When the real part of $(k - 1/2 - m)$ is not positive and $(k - 1/2 - m)$ is not an integer, the expression (237) can be transformed to:

$$w_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma\left(\frac{1}{2} - k + m\right)} \int_0^\infty t^{-k-\frac{1}{2}+m} \left(1 + \frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}\, dt, \tag{238}$$

and this formula also determines $w_{k,m}(z)$ when $(k - 1/2 - m)$ is a negative integer.

Using the results from [109] it is easy to write the asymptotic expression for the function $w_{k,m}(z)$:

$$w_{k,m}(z) = e^{-\frac{1}{2}z} z^k \times \tag{238$_1$}$$

$$\times \left\{1 + \sum_{n=1}^{\infty} \frac{\left[m^2 - \left(k - \frac{1}{2}\right)^2\right]\left[m^2 - \left(k - \frac{3}{2}\right)^2\right] \ldots \left[m^2 - \left(k - n + \frac{1}{2}\right)^2\right]}{n!\, z^n}\right\}.$$

It holds in the sector $|\arg z| \leqslant \pi - \varepsilon$, where ε is an arbitrary positive number.

Equation (236) does not change when k and z are simultaneously replaced by $(-k)$ and $(-z)$ and, as a result of this the function $w_{-k,m}(-z)$ will be a second solution of the equation (236), other than the solution (237). The linear independence of the two constructed solutions follows from the asymptotic expression (238_1).

118. Equations with periodic coefficients. Consider a linear differential equation of the second order, the coefficients of which are periodic functions of the independent variable. The theory of such equations resembles in many instances the theory of equations with analytic coefficients which we described above. Let us suppose, for the moment, that both the coefficients and the independent variables are real. We are given the equation

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0, \tag{239}$$

where $p(x)$ and $q(x)$ are real continuous functions of the real variable x with the real period ω, i.e.

$$p(x+\omega) = p(x); \quad q(x+\omega) = q(x). \tag{240}$$

The continuity of the coefficients guarantees that any solution of the equation (239), determined by some initial conditions, exists for all real values of x. Let $y_1(x)$ be one solution of the equation, i.e. we have the identity

$$y_1''(x) + p(x)\,y_1'(x) + q(x)\,y_1(x) = 0.$$

Replacing x by $x+\omega$, we can write:

$$y_1''(x+\omega) + p(x+\omega)\,y_1'(x+\omega) + q(x+\omega)\,y_1(x+\omega) = 0$$

or from (240)

$$y_1''(x+\omega) + p(x)\,y_1'(x+\omega) + q(x)\,y_1(x+\omega) = 0.$$

It follows that $y_1(x+\omega)$ will also be a solution of this equation. Let us now consider any two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the equation. The functions $y_1(x+\omega)$ and $y_2(x+\omega)$ must also be solutions of the equation (239) and, consequently, they can be expressed linearly in terms of $y_1(x)$ and $y_2(x)$, i.e.

$$\left.\begin{aligned} y_1(x+\omega) &= a_{11}\,y_1(x) + a_{12}\,y_2(x), \\ y_2(x+\omega) &= a_{21}\,y_1(x) + a_{22}\,y_2(x). \end{aligned}\right\} \tag{241}$$

where a_{ik} are constants. We can see that *if we take two linearly independent solutions of the equation* (239) *and add to the argument the period, this will be equivalent to the linear transformation* (241). Similarly, when considering equations with analytic coefficients we saw that by encircling a singularity, the linearly independent solutions are subject to a linear transformation and we can use the same arguments as in [97]. Let us give the results. The table of the constants a_{ik} depends on the choice of the linearly independent solutions, but the coefficients of the quadratic equation in ϱ:

$$\begin{vmatrix} a_{11} - \varrho, & a_{12} \\ a_{21}, & a_{22} - \varrho \end{vmatrix} = 0 \tag{242}$$

will be the same for every choice of solution. *If the equation* (242) *has two different zeros* ϱ_1 *and* ϱ_2 *then two linearly independent solutions exist which will be multiplied by* ϱ_1 *and* ϱ_2 *when* x *is replaced by* $x + \omega$, i.e. denoting these solutions by $\eta_k(x)$:

$$\eta_1(x + \omega) = \varrho_1 \eta_1(x); \quad \eta_2(x + \omega) = \varrho_2 \eta_2(x). \tag{243}$$

If the equation (242) has equal zeros, i.e. $\varrho_1 = \varrho_2$ then, in general only one solution exists which acquires the factor ϱ_1 when x is replaced by $x + \omega$, and in that case we have instead of (243) the linear transformation:

$$\eta_1(x + \omega) = \varrho_1 \eta_1(x); \quad \eta_2(x + \omega) = a_{21} \eta_1(x) + \varrho_1 \eta_2(x). \tag{244}$$

Let us also recall you that the equation (242) cannot have a zero which is zero, i.e. a determinant composed of the numbers a_{ik} will not be equal to zero.

Having recalled these results, we shall now try to determine the form of the solutions in different cases. Consider, first of all, the case (243). Let us look at the two functions

$$\varrho_1^{\frac{x}{\omega}} = e^{\frac{x}{\omega} \log \varrho_1} \quad ; \quad \varrho_2^{\frac{x}{\omega}} = e^{\frac{x}{\omega} \log \varrho_2},$$

where we take definite values for $\log \varrho_1$ and $\log \varrho_2$. When replacing x by $x + \omega$ these functions acquire the factors ϱ_1 and ϱ_2 and therefore the quotients $\eta_1(x) : \varrho_1^{x/\omega}$ and $\eta_2(x) : \varrho_2^{x/\omega}$ must be periodic functions with a period ω; consequently, in the case (243) we can write

$$\eta_1(x) = \varrho_1^{\frac{x}{\omega}} \varphi_1(x); \quad \eta_2(x) = \varrho_2^{\frac{x}{\omega}} \varphi_2(x), \tag{245}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are periodic functions with a period ω. In the case (244) we have a similar expression for $\eta_1(x)$. To investigate $\eta_2(x)$ we shall consider the quotient $\eta_2(x) : \eta_1(x)$. We have, from (244):

$$\frac{\eta_2(x+\omega)}{\eta_1(x+\omega)} = \frac{\eta_2(x)}{\eta_1(x)} + c \qquad \left(c = \frac{a_{21}}{\varrho_1}\right),$$

i.e. the quotient acquires the term c when x is replaced by $x + \omega$. The elementary function $(c/\omega)\,x$ acquires a similar term and, consequently, the difference $\eta_2(x)/\eta_1(x) - (c/\omega)\,x$ is a periodic function $\psi_1(x)$. Thus, in this case, taking into account the expression for $\eta_1(x)$, we have:

$$\eta_1(x) = \varrho_1^{\frac{x}{\omega}}\varphi_1(x); \quad \eta_2(x) = \frac{c}{\omega}x\eta_1(x) + \psi_1(x)\,\eta_1(x)$$

or

$$\eta_1(x) = \varrho_1^{\frac{x}{\omega}}\varphi_1(x); \quad \eta_2(x) = \varrho_1^{\frac{x}{\omega}}[\varphi_2(x) + x\varphi_3(x)], \tag{246}$$

where $\varphi_1(x)$, $\varphi_2(x)$ and $\varphi_3(x)$ are periodic functions. If the constant happens to be zero, then the second solution will also have the form (245).

In this case we do not, strictly speaking, have a general method for constructing the quadratic equation (242). We will, nevertheless, note some of the properties of this equation and of its zeros. Let us determine the linearly independent solutions from the following simple initial conditions:

$$y_1(0) = 1; \quad y_1'(0) = 0; \quad y_2 = 0; \quad y_2'(0) = 1. \tag{247}$$

Owing to the fact that the initial conditions and the coefficients of the equation (239) are real these solutions will be real when x is real. Assuming in the identity (241) that $x = 0$ and bearing in mind the initial conditions (247) we obtain $a_{11} = y_1(\omega)$ and $a_{21} = y_2(\omega)$. Thus for the given choice of the linearly independent solutions the quadratic equation (242) can be written in the form:

$$\begin{vmatrix} y_1(\omega) - \varrho, & y_1'(\omega) \\ y_2(\omega), & y_2'(\omega) - \varrho \end{vmatrix} = 0, \tag{248}$$

and it follows that the coefficients of this equation are real.

Let us investigate in greater detail the particular case when the term $y'(x)$ is absent in the equation (239), i.e. when the equation has the form:

$$y''(x) + q(x)\,y(x) = 0. \tag{249}$$

Consider the Wronski determinant

$$\Delta(x) = y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x).$$

We obtained for it the following formula [II, 24]:

$$\Delta(x) = \Delta(0)\, e^{-\int_{\omega}^{x} p(x)\, dx}$$

and therefore in this case, when $p(x)$ is identically zero we have

$$\Delta(x) = C,$$

where C is a constant. When the solutions satisfy the initial conditions (247) this constant must be unity. Let us now turn to the quadratic equation (248). The constant in this equation is equal to the Wronski determinant when $x = \omega$, i.e. to unity. Therefore, finally, if we take independent solutions of the equation (249) which satisfy the initial conditions (247) then the quadratic equation for ϱ must be of the form:

$$\varrho^2 - 2A\varrho + 1 = 0, \tag{250}$$

where

$$2A = y_1(\omega) + y_2'(\omega). \tag{251}$$

If the real number A satisfies the condition $|A| > 1$, then the equation has different zeros, the product of which is unity, i.e. the modulus of one of the zeros will be greater than unity and that of the second smaller than unity. When $|A| < 1$ the equation (250) will have conjugate complex zeros, the moduli of which are unity. Finally, when $A = \pm 1$, the equation (250) has a double zero ± 1. The values of A have a decisive influence upon the behaviour of the solutions when the variable x increases indefinitely. We shall now investigate the different cases mentioned above.

In the expressions (245) the factors $\varphi_1(x)$ and $\varphi_2(x)$ are periodic functions and therefore remain bounded as x increases indefinitely; the character of the solution depends essentially on the first few factors:

$$\varrho_1^{\frac{x}{\omega}} = e^{\frac{x}{\omega}\log\varrho_1}; \quad \varrho_2^{\frac{x}{\omega}} = e^{\frac{x}{\omega}\log\varrho_2}. \tag{252}$$

The real part of $\log \varrho$ is, as we know, equal to $\log |\varrho|$ and, consequently, when $|A| > 1$, this real part will be positive for one of the zeros, say for ϱ_1, and negative for the other zero; hence the modulus of the first of the functions (252) will increase indefinitely as $x \to +\infty$ while the modulus of the other will tend to zero.

Returning to the solution (245) we can say that the first solution will not remain bounded as $x \to +\infty$ but the second will tend to zero. In this case the general solution of the equation:

$$C_1 \eta_1(x) + C_2 \eta_2(x) \tag{253}$$

($C_1 \neq 0$) will, in general, remain bounded (the case of non-equilibrium). When $|A| < 1$ the real parts of $\log \varrho_1$ and $\log \varrho_2$ are equal to zero and the moduli of the functions (252) will be unity for all real values of x. In this case both solutions (245) and the general solution (253) remain bounded as $x \to +\infty$. If the initial conditions

$$y(0) = a; \quad y'(0) = b$$

are determined by the numbers a and b, the moduli of which are sufficiently small, then the constants C_1 and C_2 will also be small and therefore the modulus of the solution will remain small for every small positive x (the case of equilibrium).

The particular case when $A = \pm 1$ and the equation (250) has multiple zeros remains to be investigated. Suppose, to start with, that $A = 1$, i.e. $\varrho_1 = \varrho_2 = 1$. We can, in this case, take solutions which have the following form

$$\eta_1(x) = \varphi_1(x); \quad \eta_2(x) = \varphi_2(x) + x\varphi_3(x), \tag{254}$$

where $\varphi_k(x)$ are periodic functions. The first of the above solutions will be purely periodic and the second will, in general, be unbounded owing to the factor x. Only in the exceptional case, when $\varphi_3(x)$ is identically zero, will the second solution also be purely periodic. Finally, when $A = -1$, i.e. $\varrho_1 = \varrho_2 = -1$ we can take $\log \varrho_1 = \pi i$ and we then have in place of (254)

$$\eta_1(x) = e^{i\frac{\pi x}{\omega}} \varphi_1(x); \qquad \eta_2(x) = e^{i\frac{\pi x}{\omega}} [\varphi_2(x) + x\varphi_3(x)].$$

The first solution will, in this case, be purely periodic with a period 2ω while the other solution will, in general, be unbounded, as before.

Let us consider as an elementary example the equation with a constant coefficient

$$y''(x) + qy(x) = 0. \tag{255}$$

This constant coefficient q can be regarded as a periodic function with an arbitrary period ω. Suppose that the constant q is positive.

Writing $q = k^2$ we obtain the following two solutions of the equation:

$$\eta_1(x) = e^{ikx}; \quad \eta_2(x) = e^{-ikx}.$$

On replacing x by $x + \omega$ these solutions acquire the factors $\varrho_1 = e^{ik\omega}$ and $\varrho_2 = e^{-ik\omega}$, the moduli of which are unity. This corresponds to the case when $|A| < 1$. If the constant q in the equation (255) is negative then writing $q = -k^2$, we obtain the following two solutions of the equation:

$$\eta_1(x) = e^{kx}; \quad \eta_2(x) = e^{-kx}.$$

On replacing x by $x + \omega$ these solutions acquire real positive factors $\varrho_1 = e^{k\omega}$ and $\varrho_2 = e^{-k\omega}$ and this corresponds to the case when $|A| > 1$. Analogous circumstances also hold when the coefficient $q(x)$ in the equation (249) depends on x but does not change its sign. Suppose of all first that $q(x) < 0$. Let $y_1(x)$ be the solution of our equation which satisfies the initial conditions $y_1(0) = 1$ and $y_1'(0) = 0$. Integrating the equation (249) and remembering the initial conditions we can write

$$y_1'(x) = -\int_0^x q(x)\, y_1(x)\, dx. \tag{256}$$

When the values of x are positive and close to zero and when $y_1(x)$ is close to unity, since $q(x) < 0$, $y_1'(x) > 0$, i.e. $y_1(x)$ increases. It follows from the relationship (256) that $y_1'(x)$ could only become negative after $y_1(x)$ has become negative. But, on the other hand, if $y_1(x)$ is to become negative it must decrease to start with, i.e. this is necessary if $y_1'(x)$ is to become negative. We thus arrive at a contradiction and we can say that for every $x > 0$, $y'(x) > 0$ and $y_1(x) > 1$ and, in particular, $y_1(\omega) > 1$. Let us now consider the solution $y_2(x)$ which satisfies the initial conditions $y_2(0) = 0$ and $y_2'(0) = 1$. Integrating the equation (249) we have:

$$y_2'(x) = 1 - \int_0^x q(x)\, y_2(x)\, dx. \tag{257}$$

For values of x close to zero $y_2'(x)$ will be close to unity and it will therefore be positive; for this reason $y_2(x)$ increases and must be greater than zero, since $y_2(0) = 0$. Formula (257) shows that $y_2'(x)$ can become negative only after $y_2(x)$ has become negative. But, on the other hand, $y_2(x)$ can become negative only if it decreases to start with,

i.e. only after $y_2'(x)$ has become negative. This contradiction shows that for every positive x we have $y_2(x) > 0$ and $y_2'(x) > 1$ and, in particular, that $y_2'(\omega) > 1$. The inequalities for $y_1(\omega)$ and $y_2'(\omega)$ give

$$2A = y_1(\omega) + y_2'(\omega) > 2,$$

and we thus arrive at the following theorem; *when in the equation* (249) $q(x) < 0$ *then* $A > 1$ *and, consequently, the numbers* ϱ_1 *and* ϱ_2 *are different and positive.*

To correct somewhat the above arguments we could have substituted the condition $q(x) < 0$ by the condition $q(x) \leqslant 0$ where it is, of course, understood that $q(x)$ is not identically zero.

The case when $q(x) \geqslant 0$ presents greater difficulties. We shall only give the result here, the proof of which can be found in the work of A. M. Liapunov: *The General Problem of The Equilibrium of Movement.* When $q(x) \geqslant 0$ and when the function also satisfies the condition

$$\omega \int_0^{\omega} q(x)\,dx \leqslant 4, \tag{258}$$

then ϱ_1 and ϱ_2 are complex conjugate numbers the moduli of which are unity. This theorem gives the necessary conditions to show that $|A| < 1$.

A detailed and extensive investigation of linear (and non-linear) equations with periodic coefficients was undertaken in the above mentioned work by A. M. Liapunov and in a series of his later works. With reference to the equation (249) we can mention his work: "One series in the theory of linear differential equations of the second order with periodical coefficients" (*Notes of the Academy of Science*, physico-mathematical section, 8th series, 1902, vol. XIII).

119. The case of analytic coefficients. Let us suppose that $p(x)$ and $q(x)$, with a real period ω, are regular functions of the complex variable x in a band which contains the real axis of the x-plane. Assuming that $x = x_1 + ix_2$ we can maintain that this band is defined by the inequality $-h \leqslant x_2 \leqslant +h$. We can divide it into similar rectangles, ω in width, by straight lines parallel to the imaginary axis. In every rectangle the values of $p(x)$ and $q(x)$ will be similar owing to their periodicity. Take, for example, the rectangle defined by the inequalities:

$$0 \leqslant x_1 \leqslant \omega; \quad -h \leqslant x_2 \leqslant h.$$

We now replace x by a new variable z according to the formula

$$z = e^{i\frac{2\pi x}{\omega}}. \tag{259}$$

In the z-plane, instead of a rectangle, we obtain a circular annulus bounded by the circles, centre the origin and radii $e^{2\pi h/\omega}$ and $e^{-2\pi h/\omega}$; this annulus will be cut along the radius in the direction of the real axis; the opposite edges of the cut correspond to the sides $x_1 = 0$ and $x_1 = \omega$ of the above rectangle. Owing to their periodicity our functions have equal values along the cut and, consequently, the same can be said about their derivatives of all orders.

In other words, the functions $p(x)$ and $q(x)$, which are functions of z, will be regular and single-valued in the above annulus where they can be expanded into a Laurent series:

$$p(x) = \sum_{s=-\infty}^{+\infty} a_s z^s, \qquad q(x) = \sum_{s=-\infty}^{+\infty} b_s z^s.$$

We have from (259):

$$\frac{d}{dx} = i\frac{2\pi}{\omega} z \frac{d}{dz}; \qquad \frac{d^2}{dx^2} = -\frac{4\pi^2}{\omega^2} z^2 \frac{d^2}{dz^2} - \frac{4\pi^2}{\omega^2} z \frac{d}{dz},$$

and we obtain instead of the equation (239) the following equation:

$$-\frac{4\pi^2}{\omega^2} z^2 \frac{d^2y}{dz^2} + \left[i\frac{2\pi}{\omega} z \sum_{s=-\infty}^{+\infty} a_s z^s - \frac{4\pi^2}{\omega^2} z \right] \frac{dy}{dz} + \\ + \sum_{s=-\infty}^{+\infty} b_s z^s y = 0. \tag{260}$$

The change in x along the section ω of the real axis corresponds to the completion of a circuit inside the annulus in the z-plane. In the course of this the solution of the equation (260) undergoes a linear transformation. If $p(x)$ and $q(x)$ are integral functions of x, which frequently happens in practice, then the Laurent series in the coefficients of the equation (260) will converge for every finite z except, of course, $z = 0$. But in this case the equation (260) has, in general, an irregular singularity at $z = 0$ since the above Laurent series contain terms with negative powers of z.

Returning to the equation (239) we notice that, as a result of the regularity of its coefficients at the point $x = 0$ we can construct

its solutions $y_1(x)$ and $y_2(x)$ in the form of power series to satisfy the initial conditions (247):

$$y_1(x) = 1 + a_2 x^2 + a_3 x^3 + \ldots, \quad y_2(x) = x + \beta_2 x^2 + \beta_3 x^3 + \ldots$$

These series will converge when $|x| < h$ and, when $p(x)$ and $q(x)$ are integral functions, they will converge for every value of x. When $h > \omega$, we can use these series for the evaluation of $y_k(\omega)$ and $y'_k(\omega)$ in the quadratic equation (248).

120. Systems of linear differential equations. Previously we considered one linear differential equation of the second order. This is a particular case of a system of two linear equations of the first order [95]. In general, one linear equation of the nth order can be represented by a system of linear equations of the first order if the derivatives are accepted as the new unknown functions. We shall now deal with the general case of systems of linear equations of the first order of the form:

$$\left.\begin{aligned} y'_1 &= p_{11}(x)\, y_1 + p_{21}(x)\, y_2 + \ldots + p_{n1}(x)\, y_n \\ y'_2 &= p_{12}(x)\, y_1 + p_{22}(x)\, y_2 + \ldots + p_{n2}(x)\, y_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y'_n &= p_{1n}(x)\, y_1 + p_{2n}(x)\, y_2 + \ldots + p_{nn}(x)\, y_n, \end{aligned}\right\} \tag{261}$$

where y_i are the unknown functions, y'_i are their derivatives and $p_{ik}(x)$ are the table of the given coefficients where, in contrast to the former notation [93], we assume that the first symbol indicates the unknown function of which it is the given coefficient and the second symbol indicates the equation in which this coefficient appears. We shall apply the method of successive approximations described in [95] to the above system and, consequently, all corollaries obtained as a result of using this method will also hold. Let us recall these corrolaries. *If all the coefficients $p_{ik}(x)$ are regular in a circle $|x - a| < r$ then the system* (261) *has a unique solution which satisfies at the point $x = a$ any previously assumed initial conditions:*

$$y_1(a) = a_1; \ldots; \quad y_n(a) = a_n,$$

and this solution will be regular in the above circle $|x - a| < r$. This solution can be analytically continued in any direction except through the singularities of the singularities of the coefficients $p_{ik}(x)$ and, in the course of this continuation, it always remains a solution.

The solution of the system (261) consists of n functions. Suppose that we have n solutions for the system (261). These solutions form a square table of functions

$$Y = \left\| \begin{matrix} y_{11}, & y_{12}, \ldots, & y_{1n} \\ y_{21}, & y_{22}, \ldots, & y_{2n} \\ \cdots & \cdots & \cdots \\ y_{n1}, & y_{n2}, \ldots, & y_{nn} \end{matrix} \right\|,$$

where the first symbol denotes the number of the solution and the second the number of the function entering the solution. We can now say that the

solution of the system is a square table of the above type consisting of n solutions; we shall denote by P a table consisting of the coefficients $p_{ik}(x)$ and by Y a table determining the solution. Using the multiplication law for matrices we can write the system of linear equations as follows in the same way as we did in [93]:

$$\frac{dY}{dx} = YP. \tag{262}$$

Notice that in this case we have used a different notation for the symbols han in [93] and for this reason we have obtained a different sequence of factors on the righthand side of the formula (262). Denoting, as usual, the determinant of the matrix A by $D(A)$ we can prove the following equation for the determinant $D(Y)$ of the solution Y:

$$D(Y) = D(Y)\Big|_{x=b} e^{\int_b^x [p_{11}(x) + p_{22}(x) + \ldots + p_{nn}(x)]\,dx}, \tag{263}$$

where b is an ordinary point for the system (261), i.e. a point at which all the coefficients $p_{ik}(x)$ are regular. Formula (263) is usually known as *Jacobi's formula* and is the generalization of the formula we obtained earlier for the Van der Monde determinant.

Bearing in mind the fundamental definition of a determinant as the sum of the products of its elements, we can say that when differentiating a determinant it is sufficient to differentiate separately every column and to add subsequently all the determinants so obtained, i.e.

$$\frac{dD(Y)}{dx} = \frac{d}{dx}\begin{vmatrix} y_{11}, & y_{12} \\ y_{21}, & y_{22} \end{vmatrix} = \begin{vmatrix} y'_{11}, & y_{12} \\ y'_{21}, & y_{22} \end{vmatrix} + \begin{vmatrix} y_{11}, & y'_{12} \\ y_{21}, & y'_{22} \end{vmatrix},$$

where, to simplify notation, we have assumed that $n = 2$. Replacing the derivatives by their expressions from the equations of the system we have:

$$\frac{dD(Y)}{dx} = \begin{vmatrix} p_{11}y_{11} + p_{21}y_{12}, & y_{12} \\ p_{11}y_{21} + p_{21}y_{22}, & y_{22} \end{vmatrix} + \begin{vmatrix} y_{11}, & p_{12}y_{11} + p_{22}y_{12} \\ y_{21}, & p_{12}y_{21} + p_{22}y_{22} \end{vmatrix}.$$

Expanding the determinants into the sum of determinants and taking p_{ik} outside we can see that certain terms consist of zero determinants since they have similar columns, so that the former formula gives

$$\frac{dD(Y)}{dx} = p_{11}\begin{vmatrix} y_{11}, & y_{12} \\ y_{21}, & y_{22} \end{vmatrix} + p_{22}\begin{vmatrix} y_{11}, & y_{12} \\ y_{21}, & y_{22} \end{vmatrix}$$

or

$$\frac{dD(Y)}{dx} = (p_{11} + p_{22})\,D(Y),$$

from which Jacobi's formula (263) follows. This formula shows that if at a point $x = b$ the determinant $D(Y)$ is not zero then it will not be zero for any x which is an ordinary point of the system (261), i.e. at a point where all the coefficients of the system are regular. If this is so we say that the solution Y is

the general solution corresponding to n linearly independent solutions which form Y. We can then also consider the inverse matrix Y^{-1} where, as we know [93]:

$$\frac{dY^{-1}}{dx} = -Y^{-1}\frac{dY}{dx}Y^{-1},$$

whence, from (262), we can see that it satisfies the system

$$\frac{dY^{-1}}{dx} = -PY^{-1}. \tag{264}$$

Let Z be a solution of our system (262), i.e.

$$\frac{dZ}{dx} = ZP. \tag{265}$$

Construct the matrix

$$A = ZY^{-1}.$$

On using the usual law for differentiating a product [93], and also the equations (265) and (264) this gives:

$$\frac{dA}{dx} = 0,$$

i.e. the matrix A is a constant matrix C, the elements of which are independent of x. Hence

$$Z = CY$$

or, in other words, any solution of the system can be obtained from the general solution by multiplying, on the left, by a constant matrix. Conversely it follows from the form of the equation (262) that by multiplying the solution, on the left, by any constant matrix we can also obtain the solution. Bearing in mind that

$$D(Z) = D(C)\,D(Y),$$

we can see that $D(Z) \neq 0$ and this will be so then, and only then, when $D(C) \neq 0$, i.e. multiplying on the left the general solution Y, by the constant matrix C we obtain the general solution when, and only when $D(C) \neq 0$. It also follows from the formula (263) that in the course of analytic continuation of the general solution Y it always remains the general solution; we have already mentioned this above in the definition of the general solution. Notice that with the system of notation used in [93] we would have had to multiply by a constant matrix not on the left but on the right so as to obtain another solution.

Let us suppose that $x = a$ is a point in the plane which is either a pole or an essential singularity of the coefficient $p_{ik}(x)$. When describing this point the coefficients will return to their former values but the solution Y, in general, will become a new solution in the course of analytic continuation, which can be obtained from the former solutions by multiplying, on the left, by a constant matrix V:

$$Y^{+} = VY.$$

Let us call this matrix V an *integral matrix* when describing the point $x = a$. Bearing in mind that

$$D(Y^+) = D(V)\, D(Y)$$

and that in the course of analytic continuation a general solution always remains a general solution we can say that the determinant of the matrix V will not be zero. The matrix V depends on the particular solution Y considered. If, instead of Y, we take another general solution $Z = CY$, where C is a constant matrix, the determinant of which is not zero, we have:

$$Z^+ = CV\,Y = CVC^{-1}\,Z,$$

i.e. the integral matrix of the new solution will be a matrix, similar to the matrix V. In other words, *different complete solutions have similar integral matrices.*

121. Regular singularities. We shall now consider a singularity of a system which is a pole of the coefficients of an order not higher than the first. To simplify our notation we assume that this point is at the origin $x = 0$, when we can write our system as follows:

$$\left.\begin{aligned} xy_1' &= q_{11}(x)\, y_1 + q_{21}(x)\, y_2 + \ldots + q_{n1}(x)\, y_n \\ xy_2' &= q_{12}(x)\, y_1 + q_{22}(x)\, y_2 + \ldots + q_{n2}(x)\, y_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ xy_n' &= q_{1n}(x)\, y_1 + q_{2n}(x)\, y_2 + \ldots + q_{nn}(x)\, y_n, \end{aligned}\right\} \tag{266}$$

where $q_{ik}(x)$ are functions regular at the point $x = 0$:

$$q_{ik}(x) = a_{ik} + a_{ik}'\, x + a_{ik}''\, x^2 + \ldots \tag{267}$$

We shall try to find the solution of the system (266) in the form:

$$y_i = x^\varrho\,(c_0^{(i)} + c_1^{(i)}\, x + \ldots). \tag{268}$$

Substituting in the system (266) and comparing the coefficients of x^ϱ we obtain a system of homogeneous equations for the determination of the coefficients $c_0^{(i)}$:

$$\left.\begin{aligned} (a_{11} - \varrho)\, c_0^{(1)} + a_{21}\, c_0^{(2)} + \ldots + a_{n1}\, c_0^{(n)} &= 0 \\ a_{12}\, c_0^{(1)} + (a_{22} - \varrho)\, c_0^{(2)} + \ldots + a_{n2}\, c_0^{(n)} &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{1n}\, c_0^{(1)} + a_{2n}\, c_0^{(2)} + \ldots + (a_{nn} - \varrho)\, c_0^{(n)} &= 0 \end{aligned}\right\} \tag{269}$$

and, subsequently, comparing the coefficients of $x^{\varrho+k}$ we obtain a system of equations for the coefficients $c_k^{(i)}$ when the former coefficients $c_m^{(i)}$, where $m < k$, are known:

$$\left.\begin{aligned} (a_{11} - \varrho - k)\, c_k^{(1)} + a_{21}\, c_k^{(2)} + \ldots + a_{n1}\, c_k^{(n)} &= H_{1k} \\ a_{12}\, c_k^{(1)} + (a_{22} - \varrho - k)\, c_k^{(2)} + \ldots + a_{n2}\, c_k^{(n)} &= H_{2k} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{1n}\, c_k^{(1)} + a_{2n}\, c_k^{(2)} + \ldots + (a_{nn} - \varrho - k)\, c_k^{(n)} &= H_{nk} \end{aligned}\right\}, \tag{270}$$

where H_{ik} are linear homogeneous functions of the coefficients c_m where $m < k$. All these calculations are exactly the same, as in [98]. Denote by $f(\varrho)$ the determinant of the homogeneous system (269):

$$f(\varrho) = \begin{vmatrix} a_{11} - \varrho, & a_{21}, & \ldots, & a_{n1} \\ a_{12}, & a_{22} - \varrho, & \ldots, & a_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n}, & a_{2n}, & \ldots, & a_{nn} - \varrho \end{vmatrix}. \tag{271}$$

To obtain a solution of the system (269), other than a zero solution, we must equate this determinant to zero

$$f(\varrho) = 0; \tag{272}$$

and during the further solution of the non-homogeneous systems (270) it is necessary that the determinant of these systems should not be zero. This determinant can be obtained from the determinant of the system (269) by replacing ϱ by $\varrho + k$, i.e. it is equal to $f(\varrho + k)$. Let ϱ_1 be a zero of the equation (272) so that the numbers $\varrho_1 + k$, where k is any positive integer, should no longer be zeros of the equation (272). At the same time our earlier calculations will be formally satisfied and we can construct the series (268) which formally satisfies the system (266). It can be shown, as in [98], that these series will converge in the circle $|x| < r$ in which the series (267) converge.

When the zeros of the equation (272) are different and provided they do not differ from each other by an integer, then the above method enables us to construct n linearly independent solutions of the system (266). Otherwise, as in [98], we shall also have, in general, one solution containing $\log x$ as well as the solutions (268).

Let us write the system (266) in the form of a matrix

$$x \frac{dY}{dx} = YQ,$$

where Q is a matrix consisting of functions $q_{ik}(x)$, regular at the point $x = 0$. We can write this matrix in the form of a series in positive integral powers of x:

$$Q = A_0 + A_1 x + A_2 x^2 + \ldots,$$

where A_s are matrices with constant elements: the matrix A_0 consists of elements a_{ik}, the matrix A_1 of elements a'_{ik} etc. The system (266) can be written in the form:

$$x \frac{dY}{dx} = Y(A_0 + A_1 x + A_2 x^2 + \ldots). \tag{273}$$

We shall try to find a solution in the following form for this system:

$$Y = x^W (1 + C_1 x + C_2 x^2 + \ldots),$$

where W and C_s are the unknown matrices. We have:

$$\frac{dY}{dx} = W x^{W-1} (1 + C_1 x + C_2 x^2 + \ldots) + x^W (C_1 + 2C_2 x + \ldots).$$

Substituting in the equation (273) and multiplying on the left by x^{-W} we obtain:

$$W(1 + C_1 x + C_2 x^2 + \ldots) + x(C_1 + 2C_2 x + \ldots) = \\ = (1 + C_1 x + C_2 x^2 + \ldots)(A_0 + A_1 x + \ldots).$$

A comparison of the constant terms gives

$$W = A_0$$

and also, comparing the coefficients of x^k we obtain a system of equations of matrices for the successive determination of the matrices C_k:

$$A_0 C_k + kC_k = C_k A_0 + C_{k-1} A_1 + \ldots + C_1 A_{k-1} + A_k$$

or

$$A_0 C_k - C_k A_0 + kC_k = C_{k-1} A_1 + \ldots + C_1 A_{k-1} + A_k.$$

Without attempting to investigate this system in general we shall only deal with that particular case when the matrix A_0 can be obtained in the form of a diagonal matrix, i.e. a matrix S exists, which has constant elements and a determinant, which is not zero, such that

$$SA_0 S^{-1} = [\varrho_1, \varrho_2, \ldots, \varrho_n],$$

where ϱ_s are the zeros of the equation (272).

Replace Y by the new unknown matrix Y_1 according to the formula

$$Y = Y_1 S. \tag{274}$$

Substituting in the equation (273) and multiplying, on the right, by S^{-1} we obtain the following system for the matrix Y_1

$$x \frac{dY_1}{dx} = Y_1 (B_0 + B_1 x + B_2 x^2 + \ldots), \tag{275}$$

where

$$B_k = SA_k S^{-1},$$

and in particular

$$B_0 = [\varrho_1, \varrho_2, \ldots, \varrho_n]. \tag{276}$$

As before we are trying to find a solution of the system (275) in the form:

$$Y_1 = x^{W_1}(I + D_1 x + D_2 x^2 + \ldots).$$

On substitution we obtain $W_1 = B_0$ and the other coefficients are determined from equations of the form

$$B_0 D_k - D_k B_0 + kD_k = E_k, \tag{277}$$

where E_k is a matrix expressed in terms of the preceeding matrices D_m, where $m < k$. Bearing in mind that B_0 is a diagonal matrix (276) we obtain, from (277), the following equation for the elements of the matrix D_k:

$$\varrho_i \{D_k\}_{ij} - \{D_k\}_{ij} \varrho_j + k\{D_k\}_{ij} = \{E_k\}_{ij},$$

i.e.

$$\{D_k\}_{ij} = \frac{1}{\varrho_i - \varrho_j + k} \{E_k\}_{ij}.$$

If the difference of the zeros $(\varrho_i - \varrho_j)$ in the equation (272) is not an integer it is possible to determine all the coefficients. Notice that when there are equal zeros among the zeros of the equation (272) but the matrix A_0 can be obtained in the diagonal form (it has simple elementary dividers) then the above calculations remain valid.

In our arguments we did not touch the problem of convergence; this, as we have said already, can be done as in [98]. Notice also that we assumed above that the constant term in the unknown solution of the equation (273):

$$Y = x^W (\mathrm{I} + C_1 x + C_2 x^2 + \ldots)$$

is a unit matrix. This, however, is insignificant. It is only important that the matrix should have a determinant which is not zero. In fact, let

$$Y = x^{W'} (C_0' + C_1' x + C_2' x^2 + \ldots),$$

where $D(C_0') \neq 0$. Consider a new solution

$$C_0'^{-1} Y = C_0'^{-1} x^{W'} C_0' C_0'^{-1} (C_0' + C_1' x + C_2' x^2 + \ldots).$$

But we know that for any analytic function of a matrix

$$C_0'^{-1} f(M') C_0' = f(C_0'^{-1} W' C_0'),$$

so that, for example;

$$C_0'^{-1} \mathrm{e}^{W'} C_0' = \mathrm{e}^{W} \quad (W = C_0'^{-1} W' C_0')$$

and, consequently, the new solution will be:

$$C_0'^{-1} Y = x^W (\mathrm{I} + C_1 x + C_2 x^2 + \ldots) \qquad (C_k = C_0'^{-1} C_k').$$

Similar arguments can be used also in the solution of the equation (275).

122. Regular systems. Consider a system of simple equations the coefficients of which are rational functions with poles of the first order at a finite distance and zero poles at infinity. Let $x = a_j$ be a certain pole of the first order of the coefficients. Each of the coefficients $p_{ik}(x)$ has at this pole a residue $u_{ik}^{(j)}(t)$ and these residues form a square table U_j. We can therefore write our system as follows:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = Y \sum_{j=1}^{m} \frac{U_j}{x - a_j}, \tag{278}$$

where U_j are matrices consisting of constant elements. We shall try to find a solution for the system (278) so that at a point $x = b$, which is not one of the points a_j, it becomes a unit matrix; let us denote this solution by

$$Y(b;\ x).$$

Bearing in mind this initial condition we can rewrite the system (278) as an integral:

$$Y(b;\ x) = \mathrm{I} + \int_b^x Y(b;\ x) \sum_{j=1}^{m} \frac{U_j}{x - a_j} \mathrm{d}x, \tag{279}$$

of its elements.

We shall now use, as usual, the method of successive approximations viz. we assume that $Y_0 = 1$ and make the successive approximations according to the usual formula:

$$Y_n(x) = \mathrm{I} + \int_b^x Y_{n-1}(x) \sum_{j=1}^{m} \frac{U_j}{x - a_j} \, \mathrm{d}x. \tag{280}$$

We have from the method of successive approximations:

$$Y(b;\ x) = Y_0 + (Y_1(x) - Y_0) + (Y_2(x) - Y_1(x)) + \ldots$$

or, assuming for the sake of briefness that

$$Z_n(x) = Y_n(x) - Y_{n-1}(x) \quad (Z_0 = 1),$$

where we have, from (280),

$$Z_n(x) = \int_b^x Z_{n-1}(x) \sum_{j=1}^{m} \frac{U_j}{x - a_j} \, \mathrm{d}x, \tag{281}$$

we can write:

$$Y(b;\ x) = \mathrm{I} + Z_1(x) + Z_2(x) + \ldots \tag{282}$$

We shall determine the first few terms of this expansion by using the general formula (281). Introducing the notation

$$L_b(a_{j_1};\ x) = \int_b^x \frac{\mathrm{d}x}{x - a_{j_1}} = \log \frac{x - a_{j_1}}{b - a_{j_1}},$$

we have

$$Z_1(x) = \int_b^x \sum_{j=1}^{m} \frac{U_j}{x - a_j} \, \mathrm{d}x = \sum_{j_1=1}^{m} U_{j_1} L_b(a_{j_1};\ x).$$

Similarly, introducing the notation

$$L_b(a_{j_1},\ a_{j_2};\ x) = \int_b^x \frac{L_b(a_{j_1};\ x)}{x - a_{j_2}} \, \mathrm{d}x,$$

we have

$$Z_2(x) = \int_b^x \sum_{j_1=1}^{m} U_{j_1} L_b(a_{j_1};\ x) \sum_{j_2=1}^{m} \frac{U_{j_2}}{x - a_{j_2}} \, \mathrm{d}x,$$

or

$$Z_2(x) = \sum_{j_1, j_2}^{1,\ldots,m} U_{j_1} U_{j_2} L_b(a_{j_1}, a_{j_2};\ x),$$

m. Continuing this further and introducing the formulae:

$$L_b(a_{j_1};\ x) = \log\frac{x-a_{j_1}}{b-a_{j_1}}, \tag{283}$$

$$L_b(a_{j_1},\ldots,a_{j_\nu};\ x) = \int_b^x \frac{L_b(a_{j_1},\ldots,a_{j_{\nu-1}};\ x)}{x-a_{j_\nu}}\,dx,$$

which successively determine the coefficients $L_b(a_{j_1},\ldots,a_j^{\nu};\ x)$, we obtain:

$$Z_\nu(x) = \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1},\ldots,\ U_{j_\nu} L_b(a_{j_1},\ldots,a_{j_\nu};\ x),$$

where the summation includes all letters shown under the symbol of summation and every letter, independently of the others, runs through all the integers, from 1 to m. Finally, from (282), we obtain the following representation for our solution in the form of a power series of the matrix U_j:

$$Y(b;\ x) = \mathrm{I} + \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1}\ldots U_{j_\nu} L_b(a_{j_1},\ldots,a_{j_\nu};\ x), \tag{284}$$

where the coefficients of this series are determined by the recurrent relationship (283).

The solution $Y(b;x)$ can be analytically continued in any direction except through the singularities a_j, and the series (284) gives this solution in the whole domain of its existence, i.e. for every analytic continuation. In fact, we shall show, to start with, that the series (284) converges for every analytic continuation of the coefficients $L_b(a_{j_1},\ldots,a_j;\ x)$. Let l be a curve which originates at the point $x=b$ and remains at a finite distance from the points a_j. Let δ be the shortest distance from the points a_j to the curve l and s the length of the arc of this curve from the point b. Using the usual inequality for an integral round the contour l we obtain the following inequality for the coefficients of the series (284) on l [4]:

$$|L_b(a_{j_1};\ x)| < \int_0^s \frac{ds}{\delta} = \frac{s}{\delta},$$

whence

$$|L_b(a_{j_1}, a_{j_2};\ x)| < \int_0^s \frac{|L_b(a_{j_1};\ x)|}{\delta}\,ds < \int_0^s \frac{s\,ds}{\delta^2} = \frac{1}{2!}\left(\frac{s}{\delta}\right)^2,$$

and generally on l:

$$|L_b(a_{j_1},\ldots,a_{j_\nu};\ x)| < \frac{1}{\nu!}\left(\frac{s}{\delta}\right)^\nu.$$

But the power series

$$\sum_{\nu=0}^{\infty} \frac{1}{\nu!}\left(\frac{s}{\delta}\right)^\nu z^\nu = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\left(\frac{sz}{\delta}\right)^\nu$$

converges for every z and we can therefore say that the series (284) is absolutely convergent for every matrix U_j and for every analytic continuation of its coefficients [96]. It also follows from the above inequality that the convergence will be uniform in every finite domain (in general, in a domain with several sheets) which lies at a distance greater than zero from the point a_j. Finally, differentiating the series (284) term by term with respect to x it can readily be shown that it also satisfies the system (278). In fact we can rewrite it as follows by isolating one of the summations:

$$Y(b;x) = \mathrm{I} + \sum_{j=1}^{m} U_j L_b(a_j;\ x) + \\ + \sum_{\nu=1}^{\infty} \sum_{j=1}^{m} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1} \ldots U_{j_\nu} U_j L_b(a_{j_1},\ldots,a_{j_\nu},a_j;\ x).$$

Differentiating with respect to x and bearing in mind that from the definition

$$\frac{\mathrm{d}L_b(a_j;\ x)}{\mathrm{d}x} = \frac{1}{x-a_j}$$

and

$$\frac{\mathrm{d}L_b(a_{j_1},\ldots,a_{j_\nu},a_j;\ x)}{\mathrm{d}x} = \frac{L_b(a_{j_1},\ldots,a_{j_\nu};\ x)}{x-a_j},$$

we obtain by differentiating

$$\frac{\mathrm{d}Y(b;x)}{\mathrm{d}x} = \sum_{j=1}^{m} \frac{U_j}{x-a_j} + \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots j_\nu}^{1,\ldots,m} U_{j_1}\ldots U_{j_\nu} L_b(a_{j_1}\ldots a_{j_\nu};x) \sum_{j=1}^{m} \frac{U_j}{x-a_j}$$

or

$$\frac{\mathrm{d}Y(b;x)}{\mathrm{d}x} = \left[\mathrm{I} + \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1}\ldots U_{j_\nu} L_b(a_{j_1}\ldots a_{j_\nu};\ x)\right] \sum_{j=1}^{m} \frac{U_j}{x-a_j},$$

i.e.

$$\frac{\mathrm{d}Y(b;x)}{\mathrm{d}x} = Y(b,x) \sum_{j=1}^{m} \frac{U_j}{x-a_j}.$$

Finally it becomes clear that the constructed solution (284) becomes a unit matrix when $x = b$ because, from the definition (283), the coefficients of the series (284) vanish when $x = b$. The above considerations give rise to the following theorem.

THEOREM. *A solution of the system* (278), *which becomes a unit matrix when* $x = b$, *is determined with respect to* x *by the series* (284) *in the whole domain of its existence and for every matrix* U_j.

If we make cuts l_j in the x-plane from the points a_j to infinity, so that the cuts do not intersect, then in this cut plane, which will be a connected domain, the solution (284) will be a single-valued function of x, but on opposite edges of the cut it will have different values, viz. when encircling each of the points a_j in the positive direction our solution will be multiplied, on the left, by a constant matrix V, which we called an integral matrix corresponding to the singularity a_j. We shall now deduce an expression for the integral matrices V_j in terms of the matrices U_j which are part of the coefficients of the given system. At the initial point $x = b$ our solution is equal to I, i.e. it becomes a unit matrix and, consequently, to obtain an integral substitution V_j we must determine the value of our solution during analytic continuation round the closed contour l_j, which encircles the points a_j and returns to the point b.

This value can be obtained directly from formula (284) when the formulae (283) are integrals round the above closed contour l_j and, in this case, the coefficients obtained will obviously no longer depend on x.

We shall introduce the following notation:

$$P_j(a_j; b) = \int_{l_j} \frac{dx}{x - a_{j_1}} = \begin{cases} 2\pi i & \text{when } j = j_1 \\ 0 & \text{when } j \neq j_1 \end{cases} \tag{285}$$

and

$$P_j(a_{j_1}, \ldots, a_{j_\nu}, b) = \int_{l_j} \frac{L_b(a_{j_1}, \ldots, a_{j_{\nu-1}}; x)}{x - a_{j_\nu}} dx. \tag{286}$$

We thus obtain V_j in the form of a power series of the matrices U_j and this series converges absolutely for every choice of these matrices:

$$V_j = I + \sum_{\nu=1}^{\infty} \sum_{j_1, \ldots, j_\nu}^{1, \ldots, m} U_{j_1} \ldots U_{j_\nu} P_j(a_{j_1}, \ldots, a_{j_\nu}; b). \tag{287}$$

THEOREM. *The integral substitutions V_j are integral functions of the matrices U_j as determined by the series* (287), *the coefficients of which are determined by the formulae* (285) *and* (286).

Instead of the formulae (286) the following formulae can be proved which connect the values of P_j for adjacent values of ν:

$$P_j(a_{j_1}, \ldots, a_{j_\nu}; b) = \int_{a_j}^{b} \left[\frac{P_j(a_{j_1}, \ldots, a_{j_{\nu-1}}; b)}{b - a_{j_\nu}} - \frac{P_j(a_{j_2}, \ldots, a_{j_\nu}; b)}{b - a_{j_1}} \right] db. \tag{288}$$

We shall not give the proof of this formula.

If we analytically continue the constructed solution round any contour which originates at and returns to a point x then this closed contour is, in the analytic continuation sense, equivalent to several circuits round the points a_j in the positive or negative directions. Consequently, on returning to the point x

our solution is multiplied, on the left, by a constant matrix which is given as a product of factors V_j or V_j^{-1}. In these circumstances it is usually said that the integral matrices V_j form a group of the equation (278).

Let us explain this by a simple example. In the figure the singularities a_1, a_2 and a_3 are marked and the continuous line indicates the path of the analytic continuation l which consists of a number of circuits round the points a_j. The dotted line indicates a contour, equivalent to the contour of the analytic continuation, and it is assumed that $x = b$.

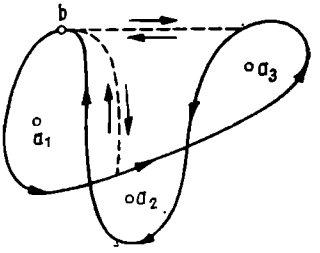

FIG. 72

The first circuit encircles the point a_1 and on completing this circuit we arrive at the point b with the solution $V_1 Y(b; x)$. The next circuit encircles the point a_3 and after completing this circuit the constant matrix V_1 remains unaltered and the matrix $Y(b; x)$ is multiplied, on the left, by V_3, i.e. after completing the second circuit we arrive at the point b with the solution $V_1 V_3 Y(b; x)$; finally, after completing the third circuit, we return to the point b with the solution

$$V_1 V_3 V_2^{-1} Y(b; x),$$

Any solution $Y(x)$ of the system (278) differs from the solution $Y(b, x)$ by the constant matrix

$$Y(x) = C Y(b; x),$$

and its integral substitutions will be as follows [120]:

$$C V_j C^{-1}.$$

Consider now the matrix $Y(b; x)^{-1}$ which is the inverse of the matrix $Y(b; x)$. This matrix, as we know from above, satisfies a system of linear equations

$$\frac{dY(b; x)^{-1}}{dx} = -\sum_{j=1}^{m} \frac{U_j}{x - a_j} Y(b; x)^{-1}.$$

Applying to this system the method of successive approximations we obtain the following representation in terms of a power series of the matrix U_j:

$$Y(b; x)^{-1} = 1 + \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1}, \ldots, U_{j_\nu} L_b^*(a_{j_1}, \ldots, a_{j_\nu}; x), \tag{289}$$

the coefficients of which are determined by the formulae:

$$L_b^*(a_{j_1}; x) = -\int_b^x \frac{dx}{x - a_{j_1}} = -\log \frac{x - a_{j_1}}{b - a_{j_1}} \tag{290}$$

and

$$L_b^*(a_{j_1}, \ldots, a_{j_\nu}; x) = -\int_b^x \frac{L_b^*(a_{j_2}, \ldots, a_{j_\nu}; x)}{x - a_{j_1}} dx. \tag{291}$$

The expansion (289) is absolutely convergent for every matrix U_j and for every analytic continuation with respect to the variable x. The results are obtained in exactly the same way as above. Bearing in mind that

$$[V_j\, Y(b;\, x)]^{-1} = Y(b;x)^{-1}\, V_j^{-1},$$

we can see that by encircling the singularity a_j the matrix $Y(b;x)^{-1}$ is multiplied, on the right, by the matrix V_j^{-1} and it is thus possible to obtain an expression for V_j^{-1} in terms of a power series of the matrix U_j by using the series (289) and by analytically continuing its coefficients round the closed contour l_j which surrounds the points a_j. This gives us the series

$$V_j^{-1} = 1 + \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1} \ldots U_{j_\nu} P_j^*(a_{j_1}, \ldots, a_{j_\nu}; b), \tag{292}$$

in which the coefficients are successively determined by the formulae:

$$P_j^*(a_{j_1};\, b) = -\int_{l_j} \frac{\mathrm{d}x}{x - a_{j_1}}; \tag{293}$$

$$P^*(a_{j_1}, \ldots, a_{j_\nu}; b) = -\int_{l_j} \frac{L_b(a_{j_2}, \ldots, a_{j_\nu}; x)}{x - a_{j_1}}\, \mathrm{d}x.$$

Notice one particular case when *the system* (278) *can be solved in final form, viz. we assume that the matrices* U_j *commute in pairs, i.e. we have for any symbols* i *and* j*:*

$$U_i\, U_j = U_j\, U_i.$$

We will show that, in this case, the solution $Y(b;x)$ of the system (278) can be written in its final form as follows:

$$Y(b;x) = \left(\frac{x - a_1}{b - a_1}\right)^{U_1} \ldots \left(\frac{x - a_m}{b - a_m}\right)^{U_m}. \tag{294}$$

It can readily be seen that the above function becomes a unit matrix when $x = b$. We will show that it satisfies the system of equations (278). Differentiating according to the usual laws for differentiating products and bearing in mind that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x - a_j}{b - a_j}\right)^{U_j} = \frac{\mathrm{d}}{\mathrm{d}x}\, \mathrm{e}^{U_j \log \frac{x - a_j}{b - a_j}} = \left(\frac{x - a_j}{b - a_j}\right)^{U_j} \frac{U_j}{x - a_j}, \tag{295}$$

we obtain

$$\frac{\mathrm{d}Y(b;x)}{\mathrm{d}x} = \sum_{j=1}^{m} \left(\frac{x - a_1}{b - a_1}\right)^{U_1} \ldots \left(\frac{x - a_{j-1}}{b - a_{j-1}}\right)^{U_{j-1}} \frac{U_j}{x - a_j} \left(\frac{x - a_j}{b - a_j}\right)^{U_j} \ldots$$

$$\ldots \left(\frac{x - a_m}{b - a_m}\right)^{U_m}.$$

Since the matrix U_j commutes with the matrix U_l it will also commute with every other function $f(U_l)$ which is given as a power series of U_l. We can therefore write the above formula as follows:

$$\frac{\mathrm{d}Y(b;x)}{\mathrm{d}x} = \sum_{j=1}^{m} \left(\frac{x-a_1}{b-a_1}\right)^{U_1} \cdots \left(\frac{x-a_m}{b-a_m}\right)^{U_m} \frac{U_j}{x-a_j},$$

whence $$\frac{\mathrm{d}Y(b;x)}{\mathrm{d}x} = Y(b;x) \sum_{j=1}^{m} \frac{U_j}{x-a_j},$$

i.e. the matrix (294) does, in fact, satisfy the system (278). The formula (294) can be obtained from the system (278) if we perform a purely formal separation of the variables in this system without taking into account the fact that we are dealing with matrices and not with numerical variables. In this case this is possible owing to the fact that the matrices U_j commute in pairs. The right hand side of formula (294) represents the sum of the series (284) on the assumption that the matrices U_j commute in pairs. It also follows from the formula (294) that in the case under consideration the matrix $Y(b;x)$ acquires on the left, the constant factor shown below, on encircling the point a_i

$$\mathrm{e}^{2\pi i U} U_j.$$

This follows directly if we write the formula

$$\left(\frac{x-a_j}{b-a_j}\right)^{U} = \mathrm{e}^{U_j \log \frac{x-a_j}{b-a_j}}$$

and use the known many-valuedness of the logarithm.

Notice also that the relative position of the factors on the right-hand side of the formula (295) is unimportant, for both factors only contain one matrix U_j and do therefore commute.

123. The form of the solution in the neighbourhood of a singularity. We shall now consider the logarithms of integral matrices with an additional numerical factor

$$W_j = \frac{1}{2\pi i} \log V_j = \frac{1}{2\pi i} \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} (V_j - 1)^{\nu}. \tag{296}$$

We have taken the principal value of the logarithm to be given by a power series which converges when the matrix V_j is sufficiently near to a unit matrix. It follows from the formula (287) that this condition will certainly be satisfied if the matrices U_j are close to a zero matrix and we shall assume this to be so in future. Substituting in the series (296) the expression for $(V_j - 1)$ from the formula (287) and grouping similar terms together we obtain an expression for W_j in the form of a power series of the matrices U_s which converges when these matrices are sufficiently close to a zero matrix:

$$W_j = \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1} \ldots U_{j_\nu} Q_{j_1}(a_{j_1}, \ldots, a_{j_\nu};\ b). \tag{297}$$

We shall not evaluate the coefficients of this expansion since it can easily be done by the direct substitution of our series into another series. Consider now the elementary function

$$\left(\frac{x-a_j}{b-a_j}\right)^{W_j} = \mathrm{e}^{W_j \log \frac{x-a_j}{b-a_j}}. \tag{298}$$

Taking the corresponding values of the logarithm which vanish when $x = b$ we can see that the function (298) becomes a unit matrix when $x = b$ and, by encircling a_j, the logarithm acquires the term $2\pi i$ and the function (298) becomes the new function

$$\mathrm{e}^{W_j\left(2\pi i + \log \frac{x-a_j}{b-a_j}\right)} = \mathrm{e}^{2\pi i W_j}\left(\frac{x-a_j}{b-a_j}\right)^{W_j} = V_j\left(\frac{x-a_j}{b-a_j}\right)^{W_j},$$

the relative order of terms in this expression is unimportant for both are power series of one and the same matrix W_j and, consequently, commute with each other. We can therefore see that the elementary function (298), after encircling a_j, acquires on the left the same factor V_j as the solution $Y(b; x)$ and it also becomes a unit matrix when $x = b$. We can therefore write:

$$Y(b; x) = \left(\frac{x-a_j}{b-a_j}\right)^{W_j} \widetilde{Y}^{(j)}(b; x), \tag{299}$$

where $\widetilde{Y}^{(j)}(b; x)$ is a matrix equal to a unit matrix when $x = b$ and which is single-valued in the neighbourhood of the point $x = a_j$. We will now show that it will not only be single-valued in the neighbourhood of the point $x = b$ but that it will also be regular at the point $x = b$, i.e. *the factor* $[(x-a_j)/(b-a_j)]$ *contains not only the branch-point of our solution but also all the characteristics of the solution at the point* a_j as was the case with regular singularities of equations of the second order.

We have from (299):

$$\widetilde{Y}^{(j)}(b; x) = \left(\frac{x-a_j}{b-a_j}\right)^{-W_j} Y(b; x), \tag{300}$$

whence, differentiating with respect to x:

$$\frac{\mathrm{d}\widetilde{Y}^{(j)}(b; x)}{\mathrm{d}x} = -\frac{W_j}{x-a_j}\left(\frac{x-a_j}{b-a_j}\right)^{-W_j} Y(b; x) + \left(\frac{x-a_j}{b-a_j}\right)^{-W_j} \frac{\mathrm{d}Y(b; x)}{\mathrm{d}x},$$

or, using the equations (278) and (300):

$$\frac{\mathrm{d}\widetilde{Y}^{(j)}(b; x)}{\mathrm{d}x} = -\frac{W_j}{x-a_j}\widetilde{Y}(b; x) + \left(\frac{x-a_j}{b-a_j}\right)^{-W_j} Y(b; x) \sum_{s=1}^{m}{}' \frac{U_s}{x-a_s},$$

i.e. the matrix $\widetilde{Y}^{(j)}(b; x)$ is a solution of the system of equations

$$\frac{\mathrm{d}\widetilde{Y}^{(j)}(b; x)}{\mathrm{d}x} = \widetilde{Y}^{(j)}(b; x) \sum_{s=1}^{\infty} \frac{U_s}{x-a_s} \frac{W_j \widetilde{Y}^{(j)}(b; x)}{x-a_j}. \tag{301}$$

Looking at the right-hand side of formula (300) we can see that both factors are power series of the matrix U_s and therefore their product is too; if all the matrices U_s are equal to zero then W_j will also be zero and the first factor to the left on the right-hand side of (300) becomes a unit matrix. Similarly for $Y(b; x)$ and therefore also for $\tilde{Y}^{(j)}(b; x)$. It is thus possible to seek $\tilde{Y}^{(j)}(b; x)$ in the form of the following power series:

$$\tilde{Y}^{(j)}(b\,;\,x) = 1 + \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1} \ldots U_{j_\nu}\, \tilde{L}_b^{(j)}(a_{j_1},\ldots,a_{j_\nu}\,;\,x). \tag{302}$$

Substituting the series (297) and (302) in equation (301) and comparing the coefficients of the product $U_{\iota} \ldots U_{j_\nu}$ we have:

$$\frac{d\tilde{L}_b^{(j)}(a_{j_1},\ldots,a_{j_\nu}\,;\,x)}{dx} = \frac{\tilde{L}_b^{(j)}(a_{j_1},\ldots,a_{j_{\nu-1}};x)}{x-a_{j_\nu}} -$$

$$- \frac{1}{x-a_j} \sum_{k=1}^{\nu} Q_j(a_{j_1},\ldots,a_{j_k}\,;\,b)\,\tilde{L}_b^{(j)}(a_{j_{k+1}},\ldots,a_{j_\nu}\,;\,x)$$

and, in particular,

$$\frac{d\tilde{L}_b^{(j)}(a_{j_1}\,;\,x)}{dx} = \frac{1}{x-a_{j_1}} - \frac{Q_j(a_{j_1}\,;\,b)}{x-a_j}\,.$$

Notice that in the above sum with respect to k the second factor becomes devoid of meaning when $k = \nu$ and, in this case, it should be replaced by unity. In future we shall often deal with analogous sums in which the factors of border terms become devoid of meaning when the accepted method of notation is used; they should then be replaced by unity.

We have already mentioned above that the matrix $\tilde{Y}^{(j)}(b; x)$ becomes a unit matrix when $x = b$ for any matrix U_j sufficiently close to zero, i.e. all the coefficients in the expansion (302) must vanish when $x = b$. Bearing this in mind and using the above formulae we can write the following formula which successively determines the coefficients in the expansion (302):

$$\tilde{L}_b^{(j)}(a_{j_1},\ldots,a_{j_\nu}\,;\,x) = \int_b^x \left[\frac{\tilde{L}_b^{(j)}(a_{j_1},\ldots,a_{j_{\nu-1}}\,;\,x)}{x-a_{j_\nu}} - \right.$$

$$\left. - \frac{1}{x-a_j} \sum_{k=1}^{\nu} Q_j(a_{j_1},\ldots,a_{j_k}\,;\,b)\,\tilde{L}_b^{(j)}(a_{j_{k+1}},\ldots,a_{j_\nu}\,;\,x) \right] dx. \tag{303}$$

In particular, when $\nu = 1$, we have:

$$\tilde{L}_b^{(j)}(a_{j_1}\,;\,x) = \int_b^x \left[\frac{1}{x-a_{j_1}} - \frac{Q_j(a_{j_1}\,;\,b)}{x-a_j} \right] dx. \tag{304}$$

The coefficients in the expansion (302) must be single-valued functions in the neighbourhood of $x = a_j$, for we know from above that the sum of the series is a single-valued function. It follows that the residue at the pole $x = a_j$

under the integral in the expression (304) vanishes and therefore the function (304) also regular at the point $x = a_j$. We shall now continue the proof from $(\nu - 1)$ to ν. Let us suppose that all the functions

$$\widetilde{L}_b^{(j)}(a_{j_1}, \ldots, a_{j_s}; x) \tag{305}$$

are regular at the point $x = a_j$ when $s < \nu$. We will show that the functions (305) will have the same property when $s = \nu$. We have the formula (303) for these functions. As a result of the regularity of the function (305) when $s < \nu$ the integrand in formula (303) can only have a pole of order one at the point $x = a_j$. However, if the residue at this pole had been other than zero then the function (303) would have been many-valued in the neighbourhood of the point $x = a_j$, which is impossible. It therefore follows that the integrand function in formula (303) and the integral itself will be regular at the point $x = a_j$. We shall not give a more detailed determination of the coefficients in the expansion (302) here.

All the above arguments applied only to the case when the matrices U_s were sufficiently close to zero. Later we shall give a representation of the matrix W_j and of connected matrices which will hold for any matrix; it will then be evident that the singularities in this representation will be those matrices U_s among the characteristic zeros of which are zeros which differ from one other by an integer other than zero.

124. Canonical solutions. The solution $Y(b; x)$ depends on the choice of the point b at which we approximate the matrix to a unit matrix. For this reason the matrix $Y(b; x)$ is known as a matrix (solution), normal at the point $x = b$. This latter point must not be one of the singularities a_j. We obviously cannot make any initial conditions at the singularity $x = a_j$ but we can attempt to construct a solution which would have the simplest form in the neighbourhood of this singularity; this can be done in exactly the same way as in the construction of a solution in the neighbourhood of a regular singularity of an equation of the second order. We shall now construct this solution and call it the *canonical solution at the singularity* $x = a_j$.

We can write:

$$Y(b; x) = \left(\frac{x - a_j}{b - a_j}\right)^{W_j} \widetilde{Y}^{(j)}(b; x) = (x - a_j)^{W_j} (b - a_j)^{-W_j} \widetilde{Y}^{(j)}(b; x),$$

where the relative position of the first two factors on the right-hand side is unimportant for both factors contain only the one matrix W_j. Combining the factor $(b - a_j)^{-W_j}$ with the factor $\widetilde{Y}^{(j)}(b; x)$ we can write:

$$Y(b; x) = (x - a_j)^{W_j} \overline{Y}^{(j)}(b; x), \tag{306}$$

where

$$\overline{Y}^{(j)}(b; x) = (b - a_j)^{-W_j} \widetilde{Y}^{(j)}(b; x)$$

is a matrix, regular at the point $x = a_j$. If all matrices U_s are zero matrices then $\widetilde{Y}^{(j)}(b; x)$ becomes a unit matrix and, consequently, the determinant of this matrix is not zero provided that U_s is sufficiently close to zero. The deter-

minant of the matrix $(b-a_j)^{-W_j} = e^{-W_j \log (b-a_j)}$ is not zero for it is the determinant of an exponential function of a matrix [93] and, consequently, the determinant of the matrix $\overline{Y}^{(j)}(b; x)$ is not zero at the point $x = a_j$ if all matrices U_s are close to zero, i.e. in this case the matrix $\overline{Y}^{(j)}(b; x)^{-1}$ will be regular at the point $x = a_j$. Every solution of our system differs from the solution $Y(b; x)$ by the constant factor C (the matrix on the left):

$$Y(x) = CY(b\,;\,x) \tag{307}$$

and we assume that the determinant of C is not zero so as to obtain a complete solution. In place of formula (307) we can write:

$$Y(x) = C(x-a_j)^{W_j} C^{-1} C\overline{Y}^{(j)}(b\,;\,x)\,;$$

but from [121]:

$$C(x-a_j)^{W_j} C^{-1} = (x-a_j)^{W_j'},$$

where

$$W_j' = CW_jC^{-1}. \tag{308}$$

We now choose a matrix C equal to

$$C = \left[\overline{Y}^{(j)}(b\,;\,a_j)\right]^{-1}, \tag{309}$$

so that we have:

$$C\overline{Y}^{(j)}(b\,;\,x) = 1 \quad \text{when} \quad x = a_j.$$

We thus obtain a solution which we denote by $\theta_j(x)$ and which we call *canonical at the point* $x = a_j$. This solution has the form:

$$\theta_j(x) = (x-a_j)^{W_j'}\bar{\theta}_j(x)\,,$$

where $\bar{\theta}_j(x)$ is a matrix, regular at the point $x = a_j$, which becomes a unit matrix at that point. We will now show that *in this canonical solution the matrix* W_j' *coincides with the matrix* U_j.

Notice, first of all, that all the matrices constructed above are given as power series of the matrix U_s provided the latter matrices are sufficiently close to zero. In this case the matrix W_j', like the matrix W_j, should have no constant term in its expansion, i.e. we should have the following expansion

$$W_j' = \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1}\ldots U_{j_\nu} J_j(a_{j_1},\ldots,a_{j_\nu}). \tag{310}$$

Differentiating with respect to x the formula

$$\bar{\theta}_j(x) = (x-a_j)^{-W_j'}\theta_j(x)\,,$$

we obtain, as in the previous section, the following system of equations for the elements of the matrix $\bar{\theta}_j(x)$:

$$\frac{d\bar{\theta}_j(x)}{dx} = \theta_j(x)\sum_{s=1}^{m}\frac{U_s}{x-a_s} - \frac{W_j\bar{\theta}_j(x)}{x-a_j}. \tag{311}$$

If all the U_s are equal to zero then $\bar{\theta}_j(x)$ becomes a constant matrix and, as a result of the condition at the point $x = a_j$, it must be a unit matrix, i.e. we must have the following expansion:

$$\bar{\theta}_j(x) = 1 + \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,m} U_{j_1} \ldots U_{j_\nu} N_j(a_{j_1} \ldots a_{j_\nu};\ x). \tag{312}$$

All coefficients in this expansion must be regular at the point a_j and must vanish at that point since the sum of the series, for any U_s, should become a unit matrix at the point $x = a_j$. Substituting the expansions (310) and (312) in the equation (311) we obtain the following equation:

$$N_j(a_{j_1}, \ldots, a_{j_\nu};\ x) = \int_{a_j}^{x} \left[\frac{N_j(a_{j_1}, \ldots, a_{j_{\nu-1}};\ x)}{x - a_{j_\nu}} - \right.$$

$$\left. - \frac{1}{x - a_j} \sum_{k=1}^{\nu} J_j(a_{j_1}, \ldots, a_{j_k})\, N_j(a_{j_{k+1}} \ldots a_{j_\nu};\ x) \right] dx, \tag{313}$$

and, in particular

$$N_j(a_{j_1};\ x) = \int_{a_j}^{x} \left[\frac{1}{x - a_{j_1}} - \frac{J_j(a_{j_1})}{x - a_{j_1}} \right] dx.$$

Owing to the regularity of the left-hand side, this latter equation shows that

$$J_j(a_{j_1}) = \begin{cases} 1 & \text{when } j_1 = j, \\ 0 & \text{when } j_1 \neq j. \end{cases} \tag{314}$$

Let us consider the equation (313) when $\nu = 2$:

$$N_j(a_{j_1}, a_{j_2};\ x) = \int_{a_j}^{x} \left\{ \frac{N_j(a_{j_1};\ x)}{x - a_{j_2}} - \frac{1}{x - a_j} \left[J_j(a_{j_1})\, N(a_{j_2};\ x) + J_j(a_{j_1}, a_{j_2}) \right] \right\} dx.$$

It is given that we must have $N_j(a_j;a_j) = 0$ and, consequently, the first term of the integrand has no pole at the point $x = a_j$. It follows that the second term does not have a pole at that point either and that the square bracket vanishes when $x = a_j$; therefore all the coefficients $J_j(a_{j_1}, a_{j_2})$ must be zero. Similarly, considering the equation (313) when $\nu = 3$ we can see that all the coefficients $J(a_{j_1}, a_{j_2}, a_{j_3})$ are zero etc. As a result of this and (314), the expansion (310) becomes the simple equation $W_j' = U_j$ and we have the following representation for the solution canonical at the point $x = a_j$:

$$\theta_j(x) = (x - a_j)^{U_j} \bar{\theta}_j(x). \tag{315}$$

Formula (313) enables us to evaluate successively the coefficients in the expansion (312). Bearing in mind that

$$J_j(a_{j_1}) = \begin{cases} 1 & \text{when } j_1 = j \\ 0 & \text{when } j_1 \neq j; \end{cases}$$

$$J_j(a_{j_1}, \ldots, a_{j_\nu}) = 0 \qquad \text{when } \nu \geqslant 2,$$

we have:

$$N_j(a_{j_1}; x) = \int_{a_j}^{x} \left[\frac{1}{x - a_{j_1}} - \frac{\delta_{j_1 j}}{x - a_j} \right] dx,$$

$$N_j(a_{j_1}, \ldots, a_{j_\nu}; x) = \int_{a_j}^{x} \left[\frac{N_j(a_{j_1} \ldots a_{j_{\nu-1}}; x)}{x - a_{j_\nu}} - \frac{\delta_{j_1 j} N_j(a_{j_2} \ldots a_{j_\nu}; x)}{x - a_j} \right] dx,$$

where $\delta_{pq} = 1$ when $p = q$, and $\delta_{pq} = 0$ when $p \neq q$.

By describing a circuit round the point a_j the solution (315) acquires a factor $e^{2\pi i U_j}$ on the left. Any other solution, as we know, will have an integral matrix, similar to the matrix $e^{2\pi i U_j}$, i.e. *by describing a circuit round the singularity a_j any solution of the system acquires on the left a factor which is a matrix similar to the matrix* $e^{2\pi i U_j}$.

Let us now return to the formula (315). The second factor, as we said before, is regular at the point $x = a_j$. The inverse matrix

$$\overline{\theta}_j(x)^{-1}$$

will, therefore, also be regular at the point $x = a_j$ for the determinant of the matrix $\overline{\theta}_j(x)$ is equal to unity at the point $x = a_j$. In general, if any solution $Y(x)$ can be represented in the neighbourhood of the point a_j in the form:

$$Y(x) = (x - a_j)^{W_j''} \overline{Y}(x),$$

where the matrix $\overline{Y}(x)$ is regular at the point a_j and its determinant at that point is not zero, then the matrix W_j'' is said to be an exponential matrix of the given solution. It can be shown that this matrix is determined in a unique way for a given solution when the matrices U_s are close to zero. In particular, for the solution, canonical at the point a_j this will be the matrix U_j itself, and, generally, for any solution, it will be a matrix similar to the matrix U_j.

Note. In all the above arguments we have used the fact that the representation of a function of a matrix in the form of a power series of these matrices is unique. This uniqueness theorem forms the basis of the method for the comparison of coefficients; we used this method above by substituting a series with unknown coefficients into both sides of the equation and comparing the coefficients of similar terms. The hypothesis that the sum of a power series of a matrix U_s is a single-valued function of x near $x = a_j$ and that all the coefficients of this series must be single-valued, is also based on this theorem of uniqueness.

We said earlier in [94] that the uniqueness theorem holds if the sums of the power series coincide for matrices of every order. In all our arguments the order of the matrices was immaterial and it was therefore permissible to use the uniqueness theorem.

125. The connection with regular solutions of Fuchs's type. Let us now consider the canonical solution at the singularity $x = a_j$:

$$\theta_j(x) = (x - a_j)^{U_j} \overline{\theta}_j(x).$$

For the sake of simplicity we assume that the order of the matrix $n = 2$, i.e. that we have a system of two equations with two unknown functions. Let S_j be a matrix which converts U_j to the diagonal form:

$$S_j U_j S_j^{-1} = [\varrho_1, \varrho_2].$$

Consider the integral matrix:

$$Z_j(x) = S_j \theta_j(x) = (x - a_j)^{S_j U_j S_j^{-1}} S_j \bar{\theta}_j(x),$$

or

$$Z_j(x) = (x - a_j)^{[\varrho_1, \varrho_2]} \bar{Z}_j(x),$$

where

$$\bar{Z}_j(x) = S_j \bar{\theta}_j(x)$$

is regular at the point $x = a_j$. Denoting by $\overline{Z}_{pq}^{(j)}(x)$ the elements of this latter matrix:

$$\bar{Z}_j(x) = \left\| \begin{matrix} \overline{Z}_{11}^{(j)}(x), & \overline{Z}_{12}^{(j)}(x) \\ \overline{Z}_{21}^{(j)}(x), & \overline{Z}_{22}^{(j)}(x) \end{matrix} \right\|,$$

where $\overline{Z}_{pq}^{(j)}(x)$ are functions which are regular when $x = a_j$ and bearing in mind that

$$(x - a_j)^{[\varrho_1, \varrho_2]} = \left\| \begin{matrix} (x - a_j)^{\varrho_1}, & 0 \\ 0, & (x - a_j)^{\varrho_2} \end{matrix} \right\|,$$

we have:

$$Z_j(x) = \left\| \begin{matrix} (x - a_j)^{\varrho_1}, & 0 \\ 0, & (x - a_j)^{\varrho_2} \end{matrix} \right\| \cdot \left\| \begin{matrix} \overline{Z}_{11}^{(j)}(x), & \overline{Z}_{12}^{(j)}(x) \\ \overline{Z}_{21}^{(j)}(x), & \overline{Z}_{22}^{(j)}(x) \end{matrix} \right\|$$

$$= \left\| \begin{matrix} (x - a_j)^{\varrho_1} \overline{Z}_{11}^{(j)}(x), & (x - a_j)^{\varrho_1} \overline{Z}_{12}^{(j)}(x) \\ (x - a_j)^{\varrho_2} \overline{Z}_{21}^{(j)}(x), & (x - a_j)^{\varrho_2} \overline{Z}_{22}^{(j)}(x) \end{matrix} \right\|.$$

Every line of this matrix contains a solution of the above system [120]. We thus have two solutions of the system which have the same form as the solutions of one regular equation in the theorem of Fuchs [99]:

$$Y_{11}(x) = (x - a_j)^{\varrho_1} \overline{Z}_{11}^{(j)}(x); \quad Y_{12}(x) = (x - a_j)^{\varrho_1} \overline{Z}_{12}^{(j)}(x);$$

$$Y_{21}(x) = (x - a_j)^{\varrho_2} \overline{Z}_{21}^{(j)}(x); \quad Y_{22}(x) = (x - a_j)^{\varrho_2} \overline{Z}_{22}^{(j)}(x).$$

In these formulae the first subscript of $Y(x)$ gives the number of the solution and the second the number of the function. Notice also that from the definition of $\bar{Z}_j(x)$ and the fact that $\bar{\theta}_j(a_j) = 1$ we have

$$\bar{Z}_j(a_j) = \left\| \begin{matrix} \overline{Z}_{11}^{(j)}(a_j), & \overline{Z}_{12}^{(j)}(a_j) \\ \overline{Z}_{21}^{(j)}(a_j), & \overline{Z}_{22}^{(j)}(a_j) \end{matrix} \right\| = S_j,$$

where S_j is a matrix, the determinant of which is not zero. The number $\overline{Z}_{pq}^{(j)}(a_j)$ must be the constant term in the expansion of $Z_{pq}^{(j)}(x)$ into the Taylor's series in powers of $(x - a_j)$.

The numbers ϱ_1 and ϱ_2 which in [98] were the zeros of the determining equation are, in this case, determined from the characteristic equation of the matrix U_j. In the works of I. A. Lappo-Danilevski the integral matrix $\theta_j(x)$ is known not as the canonical matrix but as the metacanonical matrix at the singularity $x = a_j$. In this terminology the matrix $Z_j(x)$ can be called canonical at the point $x = a_j$.

126. The case of the arbitrary U_s. The formula (297) in [123] determines an exponential substitution W_j for an integral matrix $Y(b; x)$ in the form of a power series of U_s which converges only when the U_s' are close to zero matrices. Similarly, formula (312) in [124] gives the analogous representation for the regular factor of a canonical matrix $\theta_s(x)$. We shall now examine this representation for arbitrary matrices U_s.

By definition we have for the U_s which are close to a zero matrix [123]

$$W_j = \frac{1}{2\pi i} \log V_j = \frac{1}{2\pi i} \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} (V_j - 1)^{\nu}.$$

Let us denote by $\varrho_1, \varrho_2, \ldots, \varrho_n$ the characteristic zeros of the matrix U_j. We know from [124] that the matrix V_j is similar to the matrix $e^{2\pi i U_j}$ and therefore the characteristic zeros of the matrix V_j will be:

$$\eta_1 = e^{2\pi i \varrho_1}; \quad \eta_2 = e^{2\pi i \varrho_2}; \ldots; \quad \eta_n = e^{2\pi i \varrho_n}.$$

If we suppose that the η_k are all different and use Sylvester's formula we can write:

$$W_j = \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{(V_j - \eta_1)\ldots(V_j - \eta_{k-1})(V_j - \eta_{k+1})\ldots(V_j - \eta_n)}{(\eta_k - \eta_1)\ldots(\eta_k - \eta_{k-1})(\eta_k - \eta_{k+1})\ldots(\eta_k - \eta_n)} \log \eta_k.$$

In future, for the sake of simplicity, we shall only consider the case when $n = 2$. Replacing η_k by the expression containing ϱ_k we have:

$$W_j = \frac{V_j - e^{2\pi i \varrho_2}}{e^{2\pi i \varrho_1} - e^{2\pi i \varrho_2}} \varrho_1 + \frac{V_j - e^{2\pi i \varrho_1}}{e^{2\pi i \varrho_2} - e^{2\pi i \varrho_1}} \varrho_2,$$

or

$$W_j = \frac{e^{2\pi i \varrho_2} \varrho_1 - e^{2\pi i \varrho_1} \varrho_2}{e^{2\pi i \varrho_2} - e^{2\pi i \varrho_1}} + \frac{\varrho_2 - \varrho_1}{e^{2\pi i \varrho_2} - e^{2\pi i \varrho_1}} V_j. \tag{316}$$

When $\varrho_1 = \varrho_2$ this formula becomes

$$W_j = \left(\varrho_1 - \frac{1}{2\pi i}\right) + \frac{1}{2\pi i e^{2\pi i \varrho_1}} V_j. \tag{317}$$

Above we obtained a representation of V_j in the form of a power series in U_s for any U_s. Similarly formula (316) gives a representation of W_j for any U_s. This formula becomes devoid of meaning when ϱ_1 and ϱ_2 differ by an integer other than zero, for then the denominator on the right-hand side of (316) vanishes while the numerators are other than zero. Thus for W_j as functions of U_s, those matrices U_j will be singularities, the characteristic zeros of which differ by an integer other than zero. With regard to the remaining matrices

U_s the function W_j has no other singularities. The existence of these singularities causes the series (297) to converge only when the U_s's are close to zero matrices.

We will show how the series (297) can be used for obtaining W_j in the form of a quotient of two power series which converge for any U_s. Let us construct a numerical function of U_j, i.e. a function which for a given U_j has a definite numerical value:

$$\Delta(U_j) = e^{-\pi i(\varrho_1+\varrho_2)} \frac{e^{2\pi i\varrho_1} - e^{2\pi i\varrho_2}}{2\pi i(\varrho_1 - \varrho_2)} = \frac{\sin\pi(\varrho_1 - \varrho_2)}{\pi(\varrho_1 - \varrho_2)}. \tag{318}$$

We can represent it in the form of a power series which converges for any ϱ_1 and ϱ_2:

$$\Delta(U_j) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!} \pi^{2\nu} (\varrho_1 - \varrho_2)^{2\nu}. \tag{319}$$

Denoting by $\{U_j\}_{pq}$ the elements of the matrix U_j we can write the quadratic equation which is satisfied by ϱ_1 and ϱ_2:

$$\begin{vmatrix} \{U_j\}_{11} - \varrho, & \{U_j\}_{12} \\ \{U_j\}_{21}, & \{U_j\}_{22} - \varrho \end{vmatrix} = 0.$$

We have further:

$$(\varrho_1 - \varrho_2)^2 = (\varrho_1 + \varrho_2)^2 - 4\varrho_1\varrho_2,$$

and, bearing in mind the property of the sum and product of the zeros of a quadratic equation, we obtain an for expression $(\varrho_1 - \varrho_2)^2$ in terms of the elements of the matrix U_j:

$$(\varrho_1 - \varrho_2)^2 = (\{U_j\}_{11} + \{U_j\}_{22})^2 - 4(\{U_j\}_{11}\{U_j\}_{22} - \{U_j\}_{12}\{U_j\}_{21}).$$

Substituting in (319) we obtain an expression $\Delta(U_j)$ in terms of the elements of the matrix U_j:

$$\Delta(U_j) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!} \pi^{2\nu} [(\{U_j\}_{11} + \{U_j\}_{22})^2 - 4(\{U_j\}_{11}\{U_j\}_{22} - \{U_j\}_{12}\{U_j\}_{21})]^\nu,$$

and this series converges for every U_j, i.e. it is an integral function of the elements of the matrix U_j.

For the sake of briefness we denote the terms of the above sum by $\delta_\nu(U_j)$:

$$\Delta(U_j) = \sum_{\nu=0}^{\infty} \delta_\nu(U_j), \tag{320}$$

where $\delta_0(U_j) = 1$ and, when $\nu > 0$, $\delta_\nu(U_j)$ is a homogeneous polynomial of degree ν in the elements of U_j. It follows from the formulae (316) and (318) that the elements of the product $\Delta(U_j)W_j$ are integral functions of the elements of U_j and, in general, they are integral functions of the elements of all the matrices U_s. This function can be expanded into homogeneous polynomials of the elements of U_s [83]. Bearing in mind the expansions (297) and (320) we can write the expansion into homogeneous polynomials as follows:

$$\Delta(U_j) W_j = \sum_{\nu=1}^{\infty} \sum_{s=1}^{\nu} \left(\sum_{j_1,\ldots,j_s}^{1,\ldots,m} U_{j_1} \ldots U_{j_s} \delta_{\nu-s}(U_j) Q_j(a_{j_1} \ldots a_{j_s}; b) \right)$$

The above series converges for every U_s. We thus obtain an expression for W_j in the form of a quotient of two integral functions of the elements of U_s:

$$W_j = \frac{\sum_{\nu=1}^{\infty} \sum_{s=1}^{\nu} \left(\sum_{j_1,\ldots,j_s}^{1,\ldots,m} U_{j_1} \ldots U_{j_s}\, \delta_{\nu-s}(U_j)\, Q_j(a_{j_1} \ldots a_{j_s}; b) \right)}{\sum_{\nu=0}^{\infty} \delta_\nu (U_j)}. \tag{321}$$

Notice that the numerical terms in the series in the denominator depend only on the elements of the matrix U_j. Arguing in the same way as before we can show that the products

$$\Delta(U_j)\,(x-a_j)^{W_j} \quad \text{and} \quad \Delta(U_j)\,(x-a_j)^{-W_j}$$

are integral functions of the elements of U_s. It follows from the formula (306) that

$$\Delta(U_j)\,\overline{Y}^{(j)}(b\,;\,x)^{-1} = Y(b\,;\,x)^{-1}\,\Delta(U_j)\,(x-a_j)^{W_j}.$$

The matrices $Y(b;x)$ and $Y(b;x)^{-1}$, as we know, are integral functions of the matrices U_s and, consequently, the product $\Delta(U_j)\,\overline{Y}^{(j)}(b;x)^{-1}$ is an integral function of the elements of U_s. The canonical matrix $\theta_j(x)$ can be written in the form [124]:

$$\theta_j(x) = \overline{Y}^{(j)}(b\,;\,a_j)^{-1}\,Y_j(b\,;\,x),$$

and, consequently, $\Delta(U_j)\,\theta_j(x)$ is an integral function of the elements of U_s. The same can be said about the product:

$$\Delta(U_j)\,\bar{\theta}_j(x) = (x-a_j)^{-U_j}\,\Delta(U_j)\,\theta_j(x),$$

since $(x-a_j)^{-U_j}$ is an integral function of U_j. Using the expansion (312) we can also write the canonical matrix $\theta_j(x)$ as a quotient of two integral functions of the elements of U_s:

$$\theta_j(x) = \frac{(x-a_j)^{U_j} \sum_{\nu=0}^{\infty} \sum_{s=0}^{\nu} \left(\sum_{j_1,\ldots,j_s}^{1,\ldots,m} U_{j_1} \ldots U_{j_s}\, \delta_{\nu-s}(U_j)\, N_j(a_{j_1} \ldots a_{j_s}\,;\,x) \right)}{\sum_{\nu=0}^{\infty} \delta_\nu (U_j)}. \tag{322}$$

Notice that in all the above formulae, $\Delta(U_j)$ can commute with every matrix. The numerators in the formulae (321) and (322) contain series the terms of which are matrices which because of the factors U_{jk} and the numerical factor $\delta_{\nu-s}(U_j)$ depend on the elements of U_s.

The formulae (312) and (322) give the canonical matrix in the form of a power series or as a quotient of two power series, in terms of the elements of the matrix U_s. At the same time the coefficients $N_j(a_{j_1}, \ldots, a_{j_s}; x)$ depend on x. We can, conversely, write $\theta_j(x)$ in the form of a Taylor's series in powers of $(x-a_j)$. The coefficients of this series will also depend on the elements of U_s. This series will converge in the circle $|x-a_j| < R$ which contains no singularities except $x = a_j$.

We had the equation (311) for $\bar{\theta}_j(x)$ and we have shown that $W_j' = U_j$, i.e.

$$\frac{d\bar{\theta}_j(x)}{dx} = \bar{\theta}_j(x) \sum_{s=1}^{n} \frac{U_s}{x - a_s} - \frac{U_j \bar{\theta}_j(x)}{x - a_j}.$$

Substituting this in the equation

$$\bar{\theta}_j(x) = 1 + \sum_{p=1}^{\infty} A_j^{(p)} (x - a_j)^p, \tag{323}$$

where $A_j^{(p)}$ are the unknown matrices which are independent of x, and equating the coefficients of equal powers of $(x - a_j)$ we obtain an equation for the successive determination of the matrix $A_j^{(p)}$:

$$U_j A_j^{(p)} + pA_j^{(p)} - A_j^{(p)} U_j = - \sum_{h \neq j} \sum_{q=0}^{p-1} \frac{A_j^{(q)} U_h}{(a_h - a_j)^{p-q}} \quad (p = 1, 2, \ldots). \tag{324}$$

We have already met similar systems in [121]. We are not giving here the solution of the equation (324) or the proof of the convergence of the series (323). We can use for this the same method of proof as in [98]. Notice only that the product $\Delta(U_j)A_j^{(p)}$ is an integral function of the elements of the matrix U_s and that the following formula holds for this product:

$$\Delta(U_j) A_j^{(p)} = \sum_{\nu=0}^{\infty} \left[\sum_{k=0}^{\nu} \frac{1}{p^{k+1}} \delta_{\nu-k}(U_j) \sum_{\lambda=0}^{k} \frac{(-1)^k k!}{\lambda! (k-\lambda)!} U_j^{\lambda} T_p U_j^{k-\lambda} \right].$$

where T_p denotes the right-hand side of the equation (313).

127. Expansion in the neighbourhood of an irregular singularity. We shall now consider a system of linear equations, the coefficients of which have a pole of any order at the point $x = 0$ and, for the sake of simplicity, we assume that the coefficients of this system are quotients obtained by dividing a certain polynomial by a whole positive power of the variable x. Using the matrix method of notation we can write this system as follows:

$$\frac{dY}{dx} = Y \sum_{p=-s}^{t} T_p x^p, \tag{325}$$

where T_p are the given matrices. The point $x = 0$ will, in general, be an irregular singularity of the above system but we can nevertheless apply the method of successive approximations; we thus obtain the solution in a finite form which will hold for every analytic continuation with respect to x. This solution, as always, will be given as a power series of the matrices T_p which occur in the coefficients of the system. Let us take a point b, other than $x = 0$ and construct the solution $Y(b; x)$ which becomes a unit matrix when $x = b$. For this solution we can write the usual integral equation

$$Y(b\,;\,x) = 1 + \int_b^x Y(b\,;\,x) \sum_{p=-s}^{t} T_p x^p \, dx.$$

Assuming that $Y_0 = 1$ and

$$Y_n(x) = 1 + \int_b^x Y_{n-1}(x) \sum_{p=-s}^{t} T_p(x)^p \, dx, \tag{326}$$

we have

$$Y(b\,;\,x) = Y_0 + [Y_1(x) - Y_0] + [Y_2(x) - Y_1(x)] + \dots$$

Writing, for brevity:

$$Z_\nu(x) = Y_\nu(x) - Y_{\nu-1}(x) \qquad (Z_0 = 1),$$

we can, from (326), write

$$Z_\nu(x) = \int_b^x Z_{\nu-1}(x) \sum_{p=-s}^{t} T_p x^p \, dx. \tag{327}$$

Let us introduce functions of x which are determined by the following recurrent relationships:

$$L_{p_1}(b\,;\,x) = \int_b^x x^{p_1} \, dx\,; \qquad L_{p_1,\dots,p_\nu}(b\,;\,x) = \int_b^x L_{p_1,\dots,p_{\nu-1}}(b\,;\,x)\, x^{p_\nu} \, dx, \tag{328}$$

On evaluating by the method of successive approximations we have, from (327):

$$Z_1(x) = \int_b^x \mathrm{I} \sum_{p_1=-s}^{t} T_{p_1} x^{p_1} \, dx = \sum_{p_1=-s}^{t} T_{p_1} L_{p_1}(b\,;\,x),$$

$$Z_2(x) = \int_b^x \sum_{p_1=-s}^{t} T_{p_1} L_{p_1}(b\,;\,x) \sum_{p_2=-s}^{t} T_{p_2} x^{p_2} \, dx = \sum_{p_1,p_2=-s}^{t} T_{p_1} T_{p_2} L_{p_1 p_2}(b\,;\,x),$$

and in general

$$Z_\nu(x) = \sum_{p_1,\dots,p_\nu=-s}^{t} T_{p_1} \dots T_{p_\nu} L_{p_1 \dots p_\nu}(b\,;\,x).$$

Thus the required solution is obtained as a power series of matrices

$$Y(b\,;\,x) = 1 + \sum_{\nu=1}^{\infty} \sum_{p_1,\dots,p_\nu=-s}^{t} T_{p_1} \dots T_{p_\nu} L_{p_1 \dots p_\nu}(b\,;\,x). \tag{329}$$

We can, as in [122], prove that the above series converges absolutely and uniformly and that it gives the required solution of the system. In this case we can perform the squaring shown in the formulae (328) and we can therefore write the coefficients of the series (329) in a definite form.

Instead of the functions (328) we shall consider, first of all, the above squares, but only after using a different method for determining the arbitrary constant

$$M_{p_1}(x) = \int^x x^{p_1} \, dx\,; \dots;$$

$$M_{p_1 \dots p_\nu}(x) = \int^x M_{p_1 \dots p_{\nu-1}}(x)\, x^{p_\nu} \, dx.$$

We shall find the arbitrary constants in these equations as follows: when $p_1 + \ldots + p_\nu + \nu \neq 0$, the function $M_{p_1 \ldots p_\nu}(x)$ must be of the form:

$$M_{p_1 \ldots p_\nu}(x) = x^{p_1 + \ldots + p_\nu + \nu} \sum_{\mu=0}^{\nu} a^{(\mu)}_{p_1 \ldots p_\nu} \log^\mu x, \tag{330}$$

where $a^{(\mu)}_{p_1 \ldots p_\nu}$ are numerical coefficients. If, however, $p_1 + \ldots + p_\nu + \nu = 0$ then the constant of integration will remain arbitrary. We will show, first of all, that this determination of the arbitrary constant is possible. When $\nu = 1$ we have:

$$M_{p_1}(x) = \int^x x^{p_1} \, dx = \begin{cases} x^{p_1+1} \cdot \dfrac{1}{p_1 + 1}, & \text{when } p_1 + 1 \neq 0 \\ a^{(0)}_{p_1} + \log x, & \text{when } p_1 + 1 = 0, \end{cases}$$

where $a^{(0)}_{p_1}$ is the arbitrary constant. We shall now assume that formula (330) holds for every $M_{p_1 \ldots p_\lambda}(x)$, when $\lambda < \nu$, and investigate the function $M_{p_1 \ldots p_{\nu+1}}(x)$:

$$M_{p_1 \ldots p_{\nu+1}}(x) = \int^x x^{p_1 + \ldots + p_\nu + \nu} \sum_{\mu=0}^{\nu} a^{(\mu)}_{p_1 \ldots p_\nu} \log^\mu x \cdot x^{p_{\nu+1}} \, dx.$$

There are two cases. When $p_1 + \ldots + p_{\nu+1} + \nu + 1 \neq 0$, integrating by parts we have:

$$M_{p_1 \ldots p_{\nu+1}}(x) = \frac{x^{p_1 + \ldots + p_{\nu+1} \nu + 1}}{p_1 + \ldots + p_{\nu+1} + \nu + 1} \sum_{\mu=1}^{\nu} a^{(\mu)}_{p_1 \ldots p_\nu} \log^\mu x - \\ - \int^x \frac{x^{p_1 + \ldots + p_{\nu+1} + \nu}}{p_1 + \ldots + p_{\nu+1} + \nu + 1} \sum_{\mu=1}^{\nu} \mu a^{(\mu)}_{p_1 \ldots p_\nu} \log^{\mu-1} x \, dx.$$

Continuing to integrate by parts we finally obtain the following expression:

$$M_{p_1 \ldots p_{\nu+1}}(x) = x^{p_1 + \ldots + p_{\nu+1} + \nu + 1} \sum_{\mu=1}^{\nu+1} a^{(\mu)}_{p_1 \ldots p_{\nu+1}} \log^\mu x,$$

where the coefficients $a^{(\mu)}_{p_1 \ldots p_{\nu+1}}$ can be expressed linearly in terms of the $a^{(\mu)}_{p_1 \ldots p_\nu}$ and will contain no other arbitrary constants except the constants in $a^{(\mu)}_{p_1 \ldots p_\nu}$.

If, however, $p_1 + \ldots + p_{\nu+1} + \nu + 1 = 0$ then

$$x^{p_1 + \ldots + p_\nu + \nu} x^{p_{\nu+1}} = \frac{1}{x};$$

and we have

$$M_{p_1 \ldots p_{\nu+1}}(x) = a^{(0)}_{p_1 \ldots p_{\nu+1}} + \sum_{\mu=0}^{\nu} \frac{a^{(\mu)}_{p_1 \ldots p_\nu}}{\mu + 1} \log^{\mu+1} x = \sum_{\mu=0}^{\nu+1} a^{(\mu)}_{p_1 \ldots p_{\nu+1}} \log^\mu x$$

where $a^{(0)}_{p_1 \ldots p_{\nu+1}}$ is a new arbitrary constant. We have thus shown that it is possible to determine the constants so that formula (330) holds. It also follows directly from the above that the arbitrariness of the coefficients is due to the arbitrary choice of the coefficient $a^{(0)}_{p_1 \ldots p_\nu}$ when $p_1 + \ldots + p_\nu + \nu = 0$.

We shall now write down relationships which make it possible to evaluate the $a^{(\mu)}_{p_1 \dots p_\nu}$ successively. When $\nu = 1$ our earlier calculations give:

$$a^{(0)}_{p_1} = \begin{cases} \dfrac{1}{p_1+1} \text{ when } p_1+1 \neq 0\,, \\ \text{arbitrary when } p_1+1=0\,, \end{cases} \qquad a^{(1)}_{p_1} = \begin{cases} 0 \text{ when } p_1+1 \neq 0\,, \\ 1 \text{ when } p_1+1=0\,. \end{cases}$$

It also follows from the definition of $M_{p_1 \dots p_\nu}(x)$ that :

$$\frac{\mathrm{d}}{\mathrm{d}x} M_{p_1 \dots p_\nu}(x) = M_{p_1 \dots p_{\nu-1}}(x)\, x^{p_\nu}$$

and from (330)

$$(p_1 + \dots + p_\nu + \nu) \sum_{\mu=0}^{\nu} a^{(\mu)}_{p_1 \dots p_\nu} \log^\mu x + \sum_{\mu=1}^{\nu} \mu a^{(\mu)}_{p_1 \dots p_\nu} \log^{\mu-1} x = \\ = \sum_{\mu=0}^{\nu-1} a^{(\mu)}_{p_1 \dots p_{\nu-1}} \log^\mu x,$$

whence

$$(p_1 + \dots + p_\nu + \nu)\, a^{(\nu)}_{p_1 \dots p_\nu} = 0\,,$$

$$(p_1 + \dots + p_\nu + \nu)\, a^{(\mu)}_{p_1 \dots p_\nu} + (\mu+1)\, a^{(\mu+1)}_{p_1 \dots p_\nu} = a^{(\mu)}_{p_1 \dots p_{\nu-1}} \\ (\mu = \nu-1,\ \nu-2,\ \dots,\ 1,\ 0).$$

Consider first the case when $p_1 + \dots + p_\nu + \nu \neq 0$. We then have:

$$a^{(\nu)}_{p_1 \dots p_\nu} = 0\,; \quad a^{(\mu)}_{p_1 \dots p_\nu} = \frac{1}{p_1 + \dots + p_\nu + \nu} \left[a^{(\mu)}_{p_1 \dots p_{\nu-1}} - (\mu+1)\, a^{(\mu+1)}_{p_1 \dots p_\nu} \right].$$

The successive application of this latter formula when $\mu = \nu-1,\ \nu-2,\ \dots$ gives:

$$a^{(\nu-1)}_{p_1 \dots p_\nu} = \frac{1}{p_1 + \dots + p_\nu + \nu}\, a^{(\nu-1)}_{p_1 \dots p_{\nu-1}}\,,$$

$$a^{(\nu-2)}_{p_1 \dots p_\nu} = \frac{1}{p_1 + \dots + p_\nu + \nu} \left[a^{(\nu-2)}_{p_1 \dots p_{\nu-1}} - \frac{\nu-1}{p_1 + \dots + p_\nu + \nu}\, a^{(\nu-1)}_{p_1 \dots p_{\nu-1}} \right],$$

and, in general

$$a^{(\mu)}_{p_1 \dots p_\nu} = \frac{1}{p_1 + \dots + p_\nu + \nu} \left[a^{(\mu)}_{p_1 \dots p_{\nu-1}} - \frac{\mu+1}{p_1 + \dots + p_\nu + \nu}\, a^{(\mu+1)}_{p_1 \dots p_{\nu-1}} + \right. \\ + \frac{(\mu+1)(\mu+2)}{(p_1 + \dots + p_\nu + \nu)^2}\, a^{(\mu+2)}_{p_1 \dots p_{\nu-1}} - \dots + \\ \left. + (-1)^{\nu-\mu-1} \frac{(\mu+1) \dots (\nu-1)}{(p_1 + \dots + p_\nu + \nu)^{\nu-\mu-1}}\, a^{(\nu-1)}_{p_1 \dots p_{\nu-1}} \right].$$

When $p_1 + \dots + p_\nu + \nu = 0$ the above formula gives:

$$a^{(\mu+1)}_{p_1 \dots p_\nu} = \frac{1}{\mu+1}\, a^{(\mu)}_{p_1 \dots p_{\nu-1}} \qquad (\mu = 0,\ 1, \dots,\ \nu-1)\,,$$

and $a^{(0)}_{p_1 \dots p_\nu}$ remains arbitrary. Summarising all that has been said above we obtain the following formulae for the determination of the coefficients a:

$$
\left.
\begin{aligned}
a^{(0)}_{p_1} &= \frac{1}{p_1 + 1} \quad \text{when} \quad p_1 + 1 \neq 0 \\
a^{(1)}_{p_1} &= \begin{cases} 0 & \text{when } p_1 + 1 \neq 0 \\ 1 & \text{when } p_1 + 1 = 0 \end{cases} \\
a^{(\nu)}_{p_1 \dots p_\nu} &= 0 \qquad (p_1 + \dots + p_\nu + \nu \neq 0)
\end{aligned}
\right\}
$$

$$
\left.
\begin{aligned}
a^{(\mu)}_{p_1 \dots p_\nu} = \frac{1}{p_1 + \dots p_\nu + \nu} \Big[& a^{(\mu)}_{p_1 \dots p_{\nu-1}} - \\
& - \frac{\mu + 1}{p_1 + \dots + p_\nu + \nu} a^{(\mu+1)}_{p_1 \dots p_{\nu-1}} \\
& + \frac{(\mu+1)(\mu+2)}{(p_1 + \dots + p_\nu + \nu)^2} a^{(\mu+2)}_{p_1 \dots p_{\nu-1}} - \dots + \\
& + (-1)^{\nu-\mu-1} \frac{(\mu+1) \dots (\nu-1)}{(p_1 + \dots + p_\nu + \nu)^{\nu-\mu-1}} a^{(\nu-1)}_{p_1 \dots p_{\nu-1}} \Big] \\
& (\mu = \nu - 1,\ \nu - 2,\ \dots,\ 1, 0\ ;\ p_1 + \dots + p_\nu + \nu \neq 0)\,, \\
a^{(\mu)}_{p_1 \dots p_\nu} &= a^{(\mu-1)}_{p_1 \dots p_{\nu-1}} \\
& (\mu = \nu, \nu - 1, \dots, 2,\ 1\ ;\ p_1 + \dots + p_\nu + \nu = 0).
\end{aligned}
\right\} \tag{331}
$$

To find an expression for the functions $L_{p_1 \dots p_\nu}(b; x)$ in terms of $M_{p_1 \dots p_\nu}(x)$ certain new functions $M^*_{p_1 \dots p_\nu}(x)$ have to be introduced which can be determined as single-valued functions with the aid of the following successive equations:

$$
\left.
\begin{aligned}
M^*_{p_1}(x) &= -M_{p_1}(x)\,, \\
M^*_{p_1 \dots p_\nu}(x) &= -\sum_{\mu=0}^{\nu-1} M^*_{p_1 \dots p_\mu}(x)\, M_{p_{\mu+1} \dots p_\nu}(x).
\end{aligned}
\right\} \tag{332}
$$

In the above sum in the term corresponding to $\mu = 0$ the factor $M^*_{p_1 \dots p_\mu}(x)$ becomes devoid of meaning when $\mu = 0$ and must be replaced by unity. We will now show that

$$
L_{p_1 \dots p_\nu}(b\ ;\ x) = \sum_{\mu=0}^{\nu} M^*_{p_1 \dots p_\mu}(b)\, M_{p_{\mu+1} \dots p_\nu}(x)\,, \tag{333}
$$

where, when $\mu = 0$, the first factor is replaced by unity and when $\mu = \nu$ the second factor must be replaced by unity.

When $\nu = 1$ this formula is self-explanatory since we have from the evaluation of our functions:

$$
L_{p_1}(b\ ;\ x) = M_{p_1}(x) - M_{p_1}(b) = M_{p_1}(x) + M^*_{p_1}(b).
$$

To prove the validity of the formula (333) for every ν we shall prove that if it is true for ν then it is true for $\nu + 1$.

On supposing that formula (333) is valid for all $L_{p_1 \dots p_\lambda}(x)$, when $\lambda < \nu$, we have:

$$L_{p_1 \dots p_{\nu+1}}(b\,;\,x) = \int_b^x L_{p_1 \dots p_\nu}(b\,;\,x)\, x^{p_{\nu+1}}\,\mathrm{d}x =$$

$$= \int_b^x \sum_{\mu=0}^{\nu} M^*_{p_1 \dots p_\mu}(b)\, M_{p_{\mu+1} \dots p_\nu}(x)^{p_{\nu+1}}\,\mathrm{d}x.$$

But from the definition of $M_{p_1 \dots p_\nu}(x)$:

$$\int_b^x M_{p_{\mu+1} \dots p_\nu}(x)\, x^{p_{\nu+1}}\,\mathrm{d}x = M_{p_{\mu+1} \dots p_{\nu+1}}(x) - M_{p_{\mu+1} \dots p_{\nu+1}}(b)$$

and consequently:

$$L_{p_1 \dots p_{\nu+1}}(b\,;\,x) = \sum_{\mu=0}^{\nu} M^*_{p_1 \dots p_\mu}(b)\left[M_{p_{\mu+1} \dots p_{\nu+1}}(x) - M_{p_{\mu+1} \dots p_{\nu+1}}(b)\right],$$

or, bearing in mind the usual condition for border terms in the above sums and also (332) we can write:

$$L_{p_1 \dots p_{\nu+1}}(b\,;\,x) = \sum_{\mu=0}^{\nu+1} M^*_{p_1 \dots p_\mu}(b)\, M_{p_{\mu+1} \dots p_{\nu+1}}(x),$$

i.e. formula (333) appears to be valid for $L_{p_1 \dots p_{\nu+1}}(b; x)$, also and we can therefore say that this formula will also hold in the general case. It follows directly from (332) that $M^*_{p_1 \dots p_\nu}(x)$ has the same form as $M_{p_1 \dots p_\nu}(x)$ but that coefficients are different

$$M^*_{p_1 \dots p_\nu}(x) = x^{p_1 + \dots + p_\nu + \nu} \sum_{\mu=0}^{\nu} a^{*(\mu)}_{p_1 \dots p_\nu} \log^\mu x. \tag{334}$$

To simplify the construction of the relationships which permit the evaluation of the coefficients $a^{*(\mu)}_{p_1 \dots p_\nu}$, we shall prove the validity of the following formulae for $M^*_{p_1 \dots p_\nu}$:

$$M^*_{p_1}(x) = -\int^x x^{p_1}\,\mathrm{d}x\,;\; M^*_{p_1 \dots p_\nu}(x) = -\int^x x^{p_1} M^*_{p_2 \dots p_\nu}(x)\,\mathrm{d}x. \tag{335}$$

The constants of integration are, in this case, be so chosen that the formulae (334) holds. This will, as before, determine the $a^{*(\mu)}_{p_1 \dots p_\nu}$ when $p_1 + \dots + p_\nu + \nu \neq 0$. We shall consider below the choice of $a^{*(0)}_{p_1 \dots p_\nu}$, when $p_1 + \dots + p_\nu + \nu = 0$. When $\nu = 1$ we have:

$$\frac{\mathrm{d}}{\mathrm{d}x} M^*_{p_1}(x) = -\frac{\mathrm{d}}{\mathrm{d}x} M_{p_1}(x) = -x^{p_1}$$

and, consequently:

$$M^*_{p_1}(x) = -\int^x x^{p_1}\,\mathrm{d}x.$$

Let us now suppose that the equation

$$\frac{\mathrm{d}}{\mathrm{d}x} M^*_{p_1 \dots p_\lambda}(x) = -x^{p_1} M^*_{p_2 \dots p_\lambda}(x) \tag{336}$$

holds for $\lambda < \nu - 1$. We will now show that it is also valid when $\lambda = \nu$. We have from (332):

$$\frac{\mathrm{d}}{\mathrm{d}x} M^*_{p_1 \dots p_\nu}(x) = -\sum_{\mu=0}^{\nu-1} \Big[M_{p_{\mu+1} \dots p_\nu}(x) \frac{\mathrm{d}}{\mathrm{d}x} M^*_{p_1 \dots p_\mu}(x) + \\ + M^*_{p_1 \dots p_\mu}(x) \frac{\mathrm{d}}{\mathrm{d}x} M_{p_{\mu+1} \dots p_\nu}(x) \Big]$$

or from (336) and the definition of $M_{p_1 \dots p_\nu}(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x} M^*_{p_1 \dots p_\nu}(x) = x^{p_1} \sum_{\mu=1}^{\nu-1} M^*_{p_2 \dots p_\mu}(x) M_{p_{\mu+1} \dots p_\nu}(x) - \\ - \sum_{\mu=0}^{\nu-1} M^*_{p_1 \dots p_\mu}(x) M_{p_{\mu+1} \dots p_{\nu-1}}(x) x^{p_\nu}.$$

But according to (332):

$$\sum_{\mu=0}^{\nu-1} M^*_{p_1 \dots p_\mu}(x) M_{p_{\mu+1} \dots p_{\nu-1}}(x) = 0$$

and

$$\sum_{\mu=1}^{\nu-1} M^*_{p_2 \dots p_\mu}(x) M_{p_{\mu+1} \dots p_\nu}(x) = - M^*_{p_2 \dots p_\nu}(x),$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}x} M^*_{p_1 \dots p_\nu}(x) = - x^{p_1} M^*_{p_2 \dots p_\nu}(x),$$

which we wanted to prove. Using the formulae (335) we can introduce relationships for $a^{*(\mu)}_{p_1 \dots p_\nu}$ analogous with those deduced above for $a^{(\mu)}_{p_1 \dots p_\nu}$. The proof will be exactly the same and we shall only give the final result:

$$a^{*(0)}_{p_1} = -\frac{1}{p_1 + 1} \quad \text{when} \quad p_1 + 1 \neq 0$$

$$a^{*(1)}_{p_1} = \begin{cases} 0 & \text{when} \quad p_1 + 1 \neq 0 \\ 1 & \text{when} \quad p_1 + 1 = 0 \end{cases}$$

$$a^{*(\nu)}_{p_1 \dots p} = 0 \quad (p_1 + \dots + p_\nu + \nu \neq 0)$$

$$a^{*(\mu)}_{p_1 \dots p_\nu} = \frac{-1}{p_1 + \dots + p_\nu + \nu} \Big[a^{*(\mu)}_{p_2 \dots p_\nu} - \frac{\mu + 1}{p_1 + \dots + p_\nu + \nu} a^{(\mu+1)}_{p_2 \dots p_\nu} + \\ + \frac{(\mu+1)(\mu+2)}{(p_1 + \dots + p_\nu + \nu)^2} a^{*(\mu+2)}_{p_2 \dots p_\nu} + \dots + \\ + (-1)^{\nu-\mu-1} \frac{(\mu+1) \dots (\nu-1)}{(p_1 + \dots + p_\nu + \nu)^{\nu-\mu-1}} a^{(\nu-1)}_{p_2 \dots p_\nu} \Big] \qquad (337)$$

$$(\mu = \nu - 1, \ \nu - 2, \ \dots, \ 1, 0 \ ; \ p_1 + \dots + p_\nu + \nu \neq 0) \ ;$$

$$a^{*(\mu)}_{p_1 \dots p_\nu} = -\frac{1}{\mu} a^{*(\mu-1)}_{p_2 \dots p_\nu}$$

$$(\mu = \nu, \ \nu - 1, \ \dots, \ 2, \ 1 \ ; \ p_1 + \dots + p_\nu + \nu = 0).$$

From these relationships all the a^*, except $a^{*(0)}_{p_1\dots p_\nu}$, can be determined, when $p_1 + \dots + p_\nu + \nu = 0$. As a result of the single-valued determination of $M^*_{p_1\dots p_\nu}(x)$ from the formulae (332), these latter coefficients can also be expressed in a definite way in terms of the known coefficients a and a^*. To find these expressions we replace $M_{p_1\dots p_\nu}(x)$ by $M^*_{p_1\dots p_\nu}(x)$ in formula (332) using their expressions in the formulae (330) and (334). After substitution in both sides of the equations we obtain polynomials in $\log x$:

$$\sum_{s=0}^{\nu} a^{(s)}_{p_1\dots p_\nu} \log^s x = -\sum_{\mu=0}^{\nu-1} \left(\sum_{s=0}^{\mu} a^{*(s)}_{p_1\dots p_\mu} \log^s x \right) \left(\sum_{s=0}^{\nu-\mu} a^{(s}_{p_{\mu+1}\dots p_\nu} \log^s x \right).$$

Comparing terms which do not contain $\log x$ in this formula we obtain the relationship

$$\sum_{\mu=0}^{\nu} a^{*(0)}_{p_1\dots p_\mu} a^{(0)}_{p_{\mu+1}\dots p_\nu} = 0 \tag{338}$$

or, separating the first and last terms:

$$a^{(0)}_{p_1\dots p_\nu} + \sum_{\mu=1}^{\nu-1} a^{*(0)}_{p_1\dots p_\mu} a^{(0)}_{p_{\mu+1}\dots p_\nu} + a^{*(0)}_{p_1\dots p_\nu} = 0. \tag{339}$$

These latter relationships enable us to determine the $a^{*(0)}_{p_1\dots p_\nu}$ when $p_1 + \dots + p_\nu + \nu = 0$.

Lastly, if we replace $M_{p_1\dots p_\nu}(x)$ and $M^*_{p_1\dots p_\nu}(x)$ in the expression (333) by their expressions from (330) and (334), then we obtain a definite expression for the coefficients of the series (329):

$$L_{p_1\dots p_\nu}(b\,;\,x) = \sum_{\mu=0}^{\nu} b^{p_1+\dots+p_\mu+\mu}\, x^{p_{\mu+1}+\dots+p_\nu+\nu-\mu} \times \\ \times \sum_{\lambda=0}^{\mu} a^{*(\lambda)}_{p_1\dots p_\mu} \log^\lambda b \sum_{k=0}^{\nu-\mu} a^{(k)}_{p_{\mu+1}\dots p_\nu} \log^k x.$$

Introducing these expressions into the formula (326) we obtain the final expression for the solution:

$$Y(b\,;\,x) = \\ = \mathrm{I} + \sum_{\nu=1}^{\infty} \sum_{p_1\dots p_\nu} \sum_{s=1}^{t} T_{p_1\dots p_\nu} \sum_{\mu=0}^{\nu} b^{p_1+\dots+p_\mu+\mu}\, x^{p_{\mu+1}+\dots+p_\nu+\nu-\mu} \times \\ \times \sum_{\lambda=0}^{\mu} a^{*(\lambda)}_{p_1\dots p_\mu} \log^\lambda b \sum_{k=0}^{\nu-\mu} a^{(k)}_{p_{\mu+1}\dots p_\nu} \log^k x. \tag{340}$$

From the above considerations we arrive at to the following theorem.

THEOREM. *The solution of the system* (325), *which becomes a unit matrix when* $x = b$, *can be determined for every analytic continuation with respect to* x *and for any matrix* T_p *given in the form of the series* (337); *the coefficients* a *are determined from the relationships* (331) *and when* $p_1 + \dots + p_\nu + \nu = 0$, $a^{(0)}_{p_1\dots p_\nu}$, *remain arbitrary. The coefficients* a^* *are determined from the relationships* (337) *and* (338).

Formula (340) determines the use of the method of successive approximations in its application to the system (316).

By describing a circuit round $x = 0$ the solution $Y(b; x)$ is multiplied on the left by a constant matrix V. It is not difficult to obtain a formula for the any positive integral power V^m. If m is an integer, either positive or negative, we can obtain V^m if we take the value of $Y(b; x)$ at the point $x = b$, obtained as a result of completing a circuit round the point $x = 0$, $| m |$ times in the positive direction when $m > 0$, and in the negative direction when $m < 0$. After completing this circuit the initial value of $\log x$ becomes $\log x + 2m\pi i$ and we have:

$$V^m = \mathrm{I} + \sum_{\nu=1}^{\infty} \sum_{p_1,\ldots,p_\nu} \sum_{-s}^{t} T_{p_1} \ldots T_{p_\nu} b^{p_1+\ldots+p_\nu+\nu} \sum_{\mu=0}^{\nu} \sum_{\lambda=0}^{\mu} a^{*(\lambda)}_{p_1\ldots p_\mu} \log^\lambda b \times$$

$$\times \sum_{k=0}^{\nu-\mu} a^{(k)}_{p_{\mu+1}\ldots p_\nu} (\log b + 2m\pi i)^k. \qquad (341)$$

The value of $\log b$ in the inside sum is the same as $\log b$ in the outside sum and the coefficients in the above power series are polynomials in $\log b$. It can be shown, but we will not do so here, that the coefficients of all terms containing $\log b$ will cancel each other, so that instead of formula (341) we can use the simpler formula

$$V^m = \mathrm{I} + \sum_{\nu=1}^{\infty} \sum_{p_1,\ldots,p_\nu=-s} T_{p_1} \ldots T_{p_\nu} b^{p_1+\ldots+p_\nu+\nu} \sum_{\mu=0}^{\nu} a^{*(0)}_{p_1\ldots p_k} \times$$

$$\times \sum_{k=0}^{\nu-\mu} a^{(k)}_{p_{\mu+1},\ldots,p_\nu} (2\pi m i)^k. \qquad (342)$$

Let us now return to the fundamental expansion (340). It can readily be seen that the series on the right-hand side can be formally obtained as a product of two series, viz. it can be represented as a product of two power series of matrices as follows:

$$\left[\mathrm{I} + \sum_{\nu=1}^{\infty} \sum_{p_1\ldots p_\nu} \sum_{-s}^{t} T_{p_1} \ldots T_{p_\nu} b^{p_1+\ldots+p_\nu+\nu} \sum_{\mu=0}^{\nu} a^{*(\mu)}_{p_1\ldots p_\nu} \log^\mu b\right] \times$$

$$\times \left[\mathrm{I} + \sum_{\nu=1}^{\infty} \sum_{p_1,\ldots,p_\nu} \sum_{-s}^{t} T_{p_1} \ldots T_{p_\nu} x^{p_1+\ldots+p_\nu+\nu} \sum_{\mu=0}^{\nu} a^{(\mu)}_{p_1\ldots p_\nu} \log^\mu x\right]. \qquad (343)$$

The first factor does not contain x and is a constant matrix. If we cancel this factor, in other words, if we multiply the product (343) on the left by a constant matrix, which is the reciprocal of the first factor, then we are left as a result with the series:

$$\mathrm{I} + \sum_{\nu=1}^{\infty} \sum_{p_1,\ldots,p_\nu} \sum_{-s}^{t} T_{p_1} \ldots T_{p_\nu} x^{p_1+\ldots+p_\nu+\nu} \sum_{\mu=0}^{\nu} a^{(\mu)}_{p_1\ldots p_\nu} \log^\mu x \qquad (344)$$

which should also be a solution of the system (325). The above considerations are only formal in character but it can be proved that the series (344) does, in fact, converge and give a solution of the system when the matrices T_p are close to zero. This solution (344) no longer depends on the choice of the point b at which we approximated the solution to a unit matrix. We are not going to investigate in greater detail the solution given by the series (344). A detailed investigation of the system (325) can be found in the original works of I. A. Lappo-Danilevski.

128. Expansions into uniformly convergent series. The series constructed above were uniformly convergent in a closed domain without singularities. In the neighbourhood of a regular singularity, after the isolation of this singularity, we obtained a Taylor's series, which converged uniformly in the neighbourhood of the singularity. We shall now describe the construction of series which converge uniformly on the real axis in the neighbourhood of an irregular singularity which we place at infinity. This leads us to a consideration of the asymptotic representation. Let us consider a system of two equations of the first order of the form:

$$\frac{dY}{dx} = YT = Y\left(T_0 + \frac{T_1}{x} + \frac{T_2}{x^2} + \dots\right), \tag{345}$$

where T_k are constant matrices. When $T_0 = 0$, then $x = \infty$ is a regular singularity. Assuming that $Y = Y_1 S$, where S is a constant matrix, we obtain a similar system for Y_1, but the coefficients will be $\overline{T}'_k = ST_k S^{-1}$ and we can choose S so that the matrix T'_k is given in canonical form. We shall assume in what follows that in the system (345) the matrix T_0 is already in the canonical form. Let us consider the case when T_0 has a purely diagonal form: $T_0 = [a_1, a_2]$, and the real parts of a_1 and a_2 are different. Without loss of generality we can assume that

$$\mathcal{R}(a_1) > \mathcal{R}(a_2), \tag{346}$$

where $\mathcal{R}(z)$ is the symbol of the real part of the complex number z.

Let us now introduce the elements t_{ik} of the matrix T:

$$t_{ll} = a_l + \sum_{s=1}^{\infty} t_{ll}^{(s)} \frac{1}{x^s}; \quad t_{ik} = \sum_{s=1}^{\infty} t_{ik}^{(s)} \frac{1}{x^s} \qquad (i \neq k) \tag{347}$$

and assume that

$$T = P_0 + P, \tag{348}$$

where P_0 is the diagonal matrix:

$$P_0 = \left[a_1 + t_{11}^{(1)} \frac{1}{x}, \quad a_2 + t_{22}^{(1)} \frac{1}{x}\right]. \tag{349}$$

The elements P_{ik} of the matrix P are:

$$P_{lk} = \sum_{s=1}^{\infty} t_{ik}^{(s)} \frac{1}{x^s} \quad (i \neq k); \qquad P_{ii} = \sum_{s=2}^{\infty} P_{ii}^{(s)} \frac{1}{x^s}. \tag{350}$$

We replace Y by a new unknown matrix Z according to the formula

$$Y = e^{\int^x P_0\,dx} \quad Z = e^{[a_1x + t_{11}^{(1)}\log x - a_1,\ a_2x + t_{22}^{(1)}\log x - a_2]}\,Z. \tag{351}$$

On substituting in (345) we obtain an equation for Z:

$$\frac{dZ}{dx} = ZP_0 - P_0Z + ZP, \tag{352}$$

or, introducing the parameter λ, we have:

$$\frac{dZ}{dx} = ZP_0 - P_0Z + \lambda ZP. \tag{353}$$

We shall try to find a solution of this equation in the form of the series

$$Z = \sum_{m=0}^{\infty} Z_m\lambda^m. \tag{354}$$

Substituting in (353) we obtain:

$$\frac{dZ_m}{dx} = Z_mP_0 - P_0Z_m + Z_{m-1}P \qquad (m = 1, 2, 3, \ldots), \tag{355}$$

or, for the elements $z_{ik}^{(m)}$ of the matrix Z_m:

$$\frac{dz_{ik}^{(m)}}{dx} = e^{a_kx + t_{kk}^{(1)}\log x - a_k}\ z_{ik}^{(m)} - e^{a_ix + t_{ii}^{(1)}\log x - a_i^1}\ z_{ik}^{(m)} + \sum_{s=1}^{2} z_{is}^{(m-1)}\,P_{sk}, \tag{356}$$

and it can be assumed that $Z_0 = 1$. The above equation can easily be solved and, as a result, a formula for the successive evaluation of the $z_{ik}^{(m)}$ will be obtained:

$$z_{ik}^{(m)} = e^{-r_{ik}} \int e^{r_{ik}} \sum_{s=1}^{2} z_{is}^{(m-1)}\,P_{sk}\,dx, \tag{357}$$

where

$$r_{ik} = (a_i - a_k)\,x + (P_{ii}^{(1)} - P_{kk}^{(1)})\log x = a_{ik}x + \beta_{ik}\log x. \tag{358}$$

In formula (357) the interval of integration is (x_0, x) when $i > k$, and (∞, x) when $i < k$, where x_0 is a sufficiently large real value of x, and $x > x_0$. We assume, in any case, that $x_0 > 1$. For the elements z_{ik} of the matrix Z, which must satisfy the equation (352), we obtain the series:

$$\left.\begin{aligned} z_{ik} &= \sum_{m=1}^{\infty} z_{ik}^{(m)} \qquad (i \neq k), \\ z_{ii} &= 1 + \sum_{m=1}^{\infty} z_{ii}^{(m)}. \end{aligned}\right\} \tag{359}$$

If we can prove the uniform convergence of these series in the infinite interval $x_0 \leqslant x < \infty$, then it follows from the formulae (357), that the series composed of the derivatives:

$$\sum_{m=1}^{\infty} \frac{dz_{ik}^{(m)}}{dx} \qquad (i,\ k = 1, 2),$$

converge uniformly in every finite part of the above interval and that the matrix Z satisfies the equation (352) while the matrix Y, given by the formula (351), satisfies the equation (345).

We shall now prove the uniform convergence of the series (350). It follows from (350) that:

$$|p_{ik}| < \frac{a}{x} \quad (i \neq k); \quad |p_{ii}| < \frac{a}{x^2}, \tag{360}$$

where a is a positive constant. Also, using the rule of l'Hôpital it is easy to obtain the inequalities

$$e^{-r'_{ik}} \int e^{r'_{ik}} \frac{a}{x}\, dx < \frac{a_1}{x}; \quad \int \frac{a}{x^2}\, dx < \frac{a_1}{x}, \tag{361}$$

where r'_{ik} is the real part of r_{ik} and a_1 is a positive constant. In these formulae and in the formulae below, the integrals should be understood in the same sense as above. Notice that as x_0 increases the value of the constant a_1 remains unaltered.

We have from (355) and the fact that $Z_0 = 1$:

$$z_{ik}^{(1)} = e^{-r_{ik}} \int e^{r_{ik}} p_{ik}\, dx \quad (i \neq k); \quad z_{ii}^{(1)} = \int p_{ii}\, dx, \tag{362}$$

and from (360) and (361) we obtain:

$$|z_{ik}^{(1)}| < \frac{a_1}{x} \quad (k \neq l); \quad |z_{ii}^{(1)}| < \frac{a_1}{x}. \tag{363}$$

Also from (355), (363) and the condition $x > x_0 > 1$:

$$|z_{ik}^{(2)}| < e^{-r'_{ik}} \int e^{r'_{ik}}\, 2\, \frac{aa_1}{x^2}\, dx < \frac{2a_1}{x_0} e^{-r'_{ik}} \int e^{r_{ik}} \frac{a}{x}\, dx < \frac{2a_1}{x_0} \cdot \frac{a_1}{x},$$

$$|z_{ii}^{(2)}| < \int 2\, \frac{aa_1}{x^2}\, dx < 2a_1 \cdot \frac{a_1}{x},$$

$$|z_{ik}^{(3)}| < 2a_1 e^{-r'_{ik}} \int e^{r'_{ik}}\, 2\, \frac{aa_1}{x^2}\, dx < \frac{(2a_1)^2}{x_0} \cdot \frac{a_1}{x},$$

$$|z_{ii}^{(3)}| < \int \sum_{s=1}^{2} |z_{is}^{(2)}|\, |p_{sk}|\, dx < \int \left(\frac{2a_1}{x_0} \frac{aa_1}{x^2} + 2a_1 \frac{aa_1}{x^3} \right) dx,$$

whence

$$|z_{ii}^{(3)}| < \frac{(2a_1)^2}{x_0} \cdot \frac{a_1}{x}.$$

Further inequalities are of the form:

$$|z_{ik}^{(2m)}| < \frac{(2a_1)^{2m-1}}{x_0^m} \cdot \frac{a_1}{x}; \quad |z_{ii}^{(2m)}| < \frac{(2a_1)^{2m-1}}{x_0^{m-1}} \cdot \frac{a_1}{x},$$

$$|z_{ik}^{(2m+1)}| < \frac{(2a_1)^{2m}}{x^m} \cdot \frac{a_1}{x}; \quad |z_{ii}^{(2m+1)}| < \frac{(2a_1)^{2m}}{x_0^m} \cdot \frac{a_1}{x}. \tag{364}$$

They can readily be proved, as above, by induction from $2m$ to $(2m+1)$ and from $(2m+1)$ to $(2m+2)$.

It follows from these inequalities that if we take $x_0 > (2a_1)^2$ then the series

$$\sum_{m=0}^{\infty} \left| z_{ik}^{(m)} \right|$$

will converge uniformly in the infinite interval $x_0 \leqslant x < \infty$, when either $i \neq k$ or $i = k$.

Hence, bearing in mind the formula (351), we obtain the following solution for the system (345):

$$y_{ik} = e^{a_i x} x^{t_{ii}^{(1)}} z_{ik} \qquad (i,\ k = 2)\,, \tag{365}$$

where, as always, i is the number of the solution and k denotes the number of the function. It follows from the formula (359) and the inequalities (364) that

$$z_{ik} = o\left(\frac{1}{x}\right) (i \neq k)\,; \quad z_{ii} = 1 + o\left(\frac{1}{x}\right), \tag{366}$$

and, from (346), this shows that the solutions (365) are linearly independent.

When substituting the expansions (350) in the integral (357) we obtain the following integral:

$$e^{-\alpha x - \beta \log x} \int_{x_0}^{x} e^{\alpha x + \beta \log x} \frac{dx}{x^n} \quad \text{and} \quad e^{\alpha x + \beta \log x} \int_{\infty}^{x} e^{\alpha x - \beta \log x} \frac{dx}{x^n} \tag{367}$$

where $\mathcal{R}(\alpha) > 0$ and $n \geqslant 1$.

Integrating by parts and assuming that $e^{\beta \log x} = x^{\beta} u$ we can write the asymptotic expansion of these integrals in powers of $1/x$.

Using the expansions (350), the expansions of the integrals (367) into asymptotic series and calculating the upper bounds of the first and last terms of the expansions by evaluating the inequalities (364) more accurately, we obtain the asymptotic expansion of z_{ik} in powers of $1/x$; this, according to (365), will give us the asymptotic expansions of y_{ik}. Let us separate the first terms in the asymptotic expansions of z_{ik}:

We have from (362):

$$z_{ik}^{(1)} = e^{-\alpha_{ik} x - \beta_{ik} \log x} \int_{x_0}^{x} e^{\alpha_{ik} x + \beta_{ik} \log x} \left(\frac{t_{ik}^{(1)}}{x} + \frac{t_{ik}^{(2)}}{x^2} + \frac{\varepsilon_{ik}}{x^2} \right) dx \qquad (i < k)\,,$$

$$z_{ik}^{(1)} = e^{-\alpha_{ik} x + \beta_{ik} \log x} \int_{\infty}^{x} e^{\alpha_{ik} x - \beta_{ik} \log x} \left(\frac{t_{ik}^{(1)}}{x} + \frac{t_{ik}^{(2)}}{x^2} + \frac{\varepsilon_{ik}}{x^2} \right) dx \qquad (i > k)\,,$$

$$z_{ii}^{(1)} = \int_{\infty}^{x} \left(\frac{t_{ii}^{(2)}}{x^2} + \frac{t_{ii}^{(3)}}{x^3} + \frac{\varepsilon_{ii}}{x^2} \right) dx,$$

where ε_{ik} and $\varepsilon_{ll} \to 0$ as $x \to \infty$. Integrating by parts we obtain from this:

$$z_{ik}^{(1)} = \frac{a_{ik}^{(1)}}{x} + \frac{a_{ik}^{(2)}}{x^2} + \frac{\varepsilon'_{ik}}{x^2} \quad (\varepsilon'_{ik} \to 0 \text{ as } x \to \infty ;\ i, k = 1, 2). \tag{368}$$

Substituting these expressions in the formulae (357) when $m = 2$ we have:

$$z_{ik}^{(2)} = e^{-\alpha_{ik} x - \beta_{ik} \log x} \int e^{\alpha_{ik} x + \beta_{ik} \log x} \sum_{s=1}^{2} \left(\frac{a_{is}^{(1)}}{x} + \frac{a_{is}^{(2)}}{x^2} + \frac{\varepsilon_{is}}{x^2} \right) \times$$

$$\times \left(\frac{t_{sk}^{(1)}}{x} + \frac{t_{sk}^{(2)}}{x^2} + \frac{\varepsilon_{sk}}{x^2} \right) dx.$$

When $i \neq k$ this gives the asymptotic representations

$$z_{ik}^{(2)} = \frac{b_{ik}^{(2)}}{x^2} + \frac{\varepsilon''_{ik}}{x^2} \quad \varepsilon''_{ik} \to 0 \qquad (\text{as } x \to \infty)\ (i \neq k). \tag{369}$$

Also

$$z_{ll}^{(2)} = \int_{\infty}^{x} \sum_{s=1}^{2} \left(\frac{a_{is}^{(1)}}{x} + \frac{a_{is}^{(2)}}{x^2} + \frac{\varepsilon'_{is}}{x^2} \right) \left(\frac{t_{sl}^{(1)}}{x} + \frac{t_{sl}^{(2)}}{x^2} + \frac{\varepsilon_{sl}}{x^2} \right) dx,$$

from which it follows that

$$z_{ll}^{(2)} = \frac{b_{ll}^{(1)}}{x} + \frac{b_{ll}^{(2)}}{x^2} + \frac{\varepsilon''_{ll}}{x^2}. \tag{370}$$

From (369) and (370) we have the inequalities:

$$|z_{ik}^{(2)}| < \frac{b}{x^2} ; \quad |z_{ll}^{(2)}| < \frac{b}{x} \qquad (b \text{ is a constant}). \tag{371}$$

From this, (360) and the condition $x > x_0 > 1$ we obtain:

$$|z_{ik}^{(3)}| > 2ab\, e^{-r'_{ik}} \int e^{r'_{ik}} \frac{1}{x^2} dx,$$

$$|z_{ll}^{(3)}| < 2ab \int \frac{1}{x^3} dx.$$

Ineqalities, analogous with (361), can readily be obtained:

$$e^{-r'_{ik}} \int e^{r'_{ik}} \frac{1}{x^2} dx < \frac{b_1}{x^2} ; \quad \int \frac{1}{x^3} dx < \frac{b_1}{x^2} \qquad (b_1 \text{ being a constant}),$$

from which we obtain the inequalities:

$$|z_{ik}^{(3)}| < ab\,(2b_1) \frac{1}{x^2} \qquad (i,\ k = 1, 2).$$

Substituting this in the formula (357) when $m = 4$ and bearing in mind that $x > x_0 > 1$ we have:

$$|z_{ik}^{(4)}| < \frac{abb_1 2^2}{x_0} e^{-r'_{ik}} \int e^{r'_{ik}} \frac{1}{x^2} dx < ab \frac{(2b_1)^2}{x_0} \cdot \frac{1}{x^2},$$

$$|z_{ii}^{(4)}| < abb_1 2^2 \int \frac{1}{x^3} dx < ab (2b_1)^2 \cdot \frac{1}{x^2}.$$

Continuing in this way we obtain the general inequalities:

$$|z_{ik}^{(2m)}| < ab \frac{(2b_1)^{2m-2}}{x_0^{m-1}} \cdot \frac{1}{x^2}; \quad |z_{ii}^{(2m)}| < ab \frac{(2b_1)^{2m-2}}{x_0^{m-2}} \cdot \frac{1}{x^2};$$

$$|z_{ik}^{(2m+1)}| < ab \frac{(2b_1)^{2m-1}}{x_0^{m-1}} \cdot \frac{1}{x^2}; \quad |z_{ii}^{(2m+1)}| < ab \frac{(2b_1)^{2m-1}}{x_0^{m-1}} \cdot \frac{1}{x^2},$$

from which it follows that:

$$\left.\begin{aligned} |z_{ik}^{(2m)}| + |z_{ik}^{(2m+1)}| &< ab \frac{(2b_1)^{2m-2}}{x_0^{m-1}} (1 + 2b_1) \frac{1}{x^2}, \\ |z_{ii}^{(2m)}| + |z_{ii}^{(2m+1)}| &< ab \frac{(2b_1)^{2m-2}}{x_0^{m-1}} (x_0 + 2b_1) \frac{1}{x^2}. \end{aligned}\right\} \qquad (371_1)$$

Using the series (359) and the formulae (368), (370) and (371) we obtain:

$$z_{ik} = \frac{a_{ik}^{(i)}}{x} + \frac{\eta'_{ik}}{x}; \quad z_{ii} = 1 + \frac{a_{ii}^{(1)} + b_{ii}^{(1)}}{x} + \frac{\eta_{ii}}{x}, \qquad (372)$$

where η_{ik} and $\eta_{ii} \to 0$ as $x \to \infty$. Substituting in (365) we obtain the asymptotic expansion for y_{ik}. In the same way as before it is possible to separate the following terms in the asymptotic expansion z_{ik}. The above method without modifications can also be applied to a system of n equations provided that the real parts of the characteristic zeros of the matrix T_0 are different.

Let us now assume that the numbers a_1 and a_2, which form part of the diagonal matrix $T_0 = [a_1, a_2]$, have the same real part a. Replacing Y by a new unknown matrix $Y_1 = e^{ax} Y$ we obtain an equation for Y in the form (345) in which T_0 is the diagonal matrix $[a_1 i, a_2 i]$ with purely imaginary terms on the main diagonal. We will suppose that the equation (345) already has this property. If, at the same time, the matrix T_1 is equal to zero, then without making any modifications, we can use the above method to obtain a uniformly convergent series and the asymptotic representation. When $T_0 = [a, a]$, then the substitution $Y_1 = e^{-ax} y$ gives a system with a regular singularity at infinity.

Let us apply the results we have obtained to a linear differential equation of the form:

$$y'' + \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots\right) y' + \left(b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \dots\right) y = 0. \qquad (373)$$

The usual separation of the functions $y_1 = y$ and $y_2 = y'$ gives a system in which

$$T_0 = \left\| \begin{matrix} 0, & -b_0 \\ 1, & -a_0 \end{matrix} \right\|,$$

and the characteristic equation of this matrix has the form:

$$\lambda^2 + a_0\lambda + b_0 = 0. \tag{374}$$

When the real parts of the zeros of this equation are different we can use the above method for the construction of the series. This method of successive approximations was first worked out by N. P. Erugin in his paper *Transformable Systems* (1946).

In the works of V.V. Khoroshilov (*Proc. Acad. Sci. SSSR.*, 1949) it was worked out for this case when the real parts of the zeros of the characteristic equation for the matrix T_0 were different.

CHAPTER VI

SPECIAL FUNCTIONS

§ 1. Spherical functions

129. The determination of spherical functions. In this chapter we shall study certain special classes of functions which are met with in the solution of equations in mathematical physics. All these functions are usually determined as solutions of certain linear equations with variable coefficients. For example, in problems connected with the vibration of a cord we met trigonometric functions and in problems connected with the vibration of a round membrane we met the Bessel functions.

We shall begin with the study of the so called *spherical functions* which are closely connected with the Laplace equation. We have already encountered this equation. In the Cartesian system of coordinates it has the form:

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{1}$$

We shall seek a solution of this equation in the form of homogeneous polynomials in the variables x, y and z.

Let us consider the simpler cases. The only homogeneous polynomial of zero degree is an arbitrary constant a which evidently satisfies the equation (1). The general form of homogeneous polynomials of the first degree will be:

$$U_1 = ax + by + cz.$$

This polynomial also satisfies the equation (1) for every choice of the coefficients a, b and c. In other words, we have here three linearly-independent solutions of the equation (1), viz. x, y and z and their linear combination with arbitrary constant coefficients gives a general solution of the equation (1) in the form of a homogeneous polynomial of the first degree. Consider a homogeneous polynomial of the second degree

$$U_2 = ax^2 + by^2 + cz^2 + dxy + eyz + fzx.$$

Substituting in the equation (1) we obtain a relationship for the coefficients, viz. $a + b + c = 0$. We can, for example, put $c = -a - b$ and, consequently, the general form of homogeneous polynomials of the second degree, which satisfy the equation (1), will be

$$U_2 = a(x^2 - z^2) + b(y^2 - z^2) + dxy + eyz + fzx.$$

Here we have five linearly-independent solutions of the equation: viz. $x^2 - z^2$, $y^2 - z^2$, xy, yz and zx and the linear combination of these solutions with arbitrary constant coefficients gives the general solution of the equation in the form of a homogeneous polynomial of the second degree.

Let us consider a homogeneous polynomial of the third degree:

$$U_3 = ax^3 + by^3 + cz^3 + dx^2y + ex^2z + fy^2x + gy^2z + hz^2x + kz^2y + lxyz.$$

Substituting in equation (1) we have:

$$6(ax + by + cz) + 2dy + 2ez + 2fx + 2gx + 2hx + 2ky = 0.$$

Equating to zero the coefficients of x, y, z we obtain three equations which connect these coefficients:

$$\begin{aligned} 3a + f + h &= 0 \qquad & a &= -\tfrac{1}{3}(f + h), \\ 3b + d + k &= 0 \quad \text{or} & b &= -\tfrac{1}{3}(d + k), \\ 3c + e + g &= 0 \qquad & c &= -\tfrac{1}{3}(e + g), \end{aligned}$$

so that the general solution of the equation (1) in the form of a polynomial of the third degree will be

$$U_3 = d\left(x^2y - \tfrac{1}{3}y^3\right) + e\left(x^2z - \tfrac{1}{3}z^3\right) + f\left(y^2x - \tfrac{1}{3}x^3\right) + \\ + g\left(y^2z - \tfrac{1}{3}z^3\right) + h\left(z^2x - \tfrac{1}{3}x^3\right) + k\left(z^2y - \tfrac{1}{3}y^3\right) + lxyz.$$

In this case we have seven linearly-independent solutions for the equation.

We will now show that, in general, *there are* $(2n + 1)$ *linearly-independent homogeneous polynomials of the n-th degree which satisfy the equation* (1). We shall now count the number of coefficients in a homogeneous polynomial and the number of equations which they must satisfy. A homogeneous polynomial of the nth degree in two variables

$$a_0x^n + a_1x^{n-1}y + \ldots + a_ny^n$$

has $(n+1)$ coefficients. A homogeneous polynomial of the nth degree in three variables can be written in the form:

$$a_0 z^n + \varphi_1(x, y)\, z^{n-1} + \ldots + \varphi_{n-1}(x, y)\, z + \varphi_n(x, y), \qquad (2)$$

where $\varphi_k(x, y)$ are homogeneous polynomials of degree k. Consequently the total number of coefficients in the homogeneous polynomial (2) will be:

$$1 + 2 + \ldots + n + (n+1) = \frac{(n+1)(n+2)}{2}.$$

Substituting the polynomial (2) in the left-hand side of the equation (1) we obtain a homogeneous polynomial of degree $(n-2)$ which has, in all, $(n-1)\,n/2$ terms. In this way $(n+1)(n+2)/2$ coefficients of the polynomial (2) will be connected by $(n-1)\,n/2$ homogeneous equations. If these equations are independent then the number of arbitrary coefficients will be:

$$\frac{(n+1)(n+2)}{2} - \frac{(n-1)\,n}{2} = 2n + 1,$$

which is what we had to prove. It is still not quite clear whether the above equations will, in fact, be independent. We shall therefore give another complete proof of the above proposition. We can write the polynomial (2) as follows:

$$U_n = \sum_{p+q+r=n} a_{pqr}\, x^p\, y^q\, z^r,$$

where

$$a_{pqr} = \frac{1}{p!\, q!\, r!} \frac{\partial^{p+q+r} U_n}{\partial x^p\, \partial y^q\, \partial z^r}. \qquad (3)$$

Equation (1) can be rewritten in the form

$$\frac{\partial^2 U}{\partial z^2} = -\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2}.$$

By using this equation we can eliminate in the expressions (3) any differentiation with respect to the variable z higher than the first order; we can therefore write:

$$\begin{aligned}\frac{\partial^6 U}{\partial x\, \partial y\, \partial z^4} &= -\frac{\partial^4}{\partial x\, \partial y\, \partial z^2}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = \\ &= \frac{\partial^4}{\partial x^3\, \partial y}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{\partial^4}{\partial x\, \partial y^3}\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial x^2}\right) = \\ &= \frac{\partial^6 U}{\partial x^5\, \partial y} + 2\frac{\partial^6 U}{\partial x^3\, \partial y^3} + \frac{\partial^6 U}{\partial x\, \partial y^5}.\end{aligned}$$

In this way only those coefficients a_{pqr} remain arbitrary which either cannot be differentiated with respect to z or which can only be

differentiated once. These coefficients are as follows: a_{pq0} $(p + q = n)$ or a_{pq1} $(p + q = n - 1)$ and their total number is equal to $(2n + 1)$, which is what we had to prove.

130. The definite expression for spherical functions. We shall now establish the definite expression for homogeneous polynomials mentioned in the previous section. Introducing spherical coordinates we have

$$x = r \sin\theta \cos\varphi\,; \quad y = r \sin\theta \sin\varphi\,; \quad z = r\cos\theta\,. \tag{4}$$

In this case a harmonic homogeneous polynomial of the nth degree can be represented in the form

$$U_n(x, y, z) = r^n Y_n(\theta, \varphi)\,. \tag{5}$$

This polynomial, which is a solution of the equation (1), is usually known as a *volume spherical function* and the factor $Y_n(\theta, \varphi)$, which is a polynomial in $\cos\theta$, $\sin\theta$, $\cos\varphi$ and $\sin\varphi$, is known as a *surface spherical function* or simply as a *spherical function of order n*. It is our aim to find $(2n + 1)$ linearly-independent spherical functions.

We begin by observing one simple fact about the solution of the equation (1). Let us write the following integral which depends on the parameters x, y and z:

$$U(x, y, z) = \int_{-\pi}^{\pi} f(z + ix\cos t + iy\sin t, t)\,\mathrm{d}t\,, \tag{6}$$

where we assume that above integral can be differentiated under the integral sign with respect to x, y and z. After differentiation we can readily see that the function $U(x, y, z)$ satisfies the equation (1) for every choice of the function $f(\tau, t)$ as long as the above differentiation is permissible. In fact:

$$\Delta U(x, y, z) = \int_{-\pi}^{\pi} (1 - \cos^2 t - \sin^2 t)\, f''(z + ix\cos t + iy\sin t, t)\,\mathrm{d}t\,,$$

where by $f''(\tau, t)$ we denote the second derivative of $f(\tau, t)$ with respect to τ. Notice that this argument is a complex quantity. Subsequently by using formula (6), $(2n + 1)$ homogeneous polynomials of the nth degree, which satisfy the equation (1), can readily be constructed.

They can be written as follows:

$$\int_{-\pi}^{\pi} (z + ix\cos t + iy\sin t)^n \cos mt\,\mathrm{d}t \quad (m = 0, 1, 2, \ldots, n), \tag{7}$$

$$\int_{-\pi}^{\pi} (z + ix\cos t + iy\sin t)^n \sin mt\,\mathrm{d}t \quad (m = 1, 2, \ldots, n). \tag{8}$$

Introducing spherical coordinates into the integral (7) we obtain the following expression for the spherical functions:

$$\int_{-\pi}^{\pi} [\cos\theta + i\sin\theta\cos(t - \varphi)]^n \cos mt\,\mathrm{d}t =$$

$$= \int_{-\pi-\varphi}^{\pi-\varphi} (\cos\theta + i\sin\theta\cos\psi)^n \cos m(\varphi + \psi)\,\mathrm{d}\psi.$$

Bearing in mind the fact that the integrand on the right-hand side has a period of 2π with respect to ψ we can take any interval of integration, 2π in length [II, 142]. Hence the above integral can be rewritten as follows:

$$\int_{-\pi}^{\pi} (\cos\theta + i\sin\theta\cos\psi)^n \cos m(\varphi + \psi)\,\mathrm{d}\psi.$$

Expanding $\cos m(\varphi + \psi)$ and bearing in mind that the function $\sin m\psi$ is odd we can rewrite these spherical functions in the form:

$$\cos m\varphi \int_{-\pi}^{\pi} (\cos\theta + i\sin\theta\cos\psi)^n \cos m\psi\,\mathrm{d}\psi \quad (m = 0, 1, 2, \ldots, n). \tag{9}$$

Similarly integral (8) gives the following n spherical functions:

$$\sin m\varphi \int_{-\pi}^{\pi} (\cos\theta + i\sin\theta\cos\psi)^n \cos m\psi\,\mathrm{d}\psi \quad (m = 1, 2, \ldots, n). \tag{10}$$

The linear independence of all the $(2n + 1)$ functions (9) and (10) is directly due to the fact that the dependence of these functions on φ is due to the factors $\cos m\varphi$ and $\sin m\varphi$; these functions cannot be linearly dependent for they are orthogonal in the interval $(-\pi, +\pi)$ [II, 142]. We have thus constructed all the $(2n + 1)$ spherical functions of order n. The coefficients of $\cos m\varphi$ and $\sin m\varphi$ in the expressions (9) and (10) are functions of θ. We can express them in terms of the Legendre polynomials.

We have the following expressions for the Legendre polynomial [102]:

$$P_n(x) = \frac{1}{n!\,2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(x^2-1)^n\right]. \tag{11}$$

Let us also introduce the functions $P_{n,m}(x)$, which are expressed in terms of the Legendre polynomials as follows:

$$P_{n,m}(x) = (1-x^2)^{\frac{m}{2}}\frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m} = \frac{(1-x^2)^{\frac{m}{2}}}{n!\,2^n}\frac{\mathrm{d}^{n+m}}{\mathrm{d}x^{n+m}}\left[(x^2-1)^n\right]. \tag{12}$$

We can now introduce different expressions for $P_n(x)$ and $P_{n,m}(x)$. According to Cauchy's formula we can write

$$(x^2-1)^n = \frac{1}{2\pi i}\int_C \frac{(z^2-1)^n}{z-x}\,\mathrm{d}z,$$

where C is a closed contour which is described in the counter-clockwise direction and which contains the point $z = x$. From this and from (11) we have:

$$P_n(x) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(z-1)^n(z+1)^n}{(z-x)^{n+1}}\,\mathrm{d}z. \tag{13}$$

Take as the contour C a circle, centre $z = x$ and radius $|x^2-1|^{1/2}$ (we assume that $x \neq \pm 1$). The variable z is given by the formula:

$$z = x + (x^2-1)^{\frac{1}{2}}\mathrm{e}^{i\psi},$$

where the choice of the value of $(x^2-1)^{1/2}$ is immaterial and where it can be assumed that ψ varies from $(-\pi)$ to $(+\pi)$. Replacing the variables in the integral (13) we have:

$$P_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{\left[x-1+(x^2-1)^{\frac{1}{2}}\mathrm{e}^{i\psi}\right]\left[x+1+(x^2-1)^{\frac{1}{2}}\mathrm{e}^{i\psi}\right]}{2\,(x^2-1)^{\frac{1}{2}}\mathrm{e}^{i\psi}}\right\}^n \mathrm{d}\psi.$$

Performing elementary calculations and remembering that the integrand is an even function we have:

$$P_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[x+(x^2-1)^{\frac{1}{2}}\cos\psi\right]^n \mathrm{d}\psi =$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[x+(x^2-1)^{\frac{1}{2}}\cos\psi\right]^n \mathrm{d}\psi. \tag{14}$$

Let us carry out analogous calculations for $P_{n,m}(x)$. In place of (13) we have:

$$P_{n,m}(x) = \frac{(1-x^2)^{\frac{m}{2}}(n+1)(n+2)\ldots(n+m)}{2^{n+1}\pi i}\int_C \frac{(z^2-1)^n}{(z-x)^{n+m+1}}\,dz,$$

and performing the above change of variables we obtain:

$$P_{n.m}(x) = -\,i^m\frac{(n+1)(n+2)\ldots(n+m)}{2\pi}\int_{-\pi}^{\pi}\left[x+(x^2-1)^{\frac{1}{2}}\cos\psi\right]^n e^{-im\psi}d\psi,$$

or, bearing in mind that $\sin m\psi$ is odd,

$$P_{n,m}(x) =$$

$$= -\,i^m\frac{(n+1)(n+2)\ldots(n+m)}{2\pi}\int_{-\pi}^{\pi}\left[x+(x^2-1)^{\frac{1}{2}}\cos\psi\right]^n\cos m\psi\,d\psi. \quad (15)$$

If we put $x=\cos\theta$ in either the integral (14) or (15) then the integrals which appear in the formulae (9) and (10) will be obtained. Bearing in mind that the constant factor of a harmonic polynomial or of a spherical function is irrelevant we arrive at the following conclusion: $(2n+1)$ *spherical functions of the n-th order can be written in the form:*

$$P_n(\cos\theta);\quad P_{n,m}(\cos\theta)\cos m\varphi;\quad P_{n,m}(\cos\theta)\sin m\varphi \qquad (16)$$
$$(m=1,2,\ldots,n),$$

where $P_n(x)$ *are Legendre polynomials as defined by the formula* (11) *and* $P_{n,m}(x)$ *is given by the formula* (12). Notice that the factor $(1-x^2)^{m/2}$ becomes $\sin^m\theta$ when $x=\cos\theta$. Multiplying the solutions (16) by arbitrary constants and adding we obtain the general form of a spherical function of the nth order:

$$Y_n(\theta,\varphi)=a_0P_n(\cos\theta)+\sum_{m=1}^{n}(a_m\cos m\varphi+b_m\sin m\varphi)P_{n,m}(\cos\theta). \quad (17)$$

In place of trigonometric functions we could, by constructing linear combinations of the solutions (16), use exponential functions, so that instead of taking the spherical functions (16) of the nth order we would use the following system of spherical functions of the nth order:

$$P_n(\cos\theta),\ P_{n,m}(\cos\theta)\,e^{im\varphi},\ P_{n,m}(\cos\theta)\,e^{-im\varphi}\quad(m=1,2,\ldots,n). \quad (18)$$

It follows from this construction *that the general form of homogeneous polynomials of the nth order of the variables* (x, y, z) *which satisfy the Laplace equation, will be* $r^n Y_n(\theta, \varphi)$, *where* $Y_n(\theta, \varphi)$ *is given by the formula* (17).

131. The orthogonal properties. We shall now prove the orthogonal properties of spherical functions (16) on a unit sphere and evaluate the integral of the square of these functions on the unit sphere. We begin by evaluating the integrals:

$$I_m = \int_{-1}^{1} [P_{n\,m}(x)]^2 \, dx.$$

From the definition of these functions we have:

$$I_m = \int_{-1}^{1} [P_{n.m}(x)]^2 \, dx = \int_{-1}^{1} (1 - x^2)^m \frac{d^m P_n(x)}{dx^m} \frac{d^m P_n(x)}{dx^m} \, dx,$$

and when $m = 0$ we obtain the integral of the square of the Legendre polynomial:

$$I_0 = \int_{-1}^{1} [P_n(x)]^2 \, dx.$$

We have shown earlier [102] that

$$I_0 = \int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}. \tag{19}$$

At the end of this section we shall again give the proof of this formula but at present we shall evaluate the integral I_m by using formula (19).

Integrating by parts we can write:

$$I_m = (1 - x^2)^m \frac{d^m P_n(x)}{dx^m} \frac{d^{m-1} P_n(x)}{dx^{m-1}} \Bigg|_{x=-1}^{x=+1} -$$

$$- \int_{-1}^{1} \frac{d^{m-1} P_n(x)}{dx^{m-1}} \frac{d}{dx} \left[(1 - x^2)^m \frac{d^m P_n(x)}{dx^m} \right] dx$$

or

$$I_m = - \int_{-1}^{1} \frac{d^{m-1} P_n(x)}{dx^{m-1}} \frac{d}{dx} \left[(1 - x^2)^m \frac{d^m P_n(x)}{dx^m} \right] dx. \tag{20}$$

But the function

$$z = \frac{\mathrm{d}^{m-1} P_n(x)}{\mathrm{d}x^{m-1}} = \frac{1}{2^n n!} \frac{\mathrm{d}^{n+m-1}(x^2-1)^n}{\mathrm{d}x^{n+m-1}},$$

as can readily be assertained by using the equation (84) from [102], satisfies the equation

$$(1-x^2)\frac{\mathrm{d}^{m+1} P_n(x)}{\mathrm{d}x^{m+1}} - 2mx\frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m} + \\ + (n+m)(n-m+1)\frac{\mathrm{d}^{m-1} P_n(x)}{\mathrm{d}x^{m-1}} = 0.$$

Multiplying by $(1-x^2)^{m-1}$, we can rewrite it in the form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^m \frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m}\right] = -(n+m)(n-m+1)(1-x^2)^{m-1}\frac{\mathrm{d}^{m-1} P_n(x)}{\mathrm{d}x^{m-1}}.$$

Substituting in formula (20) we have

$$I_m = (n+m)(n-m+1)\int_{-1}^{1}(1-x^2)^{m-1}\frac{\mathrm{d}^{m-1} P_n(x)}{\mathrm{d}x^{m-1}}\frac{\mathrm{d}^{m-1} P_n(x)}{\mathrm{d}x^{m-1}}\,\mathrm{d}x$$

or

$$I_m = (n+m)(n-m+1) I_{m-1}.$$

Using this as a reduction formula, we obtain

$$I_m = (n+m)(n-m+1)(n+m-1)(n-m+2) I_{m-2} = \ldots = \\ = (n+m)(n-m+1)(n+m-1)(n-m+2)\ldots(n+1) n I_0 = \\ = (n+m)(n+m-1)(n+m-2)\ldots(n-m+1) I_0 = \frac{(n+m)!}{(n-m)!} I_0.$$

This, with (19), gives the following final expression for the integrals of the squares of the functions $P_{n,m}(x)$:

$$\int_{-1}^{1}\left[P_{n,m}(x)\right]^2 \mathrm{d}x = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}. \tag{21}$$

This result makes it possible to evaluate the integral of the square of a spherical function. The spherical functions $Y_n(\theta, \varphi)$ can be regarded as being given on the surface of a sphere of unit radius; θ and φ are the usual geographical coordinates of points on the surface: φ = const. are the meridians and θ = const. the parallels.

With this choice of coordinates an element of surface are, as is well known, is expressed by the following formula [II, 59]:

$$d\sigma = \sin\theta \, d\theta \, d\varphi . \tag{22}$$

We shall prove, first of all, that two different spherical functions $Y_p(\theta, \varphi)$ and $Y_q(\theta, \varphi)$ of different orders, i.e. when $p \neq q$, will be orthogonal on the surface s of a unit sphere, i.e.

$$\iint_s Y_p(\theta, \varphi) \, Y_q(\theta, \varphi) \, d\sigma = 0 . \tag{23}$$

Let v be the volume of this sphere and s be its surface. Applying Green's formula [II, 193] to the harmonic functions:

$$U_p = r^p Y_p(\theta, \varphi) \quad \text{and} \quad U_q = r^q Y_q(\theta, \varphi) \tag{24}$$

we obtain

$$\iint_s \left(U_p \frac{\partial U_q}{\partial n} - U_q \frac{\partial U_p}{\partial n} \right) d\sigma = \iiint_v (U_p \Delta U_q - U_q \Delta U_p) \, dv ,$$

where $\Delta U_p = \Delta U_q = 0$.

In this case, differentiation along the normal coincides with differentiation along the radius r, so that the last formula and (24) give:

$$\iint_s \left[q Y_p(\theta, \varphi) \, Y_q(\theta, \varphi) - p Y_q(\theta, \varphi) \, Y_p(\theta, \varphi) \right] d\sigma = 0 ,$$

from which formula (23) follows directly.

We shall show that the spherical functions (16) which correspond to one and the same value of n, will also be orthogonal. In fact, integration over a unit sphere involves, among other things, integration with respect to φ in the interval $(0, 2\pi)$. But the functions (16) contain the following factors which depend on φ:

$$1, \cos\varphi, \sin\varphi, \cos 2\varphi, \sin 2\varphi, \ldots, \cos n\varphi, \sin n\varphi,$$

and the product of any two of these factors, when integrated over the interval $(0, 2\pi)$, is zero [II, 142]. It can be shown similarly that the functions (18) also form an orthogonal system.

Lastly we can evaluate the integral of the square of each of the constructed functions. Let us consider the spherical function $P_n(\cos\theta)$ which does not depend on φ, and construct the integral of its square on the surface of a unit sphere

$$\int_0^{\pi} \int_0^{2\pi} P_n^2(\cos\theta) \sin\theta \, d\theta \, d\varphi .$$

On introducing the new variable of integration $x = \cos\theta$ and recalling formula (19) we have:

$$\int_0^{\pi}\int_0^{2\pi} P_n^2(\cos\theta)\sin\theta\,d\theta\,d\varphi = 2\pi\int_{-1}^{1} P_n^2(x)\,dx = \frac{4\pi}{2n+1}.$$

Similarly for other functions:

$$\int_0^{\pi}\int_0^{2\pi} [P_{n,m}(\cos\theta)]^2 \sin^2 m\varphi \sin\theta\,d\theta\,d\varphi = \pi\int_{-1}^{1} [P_{n,m}(x)]^2\,dx.$$

The latter formula and (21) finally give:

$$\left.\begin{aligned} \iint_S [P_n(\cos\theta)]^2\,d\sigma &= \frac{4\pi}{2n+1}, \\ \iint_S [P_{n,m}(\cos\theta)\cos m\varphi]^2\,d\sigma &= \frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}, \\ \iint_S [P_{n,m}(\cos\theta)\sin m\varphi]^2\,d\sigma &= \frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}. \end{aligned}\right\} \qquad (25)$$

We shall use these formulae in future in connection with the expansion of an arbitrary function which is given on the surface of a sphere, in terms of spherical functions.

We shall now prove formula (19). Using the definition (11) of the Legendre polynomials we can write:

$$I_0 = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{d^n(x^2-1)^n}{dx^n}\,\frac{d^n(x^2-1)^n}{dx^n}\,dx.$$

Integrating by parts we have:

$$I_0 = \frac{1}{2^{2n}(n!)^2}\left[\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\cdot\frac{d^n(x^2-1)^n}{dx^n}\right]_{x=-1}^{x=+1} - \\ - \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{d^{n+1}(x^2-1)^n}{dx^{n+1}}\cdot\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\,dx.$$

The polynomial $(x^2-1)^n$ has the zeros $x = \pm 1$ each of order n. Its derivative of order $(n-1)$ has the same zeros which, however, are only of order one [I, 186], and, consequently, the first term on the

right-hand side in the above equation is equal to zero. Continuing to integrate by parts we obtain:

$$I_0 = \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} \frac{d^{2n}(x^2-1)^n}{dx^{2n}} \cdot (x^2-1)^n \, dx.$$

But

$$\frac{d^{2n}(x^2-1)^n}{dx^{2n}} = \frac{d^{2n}}{dx^{2n}}(x^{2n} + \ldots) = (2n)!,$$

therefore:

$$I_0 = (-1)^n \frac{(n+1)(n+2)\ldots 2n}{n!\, 2^{2n}} \int_{-1}^{1} (x^2-1)^n \, dx.$$

Introducing the new variable of integration φ according to the formula $x = \cos\varphi$ we obtain:

$$I_0 = \frac{(n+1)(n+2)\ldots 2n}{n!\, 2^{2n}} \int_0^{\pi} \sin^{2n+1}\varphi \, d\varphi =$$

$$= \frac{(n+1)(n+2)\ldots 2n}{n!\, 2^{2n}} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^{2n+1}\varphi \, d\varphi,$$

and, using the formula (28) from [I, 100] we obtain the formula (19).

132. The Legendre polynomials. We shall now study the Legendre polynomials in greater detail. Notice, first of all, that if we use the definition (11) and apply the Leibniz formula for a derivative of the nth order to the product $(x^2-1)^n = (x+1)^n(x-1)^n$ we obtain:

$$P_n(x) = \frac{1}{n!\, 2^n}\left[(x+1)^n \frac{d^n(x-1)^n}{dx^n} + \frac{n}{1}\frac{d(x+1)^n}{dx}\frac{d^{n-1}(x-1)^n}{dx^{n-1}} + \ldots + \right.$$

$$\left. + \frac{d^n(x+1)^n}{dx^n} \cdot (x-1)^n\right].$$

Bearing in mind that

$$\frac{d^n(x-1)^n}{dx^n} = n! \quad \text{and} \quad \left.\frac{d^k(x-1)^n}{dx^k}\right|_{x=1} = 0 \quad \text{when} \quad k < n,$$

we obtain directly from the previous formula:

$$P_n(1) = 1. \tag{26}$$

We shall now use a special method viz. the method of the dominating function, in the study of further properties of the Legendre polynomials. We shall also use this method in future in the study of other special functions.

Place at the North pole N of a unit sphere a positive charge $(+1)$ and let M be a variable point, the spherical coordinates of which are r, θ and φ. The Coulomb field created by this charge will have the following potential at the point M:

$$\frac{1}{d} = \frac{1}{\sqrt{1 - 2r\cos\theta + r^2}}, \tag{27}$$

where d is the distance from the charge to the variable point M.

The function (27) will be a regular function of the variable r at the point $r = 0$ and we can expand it in positive integral powers of r:

$$\frac{1}{d} = a_0(\theta) + a_1(\theta)\, r + a_2(\theta)\, r^2 + \ldots, \tag{28}$$

where the coefficients in this expansion are polynomials in $\cos\theta$. We could calculate these coefficients exactly by applying to the function

$$\frac{1}{d} = \left[1 + (r^2 - 2r\cos\theta)\right]^{-\frac{1}{2}}.$$

Newton's binomial expansion and by subsequently collecting together terms with equal powers of r. However we shall approach the problem somewhat differently.

The function (27) in the usual system of coordinates can be expressed as follows

$$\frac{1}{d} = \left[1 + (x^2 + y^2 + z^2 - 2z)\right]^{-\frac{1}{2}}. \tag{29}$$

We can obtain the series (28) if we apply Newton's binomial expansion to the function (29) and subsequently collect together, in the infinite series so obtained, all terms of equal value with respect to x, y and z, i.e. all terms of the series (28) which are homogeneous polynomials in x, y and z. As we know, the function $1/d$ is a solution of the Laplace equation [II, 119] and consequently, the same can be said about the individual terms of the series (28) i.e. the terms of this series must be volume spherical functions. These functions do not depend on the angle φ and, consequently, each term of this series

must appear in the form of a product $c_n P_n(\cos\theta)$, where c_n is a constant which has to be found. We thus have:

$$\frac{1}{\sqrt{1-2r\cos\theta+r^2}} = c_0 + c_1 P_1(\cos\theta)\, r + c_2 P_2(\cos\theta)\, r^2 + \dots$$

If we take $\theta = 0$ we obtain, since $P_n(1) = 1$:

$$\frac{1}{1-r} = c_0 + c_1 r + c_2 r^2 + \dots,$$

from which it follows that $c_n = 1$ for all n; we thus obtain the following final expansion of our elementary potential in powers of r:

$$\frac{1}{\sqrt{1-2r\cos\theta+r^2}} = 1 + P_1(\cos\theta)\, r + P_2(\cos\theta)\, r^2 + \dots \qquad (30)$$

On replacing $\cos\theta$ by x and r by z we can write:

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x)\, z^n. \qquad (31)$$

This formula can be used as a definition for the Legendre polynomials viz.: we can say that *the Legendre polynomial* $P_n(x)$ *is the coefficient of* z^n *in the expansion of the function*

$$\frac{1}{\sqrt{1-2xz+z^2}} \qquad (32)$$

in positive integral, powers of z. In other words, *the function* (32) *is the dominating function of the Legendre polynomials.*

We shall now determine the radius of convergence of the power series (31). Values of z for which the expression under the radical vanishes will be the singularities of the function (32). Solving the corresponding quadratic equation we obtain the following zeros:

$$z = x \pm \sqrt{x^2-1} = x \pm \sqrt{1-x^2}\, i. \qquad (33)$$

Since $x = \cos\theta$ we can assume that x is real and that it lies in the interval $-1 < x < +1$. In this case the zeros (33) will be conjugate complex zeros and the square of the modulus of each zero will be:

$$x^2 + (\sqrt{1-x^2})^2 = 1.$$

When $x = \pm 1$ the zeros (33) coincide and are both equal to ± 1. Thus when $-1 \leqslant x \leqslant +1$ the singularities of the function (32) will lie at a unit distance from the origin and, consequently, the series (31) will converge when $|z| < 1$. Thus the expansion (30) will be

valid when $r < 1$, i.e. for all points inside a unit sphere. For points outside a unit sphere we have another expansion. In fact, when $r > 1$, the function (27) can be rewritten as follows:

$$\frac{1}{\sqrt{1 - 2r\cos\theta + r^2}} = \frac{1}{r}\,\frac{1}{\sqrt{1 - 2\frac{1}{r}\cos\theta + \left(\frac{1}{r}\right)^2}}.$$

In this case $1/r < 1$, so we can apply the former expansion and we finally obtain the following formula for the potential (27) outside a unit sphere:

$$\frac{1}{\sqrt{1 - 2r\cos\theta + r^2}} = \sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{r^{n+1}}. \tag{34}$$

No term of this sum has singularities outside the sphere and each term vanishes at infinity.

Until now we have considered a sphere of unit radius. For a sphere of any radius R we have, by taking R^2 or r^2 outside the radical:

$$\frac{1}{\sqrt{R^2 - 2r\,R\cos\theta + r^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\,\frac{r^n}{R^{n+1}} \qquad (r < R), \tag{35}$$

$$\frac{1}{\sqrt{R^2 - 2r\,R\cos\theta + r^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\,\frac{R^n}{r^{n+1}} \qquad (r > R). \tag{36}$$

From formula (31) the fundamental properties of the Legendre polynomials can easily be deduced. Differentiating this formula with respect to z and multiplying subsequently by $(1 - 2xz + z^2)$ we obtain

$$\frac{x - z}{\sqrt{1 - 2xz + z^2}} = (1 - 2xz + z^2)\sum_{n=0}^{\infty} nP_n(x)\,z^{n-1}$$

or

$$(x - z)\sum_{n=0}^{\infty} P_n(x)\,z^n = (1 - 2xz + z^2)\sum_{n=1}^{\infty} nP_n(x)\,z^{n-1}.$$

Comparing the coefficients of like powers of z we obtain a relationship for successive Legendre polynomials:

$$(n+1)\,P_{n+1}(x) - (2n+1)\,xP_n(x) + nP_{n-1}(x) = 0 \tag{37}$$

$$(n = 1, 2, 3, \ldots)$$

$$P_1(x) - xP_0(x) = 0.$$

Similarly, differentiating formula (31) with respect to x and multiplying it subsequently by $(1 - 2xz + z^2)$ we have;

$$P_n(x) = \frac{dP_{n+1}(x)}{dx} + \frac{dP_{n-1}(x)}{dx} - 2x\frac{dP_n(x)}{dx}, \tag{38}$$

or, substituting $P_{n+1}(x)$ from (37):

$$x\frac{dP_n(x)}{dx} - \frac{dP_{n-1}(x)}{dx} = nP_n(x). \tag{39}$$

Eliminating $x\,dP_n(x)/dx$ from (38) and (39) we obtain:

$$\frac{dP_{n+1}(x)}{dx} - \frac{dP_{n-1}(x)}{dx} = (2n+1)P_n(x). \tag{40}$$

This formula remains valid when $n = 0$ if we assume that $P_{-1}(x) = 0$. On supposing in formula (40) that the symbol n is equal to $0, 1, \ldots, n$ in succession and adding we obtain the new relationship:

$$P_0(x) + 3P_1(x) + \ldots + (2n+1)P_n(x) = \frac{dP_{n+1}(x)}{dx} + \frac{dP_n(x)}{dx}. \tag{41}$$

Let us write formula (40) replacing n by $n - 2k + 1$:

$$\frac{dP_{n-2k+2}(x)}{dx} - \frac{dP_{n-2k}(x)}{dx} = (2n - 4k + 3)P_{n-2k+1}(x).$$

Adding with respect to k from $k = 1$ to $k = N$, where $N = (1/2)n$ when n is even, and $N = (1/2)(n+1)$ when n is odd, we obtain the formula:

$$\frac{dP_n(x)}{dx} = \sum_{k=1}^{N}(2n - 4k + 3)P_{n-2k+1}(x). \tag{42}$$

It follows from formula (11) that $P_n(x)$ contains only even powers of x when n is even, and only odd powers of x when n is odd. Similarly, it also follows from this formula that:

$$P_{2n}(0) = (-1)^n \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n}; \quad P_{2n+1}(0) = 0$$

$$P_n(-1) = (-1)^n. \tag{43}$$

Applying Newton's binomial expansion we can write:

$$\frac{1}{\sqrt{1 - 2r\cos\theta + r^2}} = \frac{1}{\sqrt{1 - e^{i\theta}r}} \cdot \frac{1}{\sqrt{1 - e^{-i\theta}r}} =$$

$$= \left(\sum_{n=0}^{\infty} \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n} e^{in\theta} r^n\right)\left(\sum_{m=0}^{\infty} \frac{1\cdot 3\ldots(2m-1)}{2\cdot 4\ldots 2m} e^{-im\theta} r^m\right),$$

where, when $n = 0$ and $m = 0$, the terms of the series must be assumed to be equal to unity. Multiplying the series and comparing the coefficients of like powers of r we obtain the following expression for the Legendre polynomials:

$$P_n(\cos\theta) = a_0 a_n \cos n\theta + a_1 a_{n-1} \cos(n-2)\theta + \ldots + \\ + a_n a_0 \cos n\theta, \qquad (44)$$

where all the coefficients a_k are positive and are given by the formulae

$$a_0 = 1; \quad a_k = \frac{1 \cdot 3 \ldots (2k-1)}{2 \cdot 4 \ldots 2k} \qquad (k = 1, 2, \ldots). \qquad (45)$$

From this it also follows that

$$|P_n(\cos\theta)| \leqslant a_0 a_n + a_1 a_{n-1} + \ldots + a_n a_0 = P_n(1) = 1. \qquad (46)$$

The formulae (37) make it possible to evaluate the Legendre polynomials successively. Let us write the first five of these polynomials:

$$\left.\begin{aligned} &P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1); \\ &P_3(x) = \tfrac{1}{2}(5x^3 - 3x); \qquad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned}\right\} \qquad (47)$$

When $f(x)$ is a certain function given in the interval $(-1, +1)$, the question of representing it by a series of Legendre polynomials arises

$$f(x) = a_0 + a_1 P_1(x) + a_2 P_2(x) + \ldots \qquad (48)$$

Using the fact that $P_n(x)$ is orthogonal and also the formula (19) we can see, as in the theory of trigonometric series, that the coefficients a_n are given by the formulae

$$a_n + \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \qquad (49)$$

It can be shown that for this choice of coefficients the series (48) converges in the interval $(-1, +1)$ and that its sum is equal to $f(x)$ provided this latter function satisfies certain very general conditions.

133. The expansion in terms of spherical functions. Any function given on the surface of a sphere of any radius, is a function with geographical coordinates θ and φ so that we can denote it by $f(\theta, \varphi)$. Let us suppose that it can be expanded in terms of spherical functions, i.e. that it can be represented on the sphere in the form of a series

analogous with the Fourier series:

$$f(\theta, \varphi) = a_0^{(0)} + \sum_{n=1}^{\infty} \left\{ a_0^{(n)} P_n(\cos\theta) + \right.$$
$$\left. + \sum_{m=1}^{n} \left(a_m^{(n)} \cos m\varphi + b_m^{(n)} \sin m\varphi \right) P_{n,m}(\cos\theta) \right\}. \qquad (50)$$

Using the property of orthogonality of spherical functions together with the formulae (25), we obtain the following expression for the coefficients of the series in the same way as for a Fourier series

$$\left.\begin{aligned} a_m^{(n)} &= \frac{(2n+1)(n-m)!}{2\delta_m \pi (n+m)!} \iint_S f(\theta, \varphi) P_{n,m}(\cos\theta) \cos m\varphi \, d\sigma, \\ b_m^{(n)} &= \frac{(2n+1)(n-m)!}{2\delta_m \pi (n+m)!} \iint_S f(\theta, \varphi) P_{n,m}(\cos\theta) \sin m\varphi \, d\sigma \end{aligned}\right\} \qquad (51)$$

$\left(\delta_m = 2 \text{ when } m = 0 \text{ and } \delta_m = 1 \text{ when } m > 0\,;\ P_{n,0}(x) = P_n(x)\right)$

Speaking more strictly these considerations are only preliminaries in the determination of the coefficients of the series (50). We must subsequently substitute the values of the coefficients obtained from the formulae (51) in the series (50) and prove that with certain assumptions made with regard to the function $f(\theta, \varphi)$ this series will converge and its sum will be equal to $f(\theta, \varphi)$. We shall prove this in the next section.

As a preliminary we shall explain certain integral relationships which must be satisfied by spherical functions. Let S_R be the surface of a sphere of radius R and $Y_n(\theta, \varphi)$ — a spherical function of order n. The function

$$U_n(M) = r^n Y_n(\theta, \varphi)$$

is a harmonic function and we can apply Green's formula [II, 193] to it:

$$U_n(M) = \frac{1}{4\pi} \iint_{S_R} \left(\frac{\partial U_n}{\partial \nu} \frac{1}{d} - U_n \frac{\partial \frac{1}{d}}{\partial \nu} \right) ds, \qquad (52)$$

where d is the distance from the variable point M' on the sphere S_R to the point M which lies inside the sphere, ds is an element of the surface area of the sphere and ν is the direction of the outside normal to the sphere S_R so that, in this case, $\partial/\partial\nu = \partial/\partial R$. We therefore have

$$\frac{1}{d} = \frac{1}{\sqrt{R^2 - 2Rr\cos\gamma + r^2}},$$

and also from (36)

$$\frac{1}{d} = \sum_{k=0}^{\infty} P_k(\cos\gamma)\frac{r^k}{R^{k+1}} \qquad (r < R),$$

so that

$$\frac{\partial}{\partial\nu}\left(\frac{1}{d}\right) = \frac{\partial}{\partial R}\left(\frac{1}{d}\right) = -\sum_{k=0}^{\infty}(k+1)P_k(\cos\gamma)\frac{r^k}{R^{k+2}}$$

and

$$\frac{\partial U_n}{\partial\nu} = nR^{n-1}Y_n(\theta, \varphi).$$

In these formulae γ is the angle between the radii-vectors OM and OM'. Substituting all this in formula (52) and assuming that the radius R is unity we obtain:

$$r^n Y_n(\theta, \varphi) = \frac{1}{4\pi}\iint_s \left\{ nY_n(\theta', \varphi')\sum_{k=0}^{\infty} P_k(\cos\gamma)\, r^k + \right.$$

$$\left. + Y_n(\theta', \varphi')\sum_{k=0}^{\infty}(k+1)P_k(\cos\gamma)\, r^k \right\} d\sigma,$$

where θ' and φ' denote the geographical coordinates of the variable point M' on the unit sphere. Since $r < 1$, the above series converge uniformly with respect to θ' and φ' and the Legendre polynomials satisfy the inequality (46). Integrating these series term by term we have

$$r^n Y_n(\theta, \varphi) = \sum_{k=0}^{\infty}\frac{r^k}{4\pi}\iint_s (k+n+1)\, Y_n(\theta', \varphi')\, P_k(\cos\gamma)\, d\sigma.$$

It follows directly from this sum that all its terms vanish except the term corresponding to $k = n$; this gives us the following integral formulae which are important in the applications of spherical functions:

$$\iint_s Y_n(\theta', \varphi')\, P_m(\cos\gamma)\, d\sigma = 0 \qquad \text{when} \quad m \neq n. \tag{53}$$

$$\iint_s Y_n(\theta', \varphi')\, P_n(\cos\gamma)\, d\sigma = \frac{4\pi}{2n+1}\, Y_n(\theta, \varphi). \tag{54}$$

We now introduce a formula which expresses $\cos\gamma$ in terms of trigonometric functions of the angles θ, φ, θ' and φ'. We draw for this purpose two radii OM'' and OM' of the unit sphere, the ends of

which have coordinates (θ, φ) and (θ', φ') respectively. The projections of these radii on the axes of coordinates will be:

$$\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta \quad \text{and} \quad \sin\theta'\cos\varphi',\ \sin\theta'\sin\varphi',\ \cos\theta',$$

and the cosine of the angle between these two radii will be expressed as the sum of the products of these projections, i.e. we obtain the following formula for $\cos\gamma$:

$$\cos\gamma = \sin\theta\sin\theta'\cos(\varphi - \varphi') + \cos\theta\cos\theta'. \tag{55}$$

Let us return to the series (50) again. If this series is uniformly convergent and its sum is equal to $f(\theta, \varphi)$ then we have the formulae (51) for its coefficients, in the same way as we did in the theory of trigonometric series. We can now unite into one term those terms of the series (50) which are spherical functions of any given order n, i.e. put

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi). \tag{56}$$

On replacing θ and φ by θ' and φ' in this expansion, multiplying by $P_n(\cos\gamma)$ and integrating with respect to the variables θ' and φ' we have, from (53) and (54), the following formula for the terms of the series (56):

$$Y_n(\theta, \varphi) = \frac{2n+1}{4\pi}\int\int_S f(\theta', \varphi')\, P_n(\cos\gamma)\, d\sigma. \tag{57}$$

This formula gives the sum of those terms of the series (50) which stand under the symbol of summation with respect to n and which refer to the given value of n.

Substituting the values of the coefficients (51) in a separate term of the sum (50) we have:

$$Y_n(\theta, \varphi) = \sum_{m=0}^{n} \frac{(n-m)!}{(n+m)!}\,\frac{2n+1}{2\delta_m\pi}\left[\cos m\varphi \int\int_S f(\theta', \varphi')\cos m\varphi' P_{n,m}(\cos\theta')\, d\sigma + \right.$$

$$\left. + \sin m\varphi \int\int_S f(\theta', \varphi')\sin m\varphi' P_{n,m}(\cos\theta')\, d\sigma\right] P_{n,m}(\cos\theta)$$

or

$$Y_n(\theta, \varphi)$$

$$= \int\int_S f(\theta', \varphi') \sum_{m=0}^{n} \frac{(n-m)!}{(n+m)!}\,\frac{2n+1}{2\delta_m\pi} P_{n,m}(\cos\theta')\, P_{n,m}(\cos\theta)\cos m(\varphi' - \varphi)\, d\sigma. \tag{58}$$

A comparison of the formulae (57) and (58) gives

$$\iint_S f(\theta', \varphi') \left[P_n(\cos\gamma) - \sum_{m=0}^{n} \frac{(n-m)!\,2}{(n+m)!\,\delta_m} \times \right.$$

$$\left. \times P_{n,m}(\cos\theta')\, P_{n,m}(\cos\theta) \cos m(\varphi' - \varphi) \right] d\sigma = 0. \tag{59}$$

Strictly speaking we deduced this formula on the assumption that $f(\theta, \varphi)$ is the sum of the uniformly convergent series (50). In particular this formula will certainly be valid if the sum of the series (50) can be obtained in a finite form. Notice that the angle γ is one of the geographical coordinates (latitude) if we take one of the points with geographical coordinates θ, φ for the North pole. Hence $r^n P_n(\cos\gamma)$ is a homogeneous harmonic polynomial of the nth degree and, consequently, $P_n(\cos\gamma)$ is a spherical function of the nth order of the variables θ' and φ'. We thus see that the square bracket in formula (59) is a finite sum of spherical functions and it is therefore correct to assume that $f(\theta', \varphi')$ is equal to this finite sum of spherical functions. Hence with this choice of functions we find that the integral of the square of the above square bracket is equal to zero and therefore the whole expression in the square bracket must also be equal to zero:

$$P_n(\cos\gamma) = \sum_{m=0}^{n} \frac{(n-m)!2}{(n+m)!\,\delta_m} P_{n,m}(\cos\theta')\, P_{n,m}(\cos\theta) \cos m(\varphi' - \varphi). \tag{60}$$

This formula is usually known as the *addition theorem for the Legendre poly nomials.*

134. Proof of convergence. We will now show that the arbitrary function $f(\theta, \varphi)$ which is given on the surface of a sphere and which satisfies certain conditions, can be expanded into the series (56) in terms of spherical functions.

Bearing formula (57) in mind we obtain the following expression for the sum of the first $(n + 1)$ terms of the series (156):

$$S_n = \frac{1}{4\pi} \iint_S f(\theta', \varphi') \sum_{k=0}^{n} (2k+1)\, P_k(\cos\gamma)\, d\sigma.$$

Let us introduce new geometrical coordinates γ and β, placing the North pole at a point with former geographical coordinates θ and φ. At the same time the function $f(\theta', \varphi')$ will, in this new system of coordinates, become a new function $F(\gamma, \beta)$ and we have

$$S_n = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} F(\gamma, \beta) \sum_{k=0}^{n} (2k+1)\, P_k(\cos\gamma) \sin\gamma\, d\gamma\, d\beta. \tag{61}$$

We now introduce a new function $\Phi(\gamma)$ which is the mean value of the functions $F(\gamma, \beta)$ on different parallels of the new system of coordinates

$$\Phi(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} F(\gamma, \beta)\, d\beta. \tag{62}$$

We now introduce a new variable $x = \cos\gamma$ and put

$$\Phi(\gamma) = \Psi(x). \tag{63}$$

Integrating the formula (61) with respect to β we can rewrite it in the form

$$S_n = \frac{1}{2}\int_0^{\pi} \Phi(\gamma) \sum_{k=0}^{n} (2k+1)\, P_k(\cos\gamma) \sin\gamma \, d\gamma$$

or

$$S_n = \frac{1}{2}\int_{-1}^{1} \Psi(x) \sum_{k=0}^{n} (2k+1)\, P_k(x)\, dx,$$

i.e. from (41)

$$S_n = \frac{1}{2}\int_{-1}^{1} \Psi(x) \left(\frac{dP_{n+1}(x)}{dx} + \frac{dP_n(x)}{dx}\right) dx.$$

We assume that the function $f(\theta, \varphi)$ is such that $\psi(x)$ has a continuous derivative in the interval $(-1, +1)$. Integrating by parts we have

$$S_n = \frac{1}{2}\left[\Psi(x)\left(P_{n+1}(x) + P_n(x)\right)\right]_{x=-1}^{x=1} - \frac{1}{2}\int_{-1}^{1} \left[P_{n+1}(x) + P_n(x)\right] \Psi'(x)\, dx,$$

or bearing in mind that

$$P_n(1) = P_{n+1}(1) = 1\,; \quad P_n(-1) = -P_{n+1}(1) = (-1)^n,$$

we have

$$S_n = \Psi(1) - \frac{1}{2}\int_{-1}^{1} \left[P_{n+1}(x) + P_n(x)\right] \Psi'(x)\, dx. \tag{64}$$

Let us now explain the meaning of the first term $\Psi(1)$ on the right-hand side. We have from (62) and (63)

$$\Psi(1) = \frac{1}{2\pi}\int_0^{2\pi} F(0, \beta)\, d\beta. \tag{65}$$

But, when $\gamma = 0$ and β is arbitrary, this point is the North pole of the sphere or, which comes to the same thing, the point with the former geographical coordinates θ and φ. In other words $F(0, \beta) = f(\theta, \varphi)$ does not depend on β and formula (65) gives

$$\Psi(1) = f(\theta, \varphi).$$

We can therefore rewrite the formula (61) in the form

$$S_n = f(\theta, \varphi) - \frac{1}{2}\int_{-1}^{1} \left[P_{n+1}(x) + P_n\right] \Psi'(x)\, dx. \tag{66}$$

We have to prove that

$$\lim_{n\to\infty} S_n = f(\theta, \varphi),$$

i.e. we have to prove that the integral in the formula (66) tends to zero as n increases indefinitely. Let M be the maximum value of the continuous function $\Psi'(x)$ in the interval $(-1, +1)$. The modulus of the above integral will be smaller than the following expression:

$$\frac{M}{2}\int_{-1}^{1} |P_{n+1}(x)|\,dx + \frac{M}{2}\int_{-1}^{1} |P_n(x)|\,dx.$$

We therefore only have to show that the integral

$$\int_{-1}^{1} |P_n(x)|\,dx \tag{67}$$

tends to zero as n increases. Using Buniakowski's inequality [III$_1$, 29], we have:

$$\left(\int_{-1}^{1} |P_n(x)|\,dx\right)^2 \leqslant \int_{-1}^{1} P_n^2(x)\,dx \int_{-1}^{1} 1^2\,dx = 2\int_{-1}^{1} P_n^2(x)\,dx$$

or from (19):

$$\int_{-1}^{1} |P_n(x)|\,dx < \frac{2}{\sqrt{2n+1}},$$

from whence it follows that the integral (67) tends to zero as $n \to \infty$.

The above method of proof of the expansion theorem in terms of spherical functions is taken from the book *Differential Equations in Partial Derivatives in Mathematical Physics* by Webster–Sage. The fact that an arbitrary function which satisfies the above general conditions [$\Psi(x)$ has a continuous derivative] can be expanded in terms of spherical functions shows that spherical functions form a closed system [II, 155] on the surface of a unit sphere. The fact that this system is closed was first proved by A. M. Liapunov (1899).

135. The connection between spherical functions and limit problems. We will now show the connection between the theory of spherical functions and certain limit problems of differential equations. Let us write the Laplace equation in spherical coordinates [II, 119]:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2} = 0. \tag{68}$$

We will seek its solution in the form of a product of functions of r, and θ and φ respectively:

$$U = f(r)\,Y(\theta, \varphi).$$

Substituting in the equation (68)

$$Y(\theta, \varphi)\frac{d}{dr}\left[r^2 f'(r)\right] + f(r)\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right] + \right.$$
$$\left. + \frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2}\right\} = 0;$$

on separating the variables this can be rewritten as follows

$$\frac{\frac{d}{dr}\left[r^2 f'(r)\right]}{f(r)} = -\frac{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}}{Y}.$$

The left-hand side contains only the variable r, the right-hand side only θ and φ and both sides will therefore be equal to one and the same constant. Denoting this constant by λ we obtain two equations:

$$r^2 f''(r) + 2rf'(r) - \lambda f(r) = 0 \tag{69}$$

and

$$\Delta_1 Y + \lambda Y = 0, \tag{70}$$

where, for the sake of briefness, we put

$$\Delta_1 Y = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right]. \tag{71}$$

The function $f(r)$ we know already, viz. from (5) it should be equal to r^n and therefore we will pay greater attention to equation (70). The function $Y(\theta, \varphi)$, as we have seen above, is a trigonometric polynomial and, consequently, it must be finite and continuous on the whole sphere, i.e. for any choice of the angles θ and φ and, in particular, wheni $\theta = 0$ and $\theta = \pi$ and $\sin\theta$ vanishes. We thus come to the followng limit problem: *find values of the parameter λ, for which the equation* (70) *has continuous solutions on the whole unit sphere and construct these solutions.* The first part of the problem presents no difficulties for we know that $f(r)$ is equal to r^n and, substituting this in the equation (69), we obtain an infinite number of solutions for the parameter λ viz.:

$$\lambda_n = n(n+1) \qquad (n = 0, 1, 2, \ldots). \tag{72}$$

At the same time the equation

$$r^2 f_n''(r) + 2rf_n'(r) - n(n+1) f_n(r) = 0 \tag{73}$$

will have one solution $f_n(r) = r^n$ and another solution $f_n(r) = r^{-n-1}$. Substituting $\lambda = n(n+1)$ in the equation (70) we obtain an equation

for spherical functions:

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y_n}{\partial\theta}\right)+\frac{1}{\sin\theta}\,\frac{\partial^2 Y_n}{\partial\varphi^2}\right]+n(n+1)\,Y_n=0. \qquad (74)$$

In this case the individual value of $\lambda_n = n(n+1)$ corresponds to $(2n+1)$ individual functions. These will be spherical functions of order n. Owing to the fact that spherical functions form a closed system on a unit sphere these functions give all the values of the individual functions in the equation (70). Substituting the expressions (16) in the equation (74) and putting $x = \cos\theta$ we obtain for $P_{n,m}(x)$ the following differential equation of the second order:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\,\frac{\mathrm{d}P_{n,m}(x)}{\mathrm{d}x}\right]+\left(\lambda_n-\frac{m^2}{1-x^2}\right)P_{n,m}(x)=0\,; \qquad (75)$$

when $m = 0$ an equation for the Legendre polynomials $P_n(x)$ is obtained. The individual values and the corresponding individual functions $P_{n,m}(x)$ solve the following limit problem. Find values of λ_n for which the solution of the equation (75) remains finite in the interval $-1 \leqslant x \leqslant +1$ including its ends. Notice that at the singularities $x = \pm 1$ the equation (75) has a determining equation $\varrho(\varrho-1)+\varrho-m^2/4=0$ the zeros of which are $\varrho = \pm m/2$.

The solution which corresponds to the zero $\varrho = -m/2$ becomes infinite at the corresponding singularity.

The above problem involves the finding of those values λ_n for which the solution corresponding to the zero $\varrho = m/2$ at the point $x = -1$ will still correspond to this zero at the point $x = +1$.

The values $\lambda_n = n(n+1)$ give the solutions of this problem and the corresponding individual functions are determined from formula (12).

The orthogonality of spherical functions is directly connected with the fact that these functions solve the above limit problem for the equation (70). Similarly the functions $P_{n,m}(x)$ are orthogonal on the line $(-1, +1)$:

$$\int_{-1}^{1} P_{p,m}(x)\,P_{q,m}(x)\,\mathrm{d}x=0 \qquad \text{when } p\neq q. \qquad (76)$$

This can be proved by using the equation (75) in exactly the same way as we did in [102] for the Legendre polynomials. Notice also one other fact connected with the theory of spherical functions. If we use the solution $f_n(r) = r^n$ of the equation (73) we obtain the solu-

tion $r^n Y_n(\theta, \varphi)$ of the Laplace equation. This is a harmonic polynomial of the nth degree. If we use the second solution $f_n(r) = r^{-n-1}$ of the equation (73) we arrive at the following conclusion: *the function*

$$\frac{Y_n(\theta, \varphi)}{r^{n+1}}, \tag{77}$$

where $Y_n(\theta, \varphi)$ is a spherical function of order n, is a solution of the Laplace equation. This solution becomes infinite when $r = 0$ and it is obviously not a polynomial in x, y, z.

136. The Dirichlet and Neumann problems. Spherical functions are used in problems of mathematical physics connected with the Laplace equation with reference to a sphere. As an example consider the Dirichlet and Neumann problems which we mentioned earlier [II, 192] in connection with a sphere. It is necessary to determine a harmonic function inside a sphere of radius R when its limit values are given on the surface of that sphere (the inside Dirichlet problem). We expand the given limit values in terms of spherical functions:

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi), \tag{78}$$

and construct a new series by multiplying the nth term of the above series by $(r/R)^n$, where r is the distance of a variable point from the centre of the sphere:

$$U(r, \theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi) \left(\frac{r}{R}\right)^n \quad (r < R). \tag{79}$$

Bearing in mind that $(1/R^n) Y^n(\theta, \varphi)\, r^n$ is a harmonic polynomial we can see that the function (79) is harmonic inside the sphere and also that, when $r = R$, the series (79) becomes the series (78) so that this harmonic function satisfies the necessary limit conditions.

Consider now the *exterior Dirichlet problem*, i.e. assume that it is necessary to determine a function, harmonic outside a sphere, which becomes zero at infinity [II, 192], from its limit values (78) on the surface of the sphere. Bearing in mind that $Y_n(\theta, \varphi)\, r^{-n-1}$ are harmonic functions which have no singularities outside the sphere and which are zero at infinity we obtain the solution of the exterior Dirichlet problem in the form:

$$U(r, \theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi) \left(\frac{R}{r}\right)^{n+1}. \tag{80}$$

We shall now consider the *interior Neumann problem*. Suppose that it is necessary to determine a harmonic function $U(r, \theta, \varphi)$ inside a sphere from the values of its normal derivative on the surface of the sphere

$$\frac{\partial U}{\partial \nu} = f(\theta, \varphi) \qquad (r = R). \tag{81}$$

We know that the integral of the normal derivative of a harmonic function vanishes [II, 194]:

$$\int\int_S \frac{\partial U}{\partial \nu} \, d\sigma = 0,$$

i.e. the given function $f(\theta, \varphi)$ which appears in the condition (81) must be such that

$$\int\int_S f(\theta, \varphi) \, d\sigma = 0. \tag{82}$$

On recalling the formula (57) which determined the spherical functions obtained in the expansion of $f(\theta, \varphi)$ and also that $P_n(\cos \gamma)$ is constant when $n = 0$, we can see that the condition (82) is equivalent to the fact that in the expansion of $f(\theta, \varphi)$ in terms of spherical functions all spherical functions of zero order are absent. We therefore have

$$f(\theta, \varphi) = \sum_{n=1}^{\infty} Y_n(\theta, \varphi). \tag{83}$$

It can readily be seen that the solution of the Neumann problem will be given by the following formula:

$$U(r, \theta, \varphi) = \sum_{n=1}^{\infty}{}' \frac{1}{n} Y_n(\theta, \varphi) \frac{r^n}{R^{n-1}} + C, \tag{84}$$

where C is an arbitrary constant.

In fact this series determines a harmonic function and differentiation along the normal coincides, in this case, with differentiation along the radius r. It can easily be proved that, by differentiating the series (84) with respect to r and putting $r = R$, we can obtain the series (83), i.e. the limit condition (81) will be satisfied. In the case of the *exterior Neumann problem* the function $f(\theta, \varphi)$, which appears in the condition (81) will no longer satisfy the condition (82) and we have an expansion of the general form (78) for it. It can readily be seen that

the solution of the exterior Neumann problem will be given in the form of the series

$$U(r, \theta, \varphi) = -\sum_{n=0}^{\infty} \frac{1}{n+1} Y_n(\theta, \varphi) \frac{R^{n+2}}{r^{n+1}}, \tag{85}$$

where we assume that the direction of the normal ν coincides with the direction of the radius r.

We will now consider one special case of the exterior Neumann problem. Let us suppose that a sphere of radius R moves in a limitless fluid resting at infinity with a velocity a directed along the Z axis. Take a system of coordinates with the origin as the centre of the sphere which moves with the sphere. In this case the normal component velocity of the fluid with the surface of the sphere will be given by the formula:

$$\frac{az}{r} = a \cos \theta.$$

If we suppose that the fluid is stationary and has a potential velocity, we have a problem in which the function U is to be found from the following conditions: (1) outside the sphere U must be a harmonic function, (2) at infinity the component velocities, i.e. the derivatives of the function U with respect to the coordinates, must vanish and (3) on the surface of the sphere the function U must satisfy the condition

$$\frac{\partial U}{\partial r} = -a \cos \theta.$$

In this case $f(\theta, \varphi) = -a \cos \theta$, or, remembering the expression for the Legendre polynomial we have

$$f(\theta, \varphi) = -aP_1(\cos \theta),$$

i.e. the function $f(\theta, \varphi)$ is a spherical function of the first order. The solution of the problem will be given by the formula

$$U(r, \theta, \varphi) = \frac{a}{2} P_1(\cos \theta) \frac{R^3}{r^2} = \frac{aR^3}{2r^2} \cos \theta.$$

137. The potential of voluminous masses. Let us suppose there is a bounded volume V in space which is filled with a mass of density $\varrho(M')$. The potential of this distribution will be expressed by the treble integral

$$U(M) = \iiint_V \frac{\varrho(M')}{d} \, dV, \tag{86}$$

where d is the distance from the variable point M' of the volume V to the point M at which the value of the potential is being determined. Let O be the origin and we will introduce into our considerations the lengths of the radius-vectors

$$r = |\overline{OM}|, \quad r' = |\overline{OM'}|$$

and the angle γ made by these vectors. We now consider points M at a sufficiently great distance so that the value of r is greater than the maximum value of r'. For these points we have the following expansion [132]:

$$\frac{1}{d} = \frac{1}{\sqrt{r^2 - 2rr'\cos\gamma + r'^2}} = \sum_{n=0}^{\infty} P_n(\cos\gamma)\frac{r'^n}{r^{n+1}},$$

which converges uniformly with respect to r' since $|P_n(\cos\gamma)| < 1$. Substituting this in the integral (86) we obtain the expansion of the potential $U(M)$ in negative integral powers of r:

$$U(M) = \sum_{n=0}^{\infty} \frac{Y_n(\theta, \varphi)}{r^{n+1}}, \tag{87}$$

where

$$Y_n(\theta, \varphi) = \iiint_V \varrho(M')\, r'^n P_n(\cos\gamma)\, dV. \tag{88}$$

We will now determine the first three terms in the expansion (87). On recalling the expression for the first three Legendre polynomials and also the self explanatory formula:

$$\cos\gamma = \frac{xx' + yy' + zz'}{rr'}.$$

we can write

$$P_0(\cos\gamma) = 1; \quad r' P_1(\cos\gamma) = \frac{xx' + yy' + zz'}{r};$$

$$r'^2 P_2(\cos\gamma) = \frac{1}{2}\frac{3(xx' + yy' + zz')^2 - r'^2 r^2}{r^2}.$$

Substituting these in the formula (88) we have first of all

$$Y_0(\theta, \varphi) = \iiint_V \varrho(M')\, dV = m,$$

i.e. the coefficient of $1/r$ in the expansion (87) is equal to the total mass m of the volume V. We obtain further:

$$Y_1(\theta, \varphi) = \iiint_V \varrho(M')\, r' P_1(\cos\gamma)\, dV =$$

$$= \frac{x}{r}\iiint_V \varrho(M')\, x'\, dV + \frac{y}{r}\iiint_V \varrho(M')\, y'\, dV + \frac{z}{r}\iiint_V \varrho(M')\, z'\, dV.$$

The above integrals express the product of the mass m and the coordinates of the centre of gravity. We shall assume that the origin is so chosen as to coincide with the centre of gravity of the mass. In this case we shall obviously have $Y_1(\theta, \varphi) = 0$. Lastly we shall evaluate $Y_2(\theta, \varphi)$. To do so we introduce the moments of inertia of our mass with respect to the axes:

$$A = \iiint_V \varrho(M')(y'^2 + z'^2)\, dV; \quad B = \iiint_V \varrho(M')(z'^2 + x'^2)\, dV;$$

$$C = \iiint_V \varrho(M')(x'^2 + y'^2)\, dV, \tag{89}$$

and also the products of inertia with respect to the axes

$$D = \iiint_V \varrho(M')\, y' z'\, \mathrm{d}V; \quad E = \iiint_V \varrho(M')\, z' x'\, \mathrm{d}V;$$

$$F = \iiint_V \varrho(M')\, x' y'\, \mathrm{d}V. \tag{90}$$

It can be shown, but we shall not do so here, that the system of coordinates can always be chosen so that the products of inertia (90) vanish. We will assume that the coordinates have been chosen in this way. Substituting the expression $r'^2 P_2(\cos\gamma)$ in the formula (88) we can see, as can easily be shown, that the following expression for $Y_2(\theta, \varphi)$ is obtained:

$$Y_2(\theta, \varphi) = \frac{1}{2} \frac{(B + C - 2A)\, x^2 + (C + A - 2B)\, y^2 + (A + B - 2C)\, z^2}{r^2},$$

and for the potential $U(M)$ we have up to terms of the order of $1/r^3$

$$U(M) =$$

$$= \frac{m}{r} + \frac{1}{2} \frac{(B + C - 2A)\, x^2 + (C + A - 2B)\, y^2 + (A + B - 2C)\, z^2}{r^5} + \dots \tag{91}$$

Replacing x, y and z by spherical polar coordinates we can rewrite this expression as follows:

$$U(M) = \frac{m}{r} +$$

$$+ \frac{1}{2} \frac{(B+C-2A)\cos^2\varphi \sin^2\theta + (C+A-2B)\sin^2\varphi \sin^2\theta + (A+B-2C)\cos^2\theta}{r^3} + \dots \tag{92}$$

138. The potential of a spherical shell. Suppose that on the surface S_R of a sphere of radius R a mass with surface density $\varrho(M')$ is distributed. The potential $U(M)$ of this simple layer will be expressed on the surface of the sphere by the integral

$$U(M) = \iint_{S_R} \frac{\varrho(M')}{d}\, \mathrm{d}s, \tag{93}$$

where d is the distance from the point M to a variable point M' on the surface of the sphere. The expression $1/d$ will have different forms inside and outside the sphere S_R.

If we assume, to start with, that $r < R$, we obtain [132]:

$$\frac{1}{d} = \sum_{n=1}^{\infty} P_n(\cos\varphi) \frac{r^n}{R^{n+1}}, \tag{94}$$

where γ is the angle between the radius-vectors $\overline{OM}$ and $\overline{OM'}$. The density $\varrho(M')$ is assumed to be a given function $f(\theta', \varphi')$ of the geographical coordinates of the sphere.

Substituting the expansion (94) in the integral (93) and remembering that $ds = R^2 d\sigma = R^2 \sin\theta' \, dd\varphi'\theta'$ we have:

$$U(M) = \sum_{n=0}^{\infty} \frac{r^n}{R^{n-1}} \int\int_s f(\theta', \varphi') P_n(\cos\gamma) \, d\sigma. \tag{95}$$

The above integrals are directly connected with the expansion of the function $f(\theta, \varphi)$ in terms of spherical functions, viz. if

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi), \tag{96}$$

then, as we know

$$Y_n(\theta, \varphi) = \frac{2n+1}{4\pi} \int\int_s f(\theta', \varphi') P_n(\cos\gamma) \, d\sigma,$$

and, consequently, the expansion (95) can be written as follows:

$$U(M) = 4\pi \sum_{n=0}^{\infty} \frac{r^n}{(2n+1) R^{n-1}} Y_n(\theta, \varphi) \quad (r < R). \tag{97}$$

Similarly, using the expansion (36) we have

$$U(M) = 4\pi \sum_{n=0}^{\infty} \frac{R^{n+2}}{(2n+1) r^{n+1}} Y_n(\theta, \varphi) \quad (r > R). \tag{98}$$

By using this expansion we can note certain properties of the potential of a layer. Notice, first of all, that the expansions (97) and (98) are the same when the point M falls on the surface of the sphere. In this case we put $r = R$ and obtain the following result:

$$U(M_0) = 4\pi R \sum_{n=0}^{\infty} \frac{1}{2n+1} Y_n(\theta, \varphi), \tag{99}$$

where θ and φ are the geographical coordinates of the point M_0 on the surface of the sphere. We can thus see that *the potential of a simple layer varies continuously as the point M passes through the surface of the sphere.* This property of the potential of a simple layer holds not only for a sphere but for surfaces generally.

Let us now investigate the behaviour of the normal derivative of the potential (the normal component of the force) when the point M moves across the surface of the sphere. We denote by $(\partial U(M_0)/\partial \nu)_i$ the limit of the normal derivative as the point M tends to the point M_0 along the radius from inside the sphere, and by $(\partial U(M_0)/\partial \nu)_e$ that same limit as the point M tends to the same point M_0 from outside the sphere. ν denotes the direction of the outward normal to the sphere at the point M_0. In this case this direction coincides with the radius $\overline{OM}_0$. Differentiating the formulae (97) and (98) with respect to ν, i.e. with respect to r, and putting $r = R$ we obtain the expressions for the above limits:

$$\left(\frac{\partial U(M_0)}{\partial \nu}\right)_i = 4\pi \sum_{n=1}^{\infty} \frac{n}{2n+1} Y_n(\theta, \varphi), \tag{100}$$

$$\left(\frac{\partial U(M_0)}{\partial \nu}\right)_e = -4\pi \sum_{n=0}^{\infty} \frac{n+1}{2n+1} Y_n(\theta, \varphi). \tag{101}$$

This shows that the normal derivative of the potential of a simple layer has, in general, a discontinuity on passing through the surface.

The following formulae follow directly from the formulae (100) and (101):

$$\left(\frac{\partial U(M_0)}{\partial \nu}\right)_e - \left(\frac{\partial U(M_0)}{\partial \nu}\right)_i = -4\pi \sum_{n=0}^{\infty} Y_n(\theta, \varphi),$$

$$\left(\frac{\partial U(M_0)}{\partial \nu}\right)_e + \left(\frac{\partial U(M_0)}{\partial \nu}\right)_i = -4\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} Y_n(\theta, \varphi),$$

and from (96) and (99) we can write:

$$\left(\frac{\partial U(M_0)}{\partial \nu}\right)_e - \left(\frac{\partial U(M_0)}{\partial \nu}\right)_i = -4\pi \varrho(M_0), \tag{102}$$

$$\left(\frac{\partial U(M_0)}{\partial \nu}\right)_e + \left(\frac{\partial U(M_0)}{\partial \nu}\right)_i = -\frac{U(M_0)}{R}. \tag{103}$$

Formula (102) shows, among other things, that *the discontinuity in the normal derivative is equal to the product of* (-4π) *and the density at the given point on the surface.*

Let us now explain the meaning of the right-hand side of the formula (103). Denoting, as before, a definite direction by ν, viz. the direction of the radius OM_0, and bearing in mind that in the integral

(93) only the factor $1/d$ depends on the coordinates of the point M, we have

$$\frac{\partial U(M)}{\partial \nu} = \int\int_{S_E} \varrho(M') \frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{d}\right) \mathrm{d}s. \tag{104}$$

But we have also:

$$\frac{\partial}{\partial \nu}\left(\frac{1}{d}\right) = -\frac{1}{d^2}\cos\omega,$$

where ω is the angle made by the radius-vector $\overline{M'}\,\overline{M}$ and the direction ν. Let us evaluate the integral (104) on the assumption that the point M lies on the sphere, viz. at the point M_0. In this case we have $d = 2R\cos\omega$ and, consequently:

$$\frac{\partial}{\partial \nu}\left(\frac{1}{d}\right) = -\frac{1}{2Rd}.$$

We thus obtain the following expression for the integral (104):

$$-\frac{1}{2R}\int\int_{S_E} \varrho(M')\frac{1}{d}\,\mathrm{d}s = -\frac{1}{2R}U(M_0).$$

Let us denote this by $\partial U(M_0)/\partial \nu$. We can then rewrite the formula (103) as follows:

$$\left(\frac{\partial U(M_0)}{\partial \nu}\right)_e + \left(\frac{\partial U(M_0)}{\partial \nu}\right)_i = 2\frac{\partial U(M_0)}{\partial \nu}.$$

From this follow the expressions for the limits of the normal derivative of the potential of a simple layer which are given below:

$$\left.\begin{aligned}\left(\frac{\partial U(M_0)}{\partial \nu}\right)_i &= \frac{\partial U(M_0)}{\partial \nu} + 2\pi\varrho(M_0),\\ \left(\frac{\partial U(M_0)}{\partial \nu}\right)_e &= \frac{\partial U(M_0)}{\partial \nu} - 2\pi\varrho(M_0).\end{aligned}\right\} \tag{105}$$

These formulae are also valid for bodies other than spheres.

139. The electron in a central field. When dealing with an electron in a field created by a positive nucleus we have, according to Schrödinger's theorem, the following equation:

$$-\frac{h^2}{2\mu}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - eV(r)\psi = E\psi, \tag{106}$$

where h is Planck's constant, μ is the mass of the electron and e is its charge. $V(r)$ is the given function which depends only on the distance r from the origin, and determines the potential of the field, $\psi(x, y, z)$ is the wave function and, finally, E is a constant which determines the energy level of the given

physical system. The solution of the equation (106) must be finite in the whole infinite space and remain bounded at infinity. We shall seek this solution in the form of a product of functions of r multiplied by functions which depend only on θ and φ. Expressing the Laplace operator $\Delta\psi$ in terms of spherical coordinates we can write:

$$\Delta\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\Delta_1\psi,$$

where, as before [135]:

$$\Delta_1\psi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}.$$

The equation (106) can be rewritten in the form

$$\frac{h^2}{2\mu}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\Delta_1\gamma\right] + eV(r)\psi + E\psi = 0.$$

Writing the expression $\psi = f(r)\,Y(\theta, \varphi)$ and separating the variables we have:

$$\frac{\Delta_1 Y}{Y} = \frac{-\frac{h^2}{2\mu}\left[f''(r) + \frac{2}{r}f'(r)\right] - eV(r)f(r) - Ef(r)}{\frac{h^2}{2\mu r^2}f(r)}.$$

Both sides of this equation will be equal to one and the same constant which we denote by λ. This gives the equations

$$\Delta_1 Y - \lambda Y = 0, \tag{107}$$

$$-\frac{h^2}{2\mu}\left[f''(r) + \frac{2}{r}f'(r) + \frac{\lambda}{r^2}f(r)\right] - eV(r)f(r) - Ef(r) = 0. \tag{108}$$

The equation (107) should have a continuous solution on the whole surface of the sphere. We know already that in this case the parameter λ will be equal to $-l(l+1)$ and the solution will be expressed in terms of spherical functions $Y_l(\theta, \varphi)$. Substituting the above value of λ in the equation (108) we obtain an equation for the determination of the function of r which we now denote by $f_1(r)$:

$$\frac{h^2}{2\mu}f_l''(r) + \frac{h^2}{\mu r}f_l'(r) + \left[E + eV(r) - \frac{h^2 l(l+1)}{2\mu r^2}\right]f_l(r) = 0. \tag{109}$$

The values of the parameter E are determined from the condition that the solution of the equation (109) is bounded when $r = 0$, as well as when r tends to $+\infty$. In general, we can obtain an infinite number of such solutions. They are usually numbered starting with the integer $(l+1)$, i.e. they are numbered in the following order:

$$n = l+1, \quad l+2, \quad l+3, \quad \ldots$$

Hence the values of E depend on two symbols, viz. on the integer l and on the number n. l is known as *azimuthal quantum number* and n as the *principal quantum number*. When l and n are given we have, in general, a definite function

$f_{nl}(r)$ which satisfies the equation (109) as well as the above limit conditions when $r = 0$ and $r = +\infty$. As far as the functions $Y_l(\theta, \varphi)$ are concerned, there will be $(2l + 1)$ of these:

$$Y_l^{(m)}(\theta, \varphi) \qquad (m = -l,\ -l+1,\ \ldots,\ l-1,\ l)$$

and for the full definition of the wave function we must also determine the value of the third symbol m. This is usually known as the *magnetic quantum number*. It is of importance in disturbance problems of a given physical system caused by a magnetic field acting along the Z-axis.

Let us now consider the particular case when the potential is the Coulomb potential

$$V(r) = \frac{ke}{r},$$

where k is a constant which is equal, for example in the case of a hydrogen atom, to unity. Substituting the expression of the potential in the equation (109) we obtain the following equation ($k = 1$):

$$\frac{h^2}{2\mu} f_l''(r) + \frac{h^2}{\mu r} f_l'(r) + \left[E + \frac{e^2}{r} - \frac{h^2 l(l+1)}{2\mu r^2}\right] f_l(r) = 0. \tag{110}$$

We replace r by a new variable z:

$$z = \frac{\mu e^2 r}{h^2}$$

and put

$$\varepsilon = \frac{Eh^2}{\mu e^4} \quad \text{and} \quad s = 2l + 1. \tag{111}$$

We also replace $f_l(r)$ by a new unknown function y:

$$f_l(r) = \frac{1}{\sqrt{z}} y.$$

Substituting all this in the equation (110) we obtain the equation

$$\frac{d^2 y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(2\varepsilon + \frac{2}{z} - \frac{s^2}{4z^2}\right) y = 0,$$

which we considered in [115].

Let us now consider the negative values of the parameter E. In this case, as before, we obtain an infinite number of discreet values for the constant E, viz. having put

$$\lambda = \frac{1}{\sqrt{-2\varepsilon}},$$

we obtain the following value for the parameter λ:

$$\lambda_p = \frac{s+1}{2} + p \qquad (p = 0, 1, 2, \ldots),$$

whence

$$\frac{1}{-2\varepsilon_p} = \left(\frac{s+1}{2} + p\right)^2 \quad \text{and} \quad \varepsilon_p = -\frac{1}{\left(\frac{s+1}{2} + p\right)^2} = -\frac{1}{2(p + l + 1)^2},$$

and, consequently, from (111) we obtain the following values for the parameter E:

$$E_{nl} = -\frac{\mu e^4}{2h^2 (p + l + 1)^2} = -\frac{\mu e^4}{2h^2 n^2}, \tag{112}$$

where n is the principal quantum number equal to $(p + l + 1)$.

We thus see that in the case of a Coulomb field the values of the parameter E are independent of the azimuthal quantum number l. If we fix n, and therefore also the value of the parameter E then, since $n = p + l + 1$, we can give l the following values:

$$l = n - 1, \quad n - 2, \quad \ldots, \quad 0.$$

Every value of l corresponds to $(2l + 1)$ natural functions of φ. Therefore for the parameter E, as given by (112) we obtain the following total number of natural functions:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

If, instead of Schrödinger's equation we had taken the Dirac equation for one electron then we would have obtained functions analogous with spherical functions. These "Spinning spherical functions" are dealt with in the book *The Origins of Quantum Mechanics* by Prof. V. A. Foch.

140. Spherical functions and the linear representation of rotating groups. We have already mentioned the fact that homogeneous polynomials of the variables (x, y, z) which satisfy the Laplace equation give a linear expression for a group R of the space rotating about the origin.

We thus see that a set of spherical functions of order l gives a linear expression for the group R which is of the $(2l + 1)$th order. Let us consider this problem in greater detail.

Let us consider spherical functions of order l in the same form as given by the formulae (18) and introduce for them a special notation:

$$Q_l^{(m)}(\varphi, \theta) = e^{im\varphi} P_{l,m}(\cos\theta) \quad (m = -l, \ -l+1, \ \ldots, \ l-1, \ l), \tag{113}$$

where

$$P_{l,-m}(\cos\theta) = P_l^{(m)}(\cos\theta)$$

Let $\{\alpha, \beta, \gamma\}$ be a certain rotation of the group R with Euler angles α, β and γ. As a result of this rotation a point on the sphere with coordinates (φ, θ) will move into a new position (φ', θ') and the function $Q_l^{(m)}(\varphi', \theta')$ can be expressed linearly in terms of $Q_l^{(m)}(\varphi, \theta)$. The matrix of this linear transformation will correspond to the rotation $\{\alpha, \beta, \gamma\}$ of that linear expression for the group R, which is given by the functions (113). The simple dependency of these functions on the angle φ shows that a rotation about the Z axis through an angle α, i.e. the rotation $\{\alpha, 0, 0\}$, corresponds to the diagonal matrix

$$\begin{Vmatrix} e^{-il\alpha}, & 0, & 0, & \ldots, 0 \\ 0, & e^{-i(l-1)\alpha}, & 0, & \ldots, 0 \\ 0, & 0, & e^{-i(l-2)\alpha} & \ldots, 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & \ldots, e^{il\alpha} \end{Vmatrix} \tag{114}$$

Denote by $\{D_l(R_0)\}_{ik}$ the elements of the matrix corresponding to a definite rotation R_0, where the symbols i and k take the values $-l, -l+1, \ldots, l$.

Take for R_0 the rotation about the Y-axis through an angle β, as a result of which points on the sphere with coordinates $\varphi = 0$ and θ move to points $\varphi' = 0$ and $\theta + \beta$ respectively. Bearing in mind the form of the functions (113) we can say that for the given choice of R_0 the matrix $D_l(R_0)$ transforms the functions $P_l^{(m)}(\cos\theta)$ into the functions $P_{l,m}[\cos(\theta+\beta)]$, i.e.

$$P_{l,m}[\cos(\theta+\beta)] = \sum_{s=-l}^{l} \{D_l(R_0)\}_{ms} P_{l,s}(\cos\theta) \qquad (m = -l\ -l+1,\ \ldots,\ l).$$

Returning to the formulae (12) we can see that $P_l^{(s)}(\cos\theta)$ vanishes when $\theta = 0$ if $s \neq 0$. Putting $\theta = 0$ in the above formulae we have

$$P_{l,m}(\cos\beta) = \{D_l(R_0)\}_{m0} P_l(1) = \{D_l(R_0)\}_{m0}.$$

This shows that the elements of the column $k = 0$ of the matrix $D_l(R_0)$ are, in general, other than zero, i.e. they contain zero elements only for exceptional values of β.

Thus among the matrices $D_l(R)$ are the diagonal matrices (114) with different elements, as well as matrices, all elements of a certain column of which are other than zero. As we saw in [III$_1$, 69] the matrices give, in this case, an irreducible expression for the group R and we can therefore say that the expression given by the matrices $D_l(R)$ is irreducible. The functions (113) are orthogonal in pairs but they were not compared to unity for integrals of the square of the modulus. Ascribing to these functions suitably chosen constant factors, we can construct comparable functions:

$$C_l^{(m)} Q_l^{(m)}(\varphi, \theta). \tag{115}$$

These functions now give the unitary irreducible expression for the functions $D_l(R)$ [III$_1$, 63] which is equivalent to $D_l(R)$ and in this new expression the rotation $\{\alpha, 0, 0\}$ corresponds to the old matrix (114) since the constant factors we have chosen do not change the character of the dependency of the functions (113) on φ.

Multiplying the functions (115) by arbitrary factors the moduli of which are unity we also obtain a unitary expression with the same matrix (114) which corresponds to the rotation $\{\alpha, 0, 0\}$. One of these representations is exactly the same as that which we constructed in a different way [III$_1$, 62].

The individual functions of the Schrödinger equation which we considered in the previous section, fall into groups in accordance with the values of l, and each group contains $(2l+1)$ individual functions (l being the azimuth quantum number). The functions which form such a group are numbered by m, (m being the magnetic quantum number) which runs through $m = -l, -l+1, \ldots, l$. It follows directly from the form of the functions (113) that

$$\frac{1}{i}\frac{\partial}{\partial\varphi} Q_l^{(m)}(\varphi, \theta) = m Q_l^{(m)}(\varphi, \theta),$$

i.e. the mth function of our group is the individual function of the operator

$$L_z = \frac{1}{i}\frac{\partial}{\partial \varphi}, \tag{116}$$

and m is the corresponding individual value. Also each function (113), as we know, satisfies the equation:

$$-\Delta_1 Q_l^{(m)}(\varphi, \theta) = l(l+1) Q_l^{(m)}(\varphi, \theta),$$

i.e. each function belonging to the above group of $(2l+1)$ functions is an individual function of the operator

$$L^2 = -\Delta_1 = -\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right], \tag{117}$$

and the corresponding individual value is equal to $l(l+1)$. The operator L_z differs only by the factor h from the operator of the component of the quantity of movement on the Z-axis. Similarly, the operator (117) differs only by the factor h^2 from the operator of the square of the moment of the quantity of movement.

141. The Legendre function. Let us consider the Legendre equation:

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + n(n+1)u = 0 \tag{118}$$

where we assume that x is a complex variable and where n can be any number. Both zeros of the determining equation (118) are zero at the singularities $x = \pm 1$ [102]. Hence at both these points we have one regular solution and one solution containing a logarithm; this latter solution will not be bounded in the neighbourhood of the singularity.

We shall try to satisfy the equation (118) by an integral of the form (13) which is a Legendre polynomial when n is a positive integer:

$$u(x) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(t^2-1)^n}{(t-x)^{n+1}}\,dt. \tag{119}$$

Substituting in the equation (118) we have:

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + n(n+1)u =$$

$$= \frac{n+1}{2^{n+1}\pi i}\int_C \frac{(t^2-1)^n}{(t-x)^{n+3}}\left[-(n+2)(t^2-1) + 2(n+1)t(t-x)\right]dt =$$

$$= \frac{n+1}{2^{n+1}\pi i}\int_C \frac{d}{dt}\left[\frac{(t^2-1)^{n+1}}{(t-x)^{n+2}}\right]dt,$$

which shows that formula (119) is a solution of the equation (118) provided that by describing the variable point t round the contour C, the expression

$$\frac{(t^2-1)^{n+1}}{(t-x)^{n+2}} \tag{120}$$

returns to its initial value. When n is a fraction then the integrand in (119) has three branch points: $t = x$ and $t = \pm 1$. By describing the point $t = 1$ or $t = -1$ in the counter-clockwise direction the numerator $(t^2-1)^{n+1}$ acquires the factor $e^{(2n+1)2\pi i}$ and by describing the point $t = x$ the denominator acquires the term $e^{(2n+1)2\pi i}$.

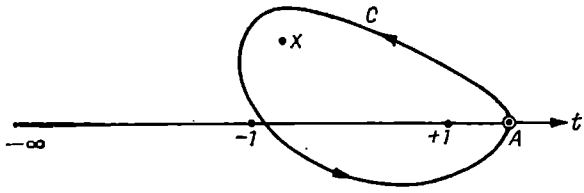

FIG. 73.

Let us cut the t-plane from $t = -1$ to $t = -\infty$ along the real axis and take as the contour C a closed contour originating at a point A on the real axis which lies to the right of the point $t = 1$ and round which the points $t = 1$ and $t = x$ are described in the counter clockwise direction (Fig. 73).

We assume that x does not lie on the cut and that the contour C does not intersect the cut. The original value of the many-valued integrand function is determined from the conditions that $\arg(t-1) = \arg(t+1) = 0$ and $|\arg(t-x)| < \pi$, when $t > 1$. As a result of this the expression (119) returns to its initial value when t describes the contour C. Notice also that, according to Cauchy's theorem, the value of the integral does not depend on the choice of the point A to the right of $t = 1$ on the real axis or on the form of the contour. It is only essential that the contour should not intersect the cut.

We thus obtain the solution of the equation (118):

$$P_n(x) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(t^2-1)^n}{(t-x)^{n+1}}\,dt, \tag{121}$$

where C is the contour we described above. This solution is a regular function of x in the whole cut plane and, in particular, at the point

$x = 1$. But, as we know from [102], the equation (118) is obtained from Gauss's equation when $\alpha = n + 1$, $\beta = -n$ and $\gamma = 1$, and when the independent variable z in the Gauss equation is substituted by $(1 - x)/2$. Since the solution (121) is regular when $x = 1$, i.e. when $z = 0$, it must be the same as the hypergeometric series given below, except for the constant term:

$$P_n(x) = CF\left(n+1,\ -n,\ 1;\ \frac{1-x}{2}\right). \tag{122}$$

To determine C we evaluate $P_n(1)$:

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(t^2-1)^n}{(t-1)^{n+1}}\,dt = \frac{1}{2^{n+1}\pi i}\int_C \frac{(t+1)^n}{t-1}\,dt,$$

and, evaluating the last integral in accordance with the theorem of residues, we obtain $P_n(1) = 1$; hence the formula (122) when $x = 1$, gives $C = 1$, i.e.:

$$P_n(x) = F\left(n+1,\ -n,\ 1;\ \frac{1-x}{2}\right). \tag{123}$$

When n is a positive integer we obtain a Legendre polynomial. It also follows from the formula (123) that owing to the fact that $F(\alpha, \beta, \gamma; z)$ does not alter when the positions of α and β are interchanged, we have for any n

$$P_n(x) = P_{-n-1}(x).$$

By using the formula (121) the relationships (37), (39) and (40) from [132] can be tested. The function $P_n(x)$, which is the solution of the equation (118) has, in general, singularities at $x = -1$ and $x = \infty$. Formula (121) defines this function in the whole cut plane.

142. The Legendre functions of the second kind. We constructed one of the solutions of the equation (118). We shall now try to construct a second solution. We know that when $y_1(x)$ is one of the solutions of the equation

$$y'' + p(x)\,y' + q(x)\,y = 0,$$

then a second solution can be constructed according to the formula

$$y_2(x) = Cy_1(x)\int e^{-\int p(x)\,dx}\frac{dx}{[y_1(x)]^2}, \tag{124}$$

where C is an arbitrary constant. Consider, first of all, the case when n is positive. At the singularity $x = \infty$ of the equation (118) the deter-

mining equation has the zeros $\varrho_1 = n + 1$ and $\varrho_2 = -n$. The solution of the equation which corresponds to the first zero vanishes when $x = \infty$. Using the formula (124) we can write this solution as follows:

$$Q_n(x) = P_n(x) \int_{\infty}^{x} \frac{dx}{(1-x^2)\,[P_n(x)]^2}\,. \tag{125}$$

The function $Q_n(x)$ has singularities at $x = \pm 1$ and is regular in the x-plane which has a cut from $x = -1$ to $x = 1$. Formula (125) defines $Q_n(x)$ in the whole plane. Notice that the zeros $P_n(x)$ lie in the interval $(-1, +1)$.

Let us express $Q_n(x)$ in terms of a Legendre polynomial and logarithms. To do this we replace $u(x)$ by a new function $v(x)$ in the equation (118) according to the formula

$$u(x) = \frac{1}{2} P_n(x) \log \frac{x+1}{x-1} - v(x). \tag{126}$$

We obtain the following equation for $v(x)$:

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + n(n+1)\,v = 2P_n'(x),$$

and from (42) we can rewrite this equation as follows:

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + n(n+1)\,v = \\ = 2\sum_{k=1}^{N}(2n-4k+3)\,P_{n-2k+1}(x), \tag{127}$$

where $N = n/2$ when n is even, and $N = (n+1)/2$ when n is odd. Bearing in mind that $P_{n-2k+1}(x)$ satisfies the equation:

$$(1-x^2)\,P''_{n-2k+1}(x) - 2x\,P'_{n-2k+1}(x) + \\ + (n-2k+1)(n-2k+2)\,P_{n-2k+1}(x) = 0,$$

we can see that the equation

$$(1-x^2)\frac{d^2w}{dx^2} - 2x\frac{dw}{dx} + n(n+1)\,w = 2(2n-4k+3)\,P_{n-2k+1}(x)$$

has a solution

$$w(x) = \frac{2n-4k+3}{(2k-1)(n-k+1)}\,P_{n-2k+1}(x).$$

This, from (126) and (127), gives the following solution of the Legendre equation (118):

$$u_0(x) = \frac{1}{2} P_n(x) \log \frac{x+1}{x-1} - \sum_{k=1}^{N} \frac{2n-4k+3}{(2k-1)(n-k+1)} P_{n-2k+1}(x). \quad (128)$$

It can be expressed in terms of $P_n(x)$ and $Q_n(x)$:

$$u_0(x) = C_1 P_n(x) + C_2 Q_n(x). \quad (129)$$

From (128) and the obvious expansion

$$\frac{1}{2} \log \frac{1+x}{1-x} = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \dots \qquad (|x| > 1)$$

it follows that $u_0(x)/x^{n-2}$ remains bounded as $x \to \infty$. On the other hand on the right-hand side of (129) $P_n(x)$ is a polynomial of the nth degree and $Q_n(x)$ tends to zero like $1/(x^{n+1})$ as $x \to \infty$. This being true, we can say that $C_1 = 0$, i.e.

$$C_2 Q_n(x) = u_0(x) = \frac{1}{2} P_n(x) \log \frac{x+1}{x-1} - R_n(x), \quad (130)$$

where $R_n(x)$ is a polynomial of the $(n-1)$th degree. It follows that

$$C_2 \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{Q_n(x)}{P_n(x)}\right] = \frac{1}{1-x^2} + \frac{S_n(x)}{[P_n(x)]^2},$$

where $S_n(x)$ is a polynomial in x. On the other hand, from (125):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{Q_n(x)}{P_n(x)}\right] = \frac{1}{(1-x^2)[P_n(x)]^2}.$$

Comparing this equation with the one above we have:

$$\frac{C_2}{(1-x^2)[P_n(x)]^2} = \frac{1}{1-x^2} + \frac{S_n(x)}{[P_n(x)]^2},$$

whence

$$C_2 = [P_n(x)]^2 + (1-x^2) S_n(x),$$

and putting $x = 1$ we obtain $C_2 = 1$, i.e. from (128) and (129) we have finally:

$$Q_n(x) = \frac{1}{2} P_n(x) \log \frac{1+x}{1-x} - \sum_{k=1}^{N} \frac{2n-4k+3}{(2k-1)(n-k+1)} P_{n-2k+1}(x). \quad (131)$$

The function $Q_n(x)$ is usually known as *the Legendre function of the second kind.*

The logarithmic terms are due to the character of the singularities $x = \pm 1$ of the equation (118). $Q_n(x)$ can easily be written in the form

of a definite integral. Notice that when n is a positive integer the expression (120) vanishes for $t = \pm 1$. Therefore, when constructing the solution of the equation (118) in the form (119) we can simply take the line $-1 \leqslant t \leqslant +1$ as the contour C and we have:

$$u_1(x) = C \int_{-1}^{1} \frac{(1-t^2)^n}{(x-t)^{n+1}}\, dt, \tag{132}$$

where C is an arbitrary constant. This integral tends to zero like $1/x^{n+1}$ as $x \to \infty$, and therefore this solution differs only by its constant term from $Q_n(x)$. Let us determine the constant C so that the solution (132) is the same as $Q_n(x)$. It follows from the formula (11) that the coefficient of x^n in the expansion of $P_n(x)$ is equal to:

$$a_n = \frac{2n(2n-1)\dots(n+1)}{n!\,2^n} = \frac{2n!}{(n!)^2\,2^n}. \tag{133}$$

Returning to the formula (125) we can see that the expansion of the integrand in whole positive powers of x^{-1} begins with the term $(-1/a_n^2 x^{2n+2})$ and the expansion of $Q_n(x)$ begins with the term $1/(2n+1)a_n x^{n+1}$. Comparing this with the formula (132) we obtain the following equation for C:

$$C \int_{-1}^{1} (1-t^2)^n\, dt = \frac{1}{a_n(2n+1)},$$

or

$$2C \int_{0}^{\frac{\pi}{2}} \sin^{2n+1}\varphi\, d\varphi = \frac{1}{a_n(2n+1)},$$

hence [I, 100]

$$2C\, \frac{2n(2n-2)\dots 4\cdot 2}{(2n+1)(2n-1)\dots 5\cdot 3} = \frac{1}{a_n(2n+1)},$$

and from (133) we obtain $C = 1/2^{n+1}$. Substituting this in the equation (132) we obtain an expression for $Q_n(x)$ in the form of the integral

$$Q_n(x) = \frac{1}{2^{n+1}} \int_{-1}^{1} \frac{(1-t^2)^n}{(x-t)^{n+1}}\, dt. \tag{134}$$

This expression holds in the whole of the x-plane except on the line $-1 \leqslant x \leqslant +1$. We shall now express $Q_n(x)$ as a hypergeometric series. To start with let us rewrite the formula (133) in terms of the function

$\Gamma(z)$ and use, for this purpose, the relationship (143) from [73] when $z = n + 1$, and the formula $\Gamma(2n + 2) = (2n + 1)\Gamma(2n + 1)$:

$$a_n = \frac{\Gamma(2n+1)}{[\Gamma(n+1)]^2 2^n} = \frac{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)}{(2n+1)\sqrt{\pi}\,\Gamma(n+1)}. \tag{135}$$

Notice also that the substitution $t = x^2$ transforms the Legendre equation (118) into the equation

$$t(t-1)\frac{d^2u}{dt^2} + \frac{3t-1}{2}\frac{du}{dt} - \frac{n(n+1)}{4}u = 0,$$

and this is the Gauss equation with parameters $\alpha = n/2 + 1/2$, $\beta = -n/2$, $\gamma = 1$. Using the first formula when $z = t$ and replacing t by x^2 we obtain the solution of the equation (118):

$$u(x) = \frac{C}{x^{n+1}} F\left(\frac{n}{2}+\frac{1}{2},\ \frac{n}{2}+1,\ n+\frac{3}{2};\ \frac{1}{x^2}\right) \quad (|x|>1), \tag{136}$$

which at infinity has the same properties as $Q_n(x)$ and only differs from it by the constant term. The constant C must be so chosen that the solution (136) coincides with $Q_n(x)$, the expansion of which in powers of $1/x$ begins with the term $1/[(2n + 1)]\ a_n x^{n+1}$. Hence $C = 1/(2n + 1)a_n$ and we have:

$$Q_n(x) = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2^{n+1}\,\Gamma\left(n+\frac{3}{2}\right)} \cdot \frac{1}{x^{n+1}} F\left(\frac{n}{2}+\frac{1}{2},\ \frac{n}{2}+1,\ \frac{n}{2}+\frac{3}{2};\ \frac{1}{x^2}\right). \tag{137}$$

Until now we have only investigated the function $Q_n(x)$ when n is a positive integer. $Q_n(x)$ can also be determined as the second solution of the equation (118) when n takes any value, in the same way as we did with $P_n(x)$. Consider the integral (134). This integral retains its meaning when the real part of $(n+1)$ is positive and it can be used for the determination of $Q_n(x)$ for these values of n. In general $Q_n(x)$ can be determined by the contour integral (119) but the contour must be suitably chosen. The expression (137) holds for all values of n except when n is a negative integer. It must be remembered that when n is not a positive integer then the point $x = \infty$ will be a branch-point for the function $Q_n(x)$. It is determined in a plane cut from $x = -\infty$ to $x = 1$. When n is a negative integer then, putting $n = -m - 1$, where m is a positive integer or zero, we can see that the equation (118) becomes

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + m(m+1)u = 0.$$

and $P_m(x)$ and $Q_m(x)$ are the solutions of the equation (118). The formulae (37), (39) and (40) from [132] can readily be tested for $Q_n(x)$.

§ 2. Bessel functions

143. The determination of Bessel functions. We first met Bessel functions in connection with the vibration of a round membrane [II, 178]. Let us recall the results which we obtained at the time and establish the connection between the wave equation and Bessel functions.

The wave equation in two dimensions is as follows:

$$\frac{\partial^2 U}{\partial t^2} = a^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right). \tag{1}$$

When dealing with the vibration of a round membrane we introduced plane polar coordinates

$$x = r \cos \varphi; \quad y = r \sin \varphi,$$

and found those solutions of the equation (1) which are in the form of a product of three functions, one of which depends on t, the second on r and the third on φ. These solutions, as we saw above, have the following form:

$$(\alpha \cos \omega t + \beta \sin \omega t)(C \cos p\varphi + D \sin p\varphi) Z_p(kr), \tag{2}$$

where α, β, C and D are arbitrary constants, while the constants ω, k and a are connected by the relationship:

$$\omega^2 = k^2 a^2. \tag{3}$$

$Z_p(z)$ in the above formula denotes an arbitrary solution of the Bessel equation

$$Z_p''(z) + \frac{1}{z} Z_p'(z) + \left(1 - \frac{p^2}{z^2}\right) Z_p(z) = 0. \tag{4}$$

Notice also that the constant p in the expression (2) can have any value. We have taken p as an integer since we wanted a solution with a period of 2π with respect to the variable φ. We also want our solution to remain finite when $r = 0$ and therefore we took as $Z_p(z)$ that solution of the equation (4) which remains finite when $z = 0$, i.e. we took the solution $J_p(z)$ ($p \geqslant 0$), which is a Bessel function. The value of the constant k, and hence of ω from (3) was determined from the limit

condition. Later we shall deal with the applications of Bessel functions. At present we shall study the properties of the functions satisfying the equation (4) and we begin with the above Bessel function.

The Bessel function, except for the constant term, is determined by an expansion of the form [II, 48]:

$$J_p(z) = Cz^p\left[1 - \frac{z^2}{2(2p+2)} + \frac{z^4}{2\cdot 4\cdot(2p+2)(2p+4)} - \dots\right]. \quad (5)$$

When $p = n$ is a positive integer or zero, then the constant factor C is equal to $1/2^n n!$ and, as always, $0! = 1$. Hence we have the following expression for a Bessel function with a positive integral subscript

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!}\left(\frac{z}{2}\right)^{n+2k}. \quad (6)$$

When p is not an integer then we take the constant term in formula (5) to be equal to

$$C = \frac{1}{2^p\,\Gamma(p+1)}$$

and we obtain the following expression for the Bessel function:

$$J_p(z) = \frac{z^p}{2^p\,\Gamma(p+1)}\left[1 - \frac{1}{1!\,(p+1)}\left(\frac{z}{2}\right)^2 + \frac{1}{2!\,(p+1)(p+2)}\left(\frac{z}{2}\right)^4 - \dots\right]$$

or, as a result of the fundamental property of the function $\Gamma(z)$:

$$J_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p+k+1)}\left(\frac{z}{2}\right)^{p+2k}. \quad (7)$$

When $p = n$ is a positive integer the latter formula is the same as formula (6). Consider now formula (7) when p is a negative integer $p = -n$. We know that $\Gamma(z)$ tends to infinity when z is a negative integer or zero. Hence in the expansion (7) all terms will vanish in which the argument of the function $\Gamma(z)$ in the denominator is equal to a negative integer or zero. These terms will correspond to the following values of the variable of summation:

$$-n + k + 1 \leqslant 0, \quad \text{i.e.} \quad k \leqslant n - 1.$$

In other words we have to start the summation from $k = n$:

$$J_{-n}(z) = \sum_{k=n}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-n+k+1)}\left(\frac{z}{2}\right)^{-n+2k}.$$

Replacing k by another variable of summation $l = k - n$ and taking $(-1)^n$ outside the summation symbol we have

$$J_{-n}(z) = (-1)^n \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+n)!\,\Gamma(l+1)} \left(\frac{z}{2}\right)^{n+2l},$$

i.e.

$$J_{-n}(z) = (-1)^n \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+n)!\,l!} \left(\frac{z}{2}\right)^{n+2l},$$

i.e.

$$J_{-n}(z) = (-1)^n J_n(z) \qquad (n \text{ being an integer}) \tag{8}$$

In other words *the Bessel function with a negative integral subscript* $(-n)$ *differs only by the factor* $(-1)^n$ *from Bessel a function with a positive integral subscript.*

When p is not an integer then the functions $J_p(z)$ and $J_{-p}(z)$ will be two linearly independent solutions of the Bessel equation [II, 48]. The series (7), as we know, converges for all finite z's.

144. Relationships between Bessel functions. We shall now determine certain fundamental relationships between Bessel functions with different subscripts. Differentiating the power series (7) we have:

$$\frac{d}{dz}\frac{J_p(z)}{z^p} = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p+k+1)} \frac{z^{2k}}{2^{p+2k}} = \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 2k}{k!\,\Gamma(p+k+1)} \frac{z^{2k-1}}{2^{p+2k}},$$

or, substituting the variable of summation k by $k+1$ and starting the summation with $k = 0$

$$\frac{d}{dz}\frac{J_p(z)}{z^p} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\,2\,(k+1)}{(k+1)!\,\Gamma(k+p+2)} \cdot \frac{z^{2k+1}}{2^{p+2k+2}}$$

or

$$\frac{d}{dz}\frac{J_p(z)}{z^p} = -\frac{1}{z^p}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p+1+k+1)} \left(\frac{z}{2}\right)^{p+1+2k}.$$

Comparing this with (7) we obtain the following formula:

$$\frac{d}{dz}\frac{J_p(z)}{z^p} = -\frac{J_{p+1}(z)}{z^p}. \tag{9}$$

Carrying out the differentiation the fraction we can rewrite this formula as follows:

$$J'_p(z) = -J_{p+1}(z) + \frac{pJ_p(z)}{z} \qquad (J'_0(z) = -J_1(z)). \tag{10}$$

Divide both sides of the formula (9) by z:

$$\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\frac{J_p(z)}{z^p} = -\frac{J_{p+1}(z)}{z^{p+1}}.$$

The above relationship can be formulated as follows: the differentiation of the fraction $J_p(z)/z^p$ and its subsequent division by z is equivalent to the addition of unity to p and to the change of sign of the above fraction.

Using this rule several times over we obtain the following formula which is valid for any positive integer m:

$$\frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\frac{J_p(z)}{z^p} = (-1)^m\frac{J_{p+m}(z)}{z^{p+m}}. \tag{11}$$

This formula can be rewritten as follows:

$$\frac{\mathrm{d}^m}{\mathrm{d}(z^2)^m}\frac{J_p(z)}{z^p} = (-1)^m\frac{J_{p+m}(z)}{2^m z^{p+m}}. \tag{12}$$

Differentiating now the product $z^p J_p(z)$ with respect to z:

$$\frac{\mathrm{d}}{\mathrm{d}z}z^p J_p(z) = \sum_{k=0}^{\infty}\frac{(-1)^k 2\,(p+k)}{k!\,\Gamma(p+k+1)}\frac{z^{2p+2k-1}}{2^{p+2k}},$$

or, bearing in mind that $\Gamma(p+k+1) = (p+k)\,\Gamma(p+k)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}z}z^p J_p(z) = z^p\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(p-1+k+1)}\left(\frac{z}{2}\right)^{p-1+2k},$$

i.e. from (7) we obtain a formula which is analogous with the formula (9):

$$\frac{\mathrm{d}}{\mathrm{d}z}z^p J_p(z) = z^p J_{p-1}(z). \tag{13}$$

Differentiating the product we can rewrite this formula as follows:

$$J_p'(z) = J_{p-1}(z) - \frac{pJ_p(z)}{z}. \tag{14}$$

Divide both sides of formula (13) by z:

$$\frac{\mathrm{d}}{z\,\mathrm{d}z}z^p J_p(z) = z^{p-1}J_{p-1}(z).$$

Applying this formula several times over we obtain a formula which is analogous with formula (11):

$$\frac{\mathrm{d}^m}{(z'\,\mathrm{d}z)^m} z^p J_p(z) = z^{p-m} J_{p-m}(z) \tag{15}$$

or

$$\frac{\mathrm{d}^m}{(\mathrm{d}z^2)^m} z^p J_p(z) = \frac{z^{p-m} J_{p-m}(z)}{2^m}\,. \tag{16}$$

In the formulae (11) and (15) we have used the following notation:

$$\frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m} f(z) = \frac{\mathrm{d}}{z\,\mathrm{d}z}\frac{\mathrm{d}}{z\,\mathrm{d}z}\cdots\frac{\mathrm{d}}{z\,\mathrm{d}z} f(z),$$

where the number of differentiations with respect to z and of subsequent divisions by z is equal to m.

Comparing the formulae (10) and (14) we obtain a relationship between three successive Bessel functions

$$\frac{pJ_p(z)}{z} - J_{p+1}(z) = J_{p-1}(z) - \frac{pJ_p(z)}{z}$$

or

$$\frac{2pJ_p(z)}{z} = J_{p-1}(z) + J_{p+1}(z). \tag{17}$$

Using the above formulae we will show that Bessel functions, the subscripts of which are equal to one half of an odd integer, i.e. $\pm(2m+1)/2$ where m is an integer, can be expressed in terms of elementary functions. To show this we consider formula (7) when $p = 1/2$:

$$J_{\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma\left(k+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{\frac{1}{2}+2k}.$$

Applying the fundamental property of the function $\Gamma(z)$ several times over we have

$$\Gamma\left(k+\tfrac{3}{2}\right) = \left(k+\tfrac{1}{2}\right)\Gamma\left(k+\tfrac{1}{2}\right) = \left(k+\tfrac{1}{2}\right)\left(k-\tfrac{1}{2}\right)\Gamma\left(k-\tfrac{1}{2}\right)$$
$$= \left(k+\tfrac{1}{2}\right)\left(k-\tfrac{1}{2}\right)\cdots\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{(2k+1)(2k-1)\ldots 3\cdot 1}{2^{k+1}}\sqrt{\pi}\,,$$

and we therefore obtain

$$J_{\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,2^k\cdot 1\cdot 3\ldots(2k+1)\sqrt{\pi}} \frac{z^{\frac{1}{2}+2k}}{2^{-\frac{1}{2}}} = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}\,,$$

i.e.

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \tag{18}$$

Applying now formula (11) we have for any positive integer m:

$$J_{\frac{2m+1}{2}}(z) = (-1)^m \sqrt{\frac{2}{\pi}} z^{\frac{2m+1}{2}} \frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\left(\frac{\sin z}{z}\right). \tag{19}$$

Analogous results are also obtained for negative subscripts. Formula (7) when $p = -1/2$ gives

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z, \tag{20}$$

and using formula (15) we obtain for any positive integer m

$$J_{-\frac{2m+1}{2}}(z) = \sqrt{\frac{2}{\pi}} z^{\frac{2m+1}{2}} \frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\left(\frac{\cos z}{z}\right). \tag{21}$$

In [II, 48] we have written Bessel functions in an expanded form when $p = \pm 3/2$ and $p = \pm 5/2$.

145. The orthogonality of Bessel functions and their zeros. As we have already said we have used Bessel functions in connection with the vibration of a round membrane. At the time we used the usual Fourier method and, in order to satisfy the initial conditions of the problem, we had to expand the given function into a series of Bessel functions. We then obtained series, analogous with the Fourier series and found that Bessel functions have the property of orthogonality [II, 178] in the usual sense. We shall now consider this problem from a more general point of view and explain some additional circumstances.

As we know the function $J_p(kz)$ satisfies the equation [II, 48]:

$$\frac{\mathrm{d}^2 J_p(kz)}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}J_p(kz)}{\mathrm{d}z} + \left(k^2 - \frac{p^2}{z^2}\right) J_p(kz) = 0,$$

or, multiplying by z we can write this equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z\frac{\mathrm{d}J_p(kz)}{\mathrm{d}z}\right] + \left(k^2 z - \frac{p^2}{z}\right) J_p(kz) = 0.$$

In future we shall assume that the symbol p is real and also that $p \geqslant 0$.

We take two different values of k and write the corresponding differential equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z\,\frac{\mathrm{d}J_p(k_1 z)}{\mathrm{d}z}\right]+\left(k_1^2 z-\frac{p^2}{z}\right)J_p(k_1 z)=0,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z\,\frac{\mathrm{d}J_p(k_2 z)}{\mathrm{d}z}\right]+\left(k_2^2 z-\frac{p^2}{z}\right)J_p(k_2 z)=0.$$

Multiplying the first equation by $J_p(k_2 z)$ and the second by $J_p(k_1 z)$, we subtract and integrate over the finite interval $(0, l)$:

$$\int_0^l\left\{J_p(k_2 z)\,\frac{\mathrm{d}}{\mathrm{d}z}\left[z\,\frac{\mathrm{d}J_p(k_1 z)}{\mathrm{d}z}\right]-J_p(k_1 z)\,\frac{\mathrm{d}}{\mathrm{d}z}\left[z\,\frac{\mathrm{d}J_p(k_2 z)}{\mathrm{d}z}\right]\right\}\mathrm{d}z+$$

$$+(k_1^2-k_2^2)\int_0^l z J_p(k_1 z)\,J_p(k_2 z)\,\mathrm{d}z=0.$$

The first integrand represents the complete derivative with respect to z of the difference

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z\,\frac{\mathrm{d}J_p(k_1 z)}{\mathrm{d}z}\,J_p(k_2 z)-z\,\frac{\mathrm{d}J_p(k_2 z)}{\mathrm{d}z}\,J_p(k_1 z)\right],$$

and the equation can therefore be written in the form

$$\left[z\,\frac{\mathrm{d}J_p(k_1 z)}{\mathrm{d}z}\,J_p(k_2 z)-z\,\frac{\mathrm{d}J_p(k_2 z)}{\mathrm{d}z}\,J_p(k_1 z)\right]_{z=0}^{z=l}+$$

$$+(k_1^2-k_2^2)\int_0^l z J_p(k_1 z)\,J_p(k_2 z)\,\mathrm{d}z=0.$$

But

$$\frac{\mathrm{d}J_p(kz)}{\mathrm{d}z}=kJ_p'(kz),$$

where we write

$$J_p'(x)=\frac{\mathrm{d}}{\mathrm{d}x}J_p(x),$$

and consequently, the above formula can be rewritten in the form

$$\left[k_1 z J_p'(k_1 z)\,J_p(k_2 z)-k_2 z J_p'(k_2 z)\,J_p(k_1 z)\right]_{z=0}^{z=l}+$$

$$+(k_1^2-k_2^2)\int_0^l z J_p(k_1 z)\,J_p(k_2 z)\,\mathrm{d}z=0. \qquad (22)$$

Let us recall the expansion of Bessel functions

$$J_p(z)=z^p\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(p+k+1)}\,\frac{z^{2k}}{2^{p+2k}}. \qquad (23)$$

Owing to the fact that $p \geqslant 0$ the term in the square brackets vanishes when $z = 0$ and we finally arrive at the following formula which is of fundamental importance to what follows:

$$l\left[k_1 J_p'(k_1 l) J_p(k_2 l) - k_2 J_p'(k_2 l) J_p(k_1 l)\right] + \\ + (k_1^2 - k_2^2) \int_0^l z J_p(k_1 z) J_p(k_2 z)\, dz = 0. \qquad (24)$$

When $l = 1$ this formula becomes:

$$k_1 J_p'(k_1) J_p(k_2) - k_2 J_p'(k_2) J_p(k_1) + \\ + (k_1^2 - k_2^2) \int_0^l z J_p(k_1 z) J_p(k_2 z)\, dz = 0. \qquad (25)$$

In the above calculations we have assumed that $p \geqslant 0$. It can readily be shown that the integrals retain their meaning and the term in the square brackets in formula (22) vanishes when $z = 0$ when we make the even wider assumption that $p > -1$.

We shall show, first of all, that the Bessel function cannot have complex zeros. Suppose, to start with, that it has such a zero $a + ib$ where $a \neq 0$. All the coefficients in the expansion (7) are real and, consequently, the function $J_p(z)$ must have the conjugate zero $a - ib$ as well as the zero $a + ib$. If we now suppose that in formula (25) $k_1 = a + ib$ and $k_2 = a - ib$, where $k_1^2 \neq k_2^2$, we have

$$\int_0^l z J_p(k_1 z) J_p(k_2 z)\, dz = 0.$$

$J_p(k_1 z)$ and $J_p(k_2 z)$ will have conjugate complex values and therefore, the integrand in the above formula is positive, so that this formula is contradicted. The case when $a = 0$ remains to be considered, i.e. it must be shown that the function $J_p(z)$ cannot have purely imaginary zeros $\pm ib$ either. In fact, substituting in the formula (23) we obtain an expansion with positive terms

$$J_p(ib) = (ib)^p \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(p + k + 1)} \frac{b^{2k}}{2^{p+2k}}.$$

This is directly due to the fact that, according to the formula (111) in [71] the function $\Gamma(z)$ is positive when $z > 0$. We thus arrive at the following result: *if p is real and $p > -1$ then all zeros of the function $J_p(z)$ are real.* Notice also that it follows directly from the ex-

pansion (23) which contains even powers only that the zeros of $J_p(z)$ will in pairs have equal absolute values of opposite sign so that it is sufficient to consider positive zeros only. In future we shall only consider these zeros. Let us write the asymptotic representation for the Bessel function [113]:

$$J_p(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + O\left(z^{-\frac{3}{2}}\right)$$

or

$$J_p(z) = \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + O\left(z^{-1}\right)\right].$$

As z moves towards infinity along the positive part of the real axis the second term in the square bracket tends to zero while the first term passes from -1 to $+1$ an infinite number of times. It follows that $J_p(z)$ *has an infinite number of real zeros.*

If $z = k_1$ and $z = k_2$ are two different positive zeros of the equation

$$J_p(zl) = 0, \tag{26}$$

then formula (24) gives directly the following *property of orthogonality of Bessel functions*

$$\int_0^l z J_p(k_1 z) J_p(k_2 z)\, \mathrm{d}z = 0. \tag{27}$$

According to Rolle's theorem, the function $J'_p(z)$ must also have an infinite number of real positive zeros, and if we now denote by k_1 and k_2 two different positive zeros of the equation

$$J'_p(zl) = 0, \tag{28}$$

then, from (24), we obtain exactly the same condition of orthogonality (27).

Let us now consider a more general equation than the one above, viz. an equation of the form

$$\alpha J_p(zl) + \beta z J'_p(z') = 0, \tag{29}$$

where α and β are given real constants. Let $z = k_1$ and $z = k_2$ be two different zeros of the equation (29), i.e.

$$\alpha J_p(k_1 l) + \beta k_1 J'_p(k_1 l) = 0; \quad \alpha J_p(k_2 l) + \beta k_2 J'_p(k_2 l) = 0.$$

It follows directly that

$$k_1 J'_p(k_1 l) J_p(k_2 l) - k_2 J'_p(k_2 l) J_p(k_1 l) = 0,$$

and, consequently, the term in square brackets in formula (24) will again vanish, and we have the same condition of orthogonality as before. The equations (26) and (27) are particular cases of the equation (29). As before, it follows from the conditions of orthogonality, that the equation (29) cannot have complex zeros $a + ib$, where $a \neq 0$.

It can be shown similarly that the equation (29) has no purely imaginary zeros provided $\alpha > 0$ and $\beta > 0$.

Let us recall the two relationships

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{J_p(z)}{z^p} = -\frac{J_{p+1}(z)}{z^p}; \quad \frac{\mathrm{d}}{\mathrm{d}z}\left[z^{p+1}J_{p+1}(z)\right] = z^{p+1}J_p(z). \tag{30}$$

According to Rolle's theorem, the first of these shows that between two successive zeros of $J_p(z)$ there can only be one zero of $J_{p+1}(z)$. The second relationship shows that between two successive zeros of $J_{p+1}(z)$ there can only be one zero of $J_p(z)$. A comparison shows that the positive zeros of $J_p(z)$ and $J_{p+1}(z)$ separate one another, i.e. *between two positive zeros of $J_p(z)$ there is one and only one, zero of $J_{p+1}(z)$ and vice versa.*

Let a and b be the smallest positive zeros $J_p(z)$ and $J_{p+1}(z)$ respectively. Remembering that $z = 0$ is a zero of $z^{p+1}J_{p-1}(z)$ and applying Rolle's theorem to the second of the formulae (30) we can see that $J_p(z)$ must have a zero in the interval $(0, b)$, i.e. $a < b$.

We thus find that *the smallest positive zero of the function $J_p(z)$ will be nearer the origin than the similar zero of $J_{p+1}(z)$.* Notice also that the function $z^{-p}J_p(z)$ is a solution of the equation [111]:

$$z\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (2p+1)\frac{\mathrm{d}y}{\mathrm{d}z} + zy = 0,$$

and therefore the functions $z^{-p}J_p(z)$ and $\mathrm{d}/\mathrm{d}z\,[z^{-p}J_p(z)]$ cannot have common positive zeros [104]. Consequently, from (30), the same can be said about the functions $J_p(z)$ and $J_{p+1}(z)$.

The orthogonality of Bessel functions is of great importance in the expansion of a given function into Bessel functions as, for example, in the vibration of a round membrane.

It is also essential to be able to evaluate an integral of the form

$$\int_0^l zJ_p^2(kz)\,\mathrm{d}z,$$

where $z = k$ is a zero of an equation of the form (29). Consider the case when k is simply a zero of the equation (26). Take the formula (24)

and assume that $k_2 = k$, while k_1 is variable and tends to k. We thus have

$$(k_1 + k) \int_0^l z J_p(k_1 z) J_p(kz)\, dz = \frac{lkJ'_p(kl) J_p(k_1 l)}{k_1 - k}.$$

When $k_1 \to k$ both the numerator and the denominator of the fraction vanish since $J_p(k_1 l)$ tends to $J_p(kl) = 0$. Expanding by the usual rule we have in the limit:

$$2k \int_0^l z J_p^2(kz)\, dz = l^2 k J'^2_p(kl)$$

or

$$\int_0^l z J_p^2(kz)\, dz = \frac{l^2}{2} J'^2_p(kl). \tag{31}$$

Now let us consider the known relationship

$$\frac{d}{dz} \frac{J_p(z)}{z^p} = -\frac{J_{p+1}(z)}{z^p}$$

and put $z = kl$. We have

$$J'_p(kl) = -J_{p+1}(kl),$$

so that the above formula can also be written as follows:

$$\int_0^l z J_p^2(kz)\, dz = \frac{l^2}{2} J^2_{p+1}(kl). \tag{32}$$

When $z = k$ is a zero of the equation (28) we obtain similarly

$$\int_0^l z J_p^2(kz)\, dz = -\frac{l^2}{2} J''_p(kl) J_p(kl). \tag{33}$$

But we have:

$$J''_p(kl) + \frac{1}{kl} J'_p(kl) + \left(1 - \frac{p^2}{k^2 l^2}\right) J_p(kl) = 0$$

and using the equation $J'_p(kl) = 0$ we can rewrite the formula (33) as follows:

$$\int_0^l z J_p^2(kz)\, dz = \frac{1}{2}\left(l^2 - \frac{p^2}{k^2}\right) J_p^2(kl). \tag{34}$$

146. Converting function and integral representation. Consider the analytic function of the complex variable t

$$e^{\frac{1}{2}z\left(t-\right.} \quad . \tag{35}$$

This function has essential singularities at the points $t = 0$ and $t=\infty$ and it can therefore be expanded into a Laurent's series in the whole t-plane; the coefficients of this expansion will be functions of the parameter z in expression (35):

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} a_n(z)\, t^n. \tag{36}$$

We will now show that these coefficients will be the Bessel functions $J_n(z)$. In fact we can represent the coefficients of the expansion (36) by the following contour integral [15]:

$$a_n(z) = \frac{1}{2\pi i}\int_{l_0} u^{-n-1} e^{\frac{1}{2}z\left(u-\frac{1}{u}\right)}\, du,$$

where l_0 is any simple closed contour which encircles the origin in the positive direction. We now replace t by another variable of integration, according to the formula $u = 2t/z$, where z has a fixed value other than zero. The point $u = 0$ corresponds to $t = 0$ and the contour l_0 will be transformed in the t-plane into another contour which also encircles the origin in the positive direction. Changing the variables we obtain the following expression for the coefficients:

$$a_n(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^n \int_{l_0} t^{-n-1} e^{t-\frac{z^2}{4t}}\, dt.$$

On the contour l_0 we can represent the exponential function by a power series which converges uniformly with respect to t:

$$e^{-\frac{z^2}{4t}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{z^{2k}}{2^{2k}t^k}$$

Substituting this in the previous formula we obtain

$$a_n(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{z}{2}\right)^{n+2k}\int_{l_0} t^{-n-k-1} e^t\, dt.$$

When $n + k$ is a negative integer then the point $t = 0$ will not be a singularity of the integrand in the above formula as this will

be equal to zero. When $(n + k)$ is a positive integer or zero then, remembering the expansion of e^t, we can see that the residue of the integrand at the point $t = 0$ is equal to $1/(n + k)!$. Thus when n is a positive integer we have

$$a_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{z}{2}\right)^{n+2k},$$

i.e. $a_n(z)$ is, in fact, the same as $J_n(z)$. If we replace t by $-1/t$ in formula (36) then the left-hand side remains unaltered, and this shows that $a_{-n}(z) = (-1)^n a_n(z)$, i.e. when n is negative we have from (8)

$$a_{-n}(z) = (-1)^n J_n(z) = J_{-n}(z).$$

Hence, instead of the formula (36), we can write the following expansion:

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} J_n(z)\, t^n. \tag{37}$$

In other words the *function* (35) *is the converting function for Bessel functions when n is an integer.* Formula (37) is convenient for deriving the properties of Bessel functions when n is an integer. We shall use this function for deducing the integral representation of Bessel functions when n is an integer.

On putting $t = e^{i\varphi}$ in formula (37) we have

$$e^{tiz\sin\varphi} = \sum_{n=-\infty}^{+\infty} J_n(z)\, e^{in\varphi}$$

or, separating the real and imaginary parts, where we assume that z and φ are real:

$$\cos(z\sin\varphi) = J_0(z) + \sum_{n=1}^{\infty} J_n(z)\cos n\varphi + \sum_{n=-1}^{-\infty} J_n(z)\cos n\varphi,$$

$$\sin(z\sin\varphi) = \sum_{n=1}^{\infty} J_n(z)\sin n\varphi + \sum_{n=-1}^{-\infty} J_n(z)\sin n\varphi,$$

or, from (8), we have:

$$\left.\begin{aligned} \cos(z\sin\varphi) &= J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\varphi, \\ \sin(z\sin\varphi) &= 2\sum_{n=1}^{\infty} J_{2n-1}(z)\sin(2n-1)\varphi. \end{aligned}\right\} \tag{38}$$

The formulae (38) represent the expansion of the functions into a Fourier series; applying the usual method for determining the coefficients we obtain the following integral representation for Bessel functions:

$$\left.\begin{aligned} J_{2n}(z) &= \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\varphi)\cos 2n\varphi\,\mathrm{d}\varphi \quad (n = 0, 1, \ldots), \\ J_{2n-1}(z) &= \frac{1}{\pi}\int_0^{\pi} \sin(z\sin\varphi)\sin(2n-1)\varphi\,\mathrm{d}\varphi \quad (n = 1,2, \ldots). \end{aligned}\right\} \tag{39}$$

The same method for determining coefficients gives the following two equations:

$$\frac{1}{\pi}\int_0^{\pi} \cos(z\sin\varphi)\cos(2n-1)\varphi\,\mathrm{d}\varphi = 0,$$

$$\frac{1}{\pi}\int_0^{\pi} \sin(z\sin\varphi)\sin 2n\varphi\,\mathrm{d}\varphi = 0.$$

The formulae (39) can be combined into one formula which will be valid whether n is even or odd. Consider in connection with this the integral

$$\frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - z\sin\varphi)\,\mathrm{d}\varphi =$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\varphi)\cos n\varphi\,\mathrm{d}\varphi + \frac{1}{\pi}\int_0^{\pi} \sin(z\sin\varphi)\sin n\varphi\,\mathrm{d}\varphi.$$

When n is even, the first term on the right-hand side is $J_n(z)$ and the second is zero so that the sum is equal to $J_n(z)$. When n is odd, the first term will be zero and the second term will be $J_n(z)$ so that for any positive integer n we have the integral representation

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - z\sin\varphi)\,\mathrm{d}\varphi \quad (n = 0, 1, 2, \ldots). \tag{40}$$

Strictly speaking the proof of the above equation is only valid when z is real. As a result of the principle of analytic continuation we can maintain that it is also valid for every complex z. Bearing in mind

the fact that the integrand is even we can write the above formula as follows:

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\varphi - z\sin\varphi)\, d\varphi. \tag{41}$$

This equation can also be written in the form:

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n\varphi - z\sin\varphi)}\, d\varphi. \tag{42}$$

In fact, applying Euler's formula to the exponential function we obtain two terms, one of which is equal to the integral (41) and the other being zero since the integrand function is odd.

Notice that the formula (40) no longer applies when n is not an integer. In this case we have a more complicated formula, viz.:

$$J_p(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\varphi - z\sin\varphi)\, d\varphi - \frac{\sin p\pi}{\pi} \int_0^{\infty} e^{-p\varphi - z\sinh\varphi}\, d\varphi. \tag{43}$$

and this formula is valid for values of z which lie to the right of the imaginary axis. Notice also the formula for $\sinh\varphi$:

$$\sinh\varphi = \frac{e^{\varphi} - e^{-\varphi}}{2}.$$

The proof of this formula will be given in [151].

Applying the formula (37) and using the obvious equality

$$e^{\frac{1}{2}a\left(t-\frac{1}{t}\right)} \cdot e^{\frac{1}{2}b\left(t-\frac{1}{t}\right)} = e^{\frac{1}{2}(a+b)\left(t-\frac{1}{t}\right)}.$$

we have

$$\sum_{n=-\infty}^{\infty} J_n(a+b)\, t^n = \sum_{k=-\infty}^{\infty} J_k(a)\, t^k \cdot \sum_{k=-\infty}^{\infty} J_k(b)\, t^k.$$

Multiplying the power series on the right and collecting terms in t^n we have

$$J_n(a+b) = \sum_{k=-\infty}^{+\infty} J_k(a)\, J_{n-k}(b). \tag{44}$$

This formula expresses *the addition theorem for Bessel functions when n is an integer.*

When n is zero a more general addition theorem must be applied, viz.:

$$J_0\left(\sqrt{a^2 + b^2 + 2ab\cos\alpha}\right) = J_0(a)\, J_0(b) + 2\sum_{k=1}^{\infty} J_k(a)\, J_k(b)\cos k\alpha. \tag{45}$$

The formulae (38) represent the expansion of the functions into a Fourier series; applying the usual method for determining the coefficients we obtain the following integral representation for Bessel functions:

$$\left.\begin{aligned} J_{2n}(z) &= \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\varphi)\cos 2n\varphi\,\mathrm{d}\varphi \quad (n = 0, 1, \ldots), \\ J_{2n-1}(z) &= \frac{1}{\pi}\int_0^{\pi} \sin(z\sin\varphi)\sin(2n-1)\varphi\,\mathrm{d}\varphi \quad (n = 1,2, \ldots). \end{aligned}\right\} \tag{39}$$

The same method for determining coefficients gives the following two equations:

$$\frac{1}{\pi}\int_0^{\pi} \cos(z\sin\varphi)\cos(2n-1)\varphi\,\mathrm{d}\varphi = 0,$$

$$\frac{1}{\pi}\int_0^{\pi} \sin(z\sin\varphi)\sin 2n\varphi\,\mathrm{d}\varphi = 0.$$

The formulae (39) can be combined into one formula which will be valid whether n is even or odd. Consider in connection with this the integral

$$\frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - z\sin\varphi)\,\mathrm{d}\varphi =$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\varphi)\cos n\varphi\,\mathrm{d}\varphi + \frac{1}{\pi}\int_0^{\pi} \sin(z\sin\varphi)\sin n\varphi\,\mathrm{d}\varphi.$$

When n is even, the first term on the right-hand side is $J_n(z)$ and the second is zero so that the sum is equal to $J_n(z)$. When n is odd, the first term will be zero and the second term will be $J_n(z)$ so that for any positive integer n we have the integral representation

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - z\sin\varphi)\,\mathrm{d}\varphi \quad (n = 0, 1, 2, \ldots). \tag{40}$$

Strictly speaking the proof of the above equation is only valid when z is real. As a result of the principle of analytic continuation we can maintain that it is also valid for every complex z. Bearing in mind

the fact that the integrand is even we can write the above formula as follows:

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\varphi - z \sin \varphi)\, d\varphi. \tag{41}$$

This equation can also be written in the form:

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n\varphi - z\sin\varphi)}\, d\varphi. \tag{42}$$

In fact, applying Euler's formula to the exponential function we obtain two terms, one of which is equal to the integral (41) and the other being zero since the integrand function is odd.

Notice that the formula (40) no longer applies when n is not an integer. In this case we have a more complicated formula, viz.:

$$J_p(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\varphi - z \sin \varphi)\, d\varphi - \frac{\sin p\pi}{\pi} \int_0^{\infty} e^{-p\varphi - z \sinh \varphi}\, d\varphi. \tag{43}$$

and this formula is valid for values of z which lie to the right of the imaginary axis. Notice also the formula for $\sinh \varphi$:

$$\sinh \varphi = \frac{e^{\varphi} - e^{-\varphi}}{2}.$$

The proof of this formula will be given in [151].

Applying the formula (37) and using the obvious equality

$$e^{\frac{1}{2}a\left(t - \frac{1}{t}\right)} \cdot e^{\frac{1}{2}b\left(t - \frac{1}{t}\right)} = e^{\frac{1}{2}(a+b)\left(t - \frac{1}{t}\right)}.$$

we have

$$\sum_{n=-\infty}^{\infty} J_n(a+b)\, t^n = \sum_{k=-\infty}^{\infty} J_k(a)\, t^k \cdot \sum_{k=-\infty}^{\infty} J_k(b)\, t^k.$$

Multiplying the power series on the right and collecting terms in t^n we have

$$J_n(a+b) = \sum_{k=-\infty}^{+\infty} J_k(a)\, J_{n-k}(b). \tag{44}$$

This formula expresses *the addition theorem for Bessel functions when n is an integer.*

When n is zero a more general addition theorem must be applied, viz.:

$$J_0\left(\sqrt{a^2 + b^2 + 2ab\cos\alpha}\right) = J_0(a)\, J_0(b) + 2 \sum_{k=1}^{\infty} J_k(a)\, J_k(b) \cos k\alpha. \tag{45}$$

147. The Fourier—Bessel formula. Arbitrary functions which are defined in the interval $(0, \infty)$ and which satisfy an additional condition in this interval can be represented by an integral, analogous with the Fourier integral but containing Bessel functions instead of trigonometric functions, viz.: if $f(\varrho)$ is continuous in the interval $(0, \infty)$ and satisfies the Dirichlet condition [II, 143] in any finite interval and if the integral given below also exists

$$\int_0^\infty \varrho \, |f(\varrho)| \, d\varrho,$$

then for any integer n and $\varrho > 0$ the following formula applies:

$$f(\varrho) = \int_0^\infty s J_n(s\varrho) \, ds \int_0^\infty t f(t) J_n(st) \, dt. \tag{46}$$

We shall give the formal proof of the relationship (46) without going into greater detail. Supposing that ϱ is the radius-vector, we introduce polar co-ordinates and apply to the function

$$g(x, y) = f(\varrho) e^{in\varphi} \qquad \begin{pmatrix} x = \varrho \cos\varphi \\ y = \varrho \sin\varphi \end{pmatrix} \tag{47}$$

the Fourier formula [II, 160], changing the order of the inside integrals:

$$g(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(ux+vy)} \, du \, dv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(\xi, \eta) \, e^{-i(u\xi+v\eta)} \, d\xi \, d\eta.$$

We replace the variables (u, v) and (ξ, η) by the polar coordinates

$$\xi = s \cos\alpha; \qquad u = t \cos\beta;$$

$$\eta = s \sin\alpha; \qquad v = t \sin\beta.$$

Using the formula (47) we can write:

$$f(\varrho) e^{in\varphi} = \frac{1}{4\pi^2} \int_0^\infty t \, dt \int_{-\pi}^{\pi} e^{i\varrho t \cos(\beta-\varphi)} \, d\beta \int_0^\infty s f(s) \, ds \int_{-\pi}^{\pi} e^{ina} \, e^{-ist\cos(\alpha-\beta)} \, d\alpha.$$

Replacing β by a new variable of integration β' according to the formula

$$\beta - \varphi = \frac{\pi}{2} + \beta',$$

we obtain

$$(f\varrho) e^{in\varphi} = \frac{1}{4\pi^2} \int_0^\infty t \, dt \int_{-\frac{3\pi}{2}-\varphi}^{\frac{\pi}{2}-\varphi} e^{-i\varrho t \sin\beta'} \, d\beta' \int_0^\infty s f(s) \, ds \int_{-\pi}^{\pi} e^{ina} \, e^{-ist\cos\left(\alpha-\varphi-\beta'-\frac{\pi}{2}\right)} \, d\alpha.$$

Bearing in mind the periodicity of trigonometric functions we can change the interval of integration to the former interval $(-\pi, +\pi)$. Similarly, replacing α by a new variable α', according to the formula

$$\alpha - \varphi - \beta' = \alpha',$$

we obtain

$$f(\varrho)\, e^{in\varphi} = \frac{e^{in\varphi}}{4\pi^2} \int_0^\infty t\, dt \int_{-\pi}^{\pi} e^{-i\varrho t \sin\beta' + in\beta'}\, d\beta' \int_0^\infty sf(s)\, ds \int_{-\pi}^{\pi} e^{-ist\sin\alpha' + in\alpha'}\, d\alpha',$$

which, with (42), gives the formula (46).

When a function is given in a finite interval $(0, l)$ then instead of the formula (46) we can consider an expansion into a series, analogous with the Fourier series, in terms of orthogonal functions with which we dealt in the previous section.

Notice that formula (46) can be proved for all real n's greater than $(-1/2)$ and also when less strict assumptions are made with regard to the function $f(\varrho)$.

148. The Hankel and Neumann functions. We obtained in [112] two solutions for the Bessel equation:

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{p^2}{z^2}\right) w = 0 \tag{48}$$

which were given by the following formulae

$$\left.\begin{aligned} H_p^{(1)}(z) &= \frac{\Gamma\left(\frac{1}{2} - p\right)}{\pi^{\frac{3}{2}} i} \left(\frac{z}{2}\right)^p \int_{\lambda_1} (\tau^2 - 1)^{p-\frac{1}{2}} e^{iz\tau}\, d\tau, \\ H_p^{(2)}(z) &= -\frac{\Gamma\left(\frac{1}{2} - p\right)}{\pi^{\frac{3}{2}} i} \left(\frac{z}{2}\right)^p \int_{\lambda_2} (\tau^2 - 1)^{p-\frac{1}{2}} e^{iz\tau}\, d\tau. \end{aligned}\right\} \tag{49}$$

In these formulae the integrand is single-valued in the τ-plane cut parallel to the imaginary axis from $\tau = \pm 1$ to $\tau = +i\infty$ viz. wo assume in the first of these formulae that $\arg(\tau^2 - 1) = 0$ when $\tau > 1$ and in the second that $\arg(\tau^2 - 1) = 2\pi$ when $\tau > 1$. On going from the line $(1, +\infty)$ on the real axis to the line $(-\infty, -1)$ through the lower half-plane, and thus describing half the circuit round the points $\tau = \pm 1$ while omitting the cuts, the amplitude in the expression $(\tau^2 - 1) = (\tau - 1)(\tau + 1)$ increases by (-2π), in other words, in the second of the formulae (49) we can assume that $\arg(\tau^2 - 1) = 0$ when $\tau < -1$. Formula (49) gives *the Hankel functions* for values of z to the right of the imaginary axis, i.e. the real part of which is greater than zero. Notice also that the integrand in the integrals (49)

is an integral function of the parameter p for a fixed value of z and, bearing in mind its rapid decrease at infinity, we can maintain that the Hankel functions $H_p^{(k)}(z)$ are also integral functions of the parameter p for fixed values of z. It follows directly from the asymptotic expressions for Hankel functions [112] that these functions are two linearly-independent solutions of the Bessel equation. We also saw that the Bessel function is equal to half the sum of the Hankel functions

$$J_p(z) = \frac{H_p^{(1)}(z) + H_p^{(2)}(z)}{2}. \tag{50}$$

There is a close connection between the Bessel equation (48) and the equation

$$\frac{d^2w}{dz^2} + p^2 w = 0, \tag{51}$$

which is satisfied by the usual trigonometric functions $\cos pz$ and $\sin pz$. The Hankel functions are in this case analogous with the solutions e^{ipz} and e^{-ipz} while the Bessel functions $J_p(z)$ are analogous with the solution $\cos pz$ of the equation (51). Let us also consider the solution of the equation (48) which is equal to the difference of the Hankel functions divided by $2i$:

$$N_p(z) = \frac{H_p^{(1)}(z) - H_p^{(2)}(z)}{2i}. \tag{52}$$

This solution, usually known as the *Neumann function*, is analogous with the solution $\sin pz$ of the equation (51). From the formulae (50) and (52) we obtain directly the following expressions for the Hankel functions in terms of the Bessel and Neumann functions:

$$H_p^{(1)}(z) = J_p(z) + iN_p(z); \quad H_p^{(2)}(z) = J_p(z) - iN_p(z). \tag{53}$$

This shows that the functions $J_p(z)$ and $N_p(z)$ determine two linearly independent solutions of the equation (48).

The Hankel functions have the following asymptotic representations:

$$\left.\begin{aligned} H_p^{(1)}(z) &= \sqrt{\frac{2}{\pi z}}\, e^{i\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right)}\left[1 + O(z^{-1})\right], \\ H_p^{(2)}(z) &= \sqrt{\frac{2}{\pi z}}\, e^{-i\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right)}\left[1 + O(z^{-1})\right], \end{aligned}\right\} \tag{54}$$

which we have proved for $z > 0$. Using formula (50) we can, as in [113], obtain the asymptotic representation for the Bessel function

$$J_p(z) = \sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{p\pi}{4} - \frac{\pi}{4}\right) + O(z^{-1})\right], \tag{55}$$

and similarly, using formula (52) we can obtain the asymptotic representation for the Neuman function when $z > 0$

$$N_p(z) = \sqrt{\frac{2}{\pi z}}\left[\sin\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + O(z^{-1})\right]. \tag{56}$$

In all the above formulae it must be assumed that $z > 0$ and the positive radical must be taken.

We will now introduce a formula which expresses the Neumann function in terms of the Bessel function. To start with we consider the case when p is not an integer. We know that the equation (48) then has two linearly independent solutions $J_p(z)$ and $J_{-p}(z)$. The second of these can be expressed linearly in terms of the solutions $J_p(z)$ and $N_p(z)$, which as we have already said above, are also linearly independent solutions, i.e. we can write

$$J_{-p}(z) = C_1 J_p(z) + C_2 N_p(z), \tag{57}$$

where C_1 and C_2 are constant coefficients which we shall now determine. Bearing in mind the asymptotic expressions (55) and (56) we can write

$$\cos\left(z + \frac{p\pi}{2} - \frac{\pi}{4}\right) = C_1 \cos\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + \\ + C_2 \sin\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + C_1 O(z^{-1}) + C_2 O(z^{-1}).$$

Notice that the product of a constant, or of any bounded function, and $0(z^{-1})$ of the order $1/z$ also gives $0(z^{-1})$ of the order $1/z$. We thus obtain

$$\cos\left(z + \frac{p\pi}{2} - \frac{\pi}{4}\right) = \\ = C_1 \cos\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + C_2 \sin\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + O(z^{-1}). \tag{58}$$

From this the values of the constants can be deduced by comparing the principal terms in the above expansions. In fact, let us suppose that

$$C_1 = \cos p\pi - A_1; \quad C_2 = -\sin p\pi - A_2,$$

where A_1 and A_2 are the new unknown constants. Substituting in the formula (58) we have

$$\cos\left(z + \frac{p\pi}{2} - \frac{\pi}{4}\right) = \cos\left(z + \frac{p\pi}{2} - \frac{\pi}{4}\right) - A_1 \cos\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) - \\ - A_2 \sin\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + O(z^{-1})$$

or

$$A_1 \cos\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) + A_2 \sin\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right) = O(z^{-1}),$$

i.e. the left-hand side of the above equation which is a periodic function of period 2π must tend to zero as $z \to +\infty$. It follows directly that $A_1 = A_2 = 0$, i.e.

$$C_1 = \cos p\pi; \quad C_2 = -\sin p\pi.$$

Substituting these constants in the formula (57) and solving it with respect to $N_p(z)$ we arrive at the required *formula which expresses the Neumann function in terms of the Bessel functions:*

$$N_p(z) = \frac{J_p(z)\cos p\pi - J_{-p}(z)}{\sin p\pi}. \tag{59}$$

The Neumann and Hankel functions are integral functions of the parameter p. Formula (59) is valid as long as p is not an integer. When p is an integer the denominator in formula (59) vanishes. But, from (8), the numerator will also be zero. Hence to obtain the value of the fraction (59) when p is an integer we simply have to abolish the indefiniteness and replace the numerator and the denominator by their derivatives with respect to the parameter p and assume subsequently that p is equal to an integer n:

$$N_n(z) = \left.\frac{\frac{\partial J_p(z)}{\partial p}\cos p\pi - \pi J_p(z)\sin p\pi - \frac{\partial J_{-p}(z)}{\partial p}}{\pi\cos p\pi}\right|_{p=n}.$$

We thus obtain the following expression for the Neumann function with an integral n:

$$N_n(z) = \frac{1}{\pi}\left[\frac{\partial J_p(z)}{\partial p} - (-1)^n \frac{\partial J_{-p}(z)}{\partial p}\right]_{p=n}. \tag{60}$$

Substituting the expression (59) in the formulae (53) we obtain formulae in which the Hankel functions are expressed in terms of Bessel functions in which p is not an integer:

$$\left.\begin{aligned} H_p^{(1)}(z) &= i\,\frac{J_p(z)\,e^{-ip\pi} - J_{-p}(z)}{\sin p\pi}, \\ H_p^{(2)}(z) &= -i\,\frac{J_p(z)\,e^{ip\pi} - J_{-p}(z)}{\sin p\pi}. \end{aligned}\right\} \tag{61}$$

This leads us directly to the following relationship between Hankel functions, the symbols of which differ only by the symbol

$$H_{-p}^{(1)}(z) = e^{ip\pi} H_p^{(1)}(z); \quad H_{-p}^{(2)}(z) = e^{-ip\pi} H_p^{(2)}(z). \tag{62}$$

Strictly speaking this formula can be proved on the assumption that p is not an integer. But the left and right-hand sides of the formulae (62) are integral functions of p and therefore the formula holds for any p. When p is an integer both the numerator and the denominator in the formulae (61) vanish. Abolishing the indefiniteness, as before, we can also obtain a formula when $p = n$ is an integer.

Consider lastly the case when p has the form $p = (2m + 1)/2$, where m is a positive integer or zero. If we substitute this expression for p in the formulae (49) which determine the Hankel functions, then the integrand will be regular in the whole plane including the points $\tau = \pm 1$ and therefore the integrals will be equal to zero. But at the same time the factor $\Gamma(1/2 - p)$ becomes infinity and the formulae (49) will be devoid of meaning. Instead of these formulae we take the expansions (195) and (196) from [112]. These expansions are, in general, divergent but they do formally satisfied equations as we have proved before. In the case under consideration they will not only be convergent but they will simply become finite sums and give Hankel functions in the finite form. Consider, for example, the first Hankel function with the value of $p = (2m + 1)/2$:

$$H^{(1)}_{\frac{2m+1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\frac{e^{i\left(z-\frac{(m+1)\pi}{2}\right)}}{\Gamma(m+1)}\sum_{k=0}^{\infty}\binom{m}{k}\Gamma(m+1+k)\left(\frac{i}{2z}\right)^k$$

or

$$H^{(1)}_{\frac{2m+1}{2}}(z) =$$

$$= \sqrt{\frac{2}{\pi z}}\,\frac{e^{i\left(z-\frac{(m+1)\pi}{2}\right)}}{m!}\sum_{k=0}^{\infty}\frac{m(m-1)\dots(m-k+1)}{k!}(m+k)!\left(\frac{i}{2z}\right)^k.$$

This shows that all terms corresponding to $k \geqslant m + 1$ will vanish and we obtain the following formula for the Hankel functions

$$H^{(1)}_{\frac{2m+1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\frac{e^{i\left(z-\frac{(m+1)\pi}{2}\right)}}{m!}\sum_{k=0}^{m}\binom{m}{k}(m+k)!\left(\frac{i}{2z}\right)^k. \qquad (63)$$

Similarly, for the second Hankel function we obtain the following finite formula:

$$H^{(2)}_{\frac{2m+1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\frac{e^{-i\left(z-\frac{m+1}{2}\pi\right)}}{m!}\sum_{k=0}^{m}\binom{m}{k}(m+k)!\left(-\frac{i}{2z}\right)^k. \qquad (64)$$

The formulae (59), (61) and (62) remain valid for the values $p = (2m + 1)/2$. Notice that formula (61) determines the Hankel functions when $p = (2m + 1)/2$; from (19) and (21) we have:

$$H^{(1)}_{\frac{2m+1}{2}}(z) =$$

$$= \frac{i}{\sin\left(m+\frac{1}{2}\right)\pi}\sqrt{\frac{2}{\pi}}\, z^{\frac{2m+1}{2}} \frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\left[(-1)^m \mathrm{e}^{-i\left(m+\frac{1}{2}\right)\pi}\frac{\sin z}{z} - \frac{\cos z}{z}\right]$$

or

$$H^{(1)}_{\frac{2m+1}{2}}(z) = (-1)^m i\sqrt{\frac{2}{\pi}}\, z^{\frac{2m+1}{2}} \frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\left(\frac{-i\sin z - \cos z}{z}\right),$$

so that we can finally write

$$H^{(1)}_{\frac{2m+1}{2}}(z) = (-1)^{m+1} i\sqrt{\frac{2}{\pi}}\, z^{\frac{2m+1}{2}} \frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\left(\frac{\mathrm{e}^{iz}}{z}\right) \tag{65}$$

and similarly

$$H^{(2)}_{\frac{2m+1}{2}}(z) = (-1)^m i\sqrt{\frac{2}{\pi}}\, z^{\frac{2m+1}{2}} \frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\left(\frac{\mathrm{e}^{-iz}}{z}\right). \tag{66}$$

The expansions (63) and (64) can also be obtained from this. Using (61) when $p = 1/2$ and also the expressions

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z; \qquad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos z,$$

we have

$$H^{(1)}_{\frac{1}{2}}(z) = -i\sqrt{\frac{2}{\pi z}}\,\mathrm{e}^{iz}; \qquad H^{(2)}_{\frac{1}{2}}(z) = i\sqrt{\frac{2}{\pi z}}\,\mathrm{e}^{-iz}.$$

A number of relationships which we proved earlier for Bessel functions can also be easily proved for Hankel functions. We give here some of these relationships:

$$\frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\left(\frac{H^{(1)}_p(z)}{z^p}\right) = (-1)^m \frac{H^{(1)}_{p+m}(z)}{z^{p+m}}; \quad \frac{\mathrm{d}^m}{(z\,\mathrm{d}z)^m}\left(\frac{H^{(2)}_p(z)}{z^p}\right) = (-1)^m \frac{H^{(2)}_{p+m}(z)}{z^{p+m}},$$

$$\frac{2p}{z}H^{(1)}_p(z) = H^{(1)}_{p-1}(z) + H^{(1)}_{p+1}(z); \quad \frac{2p}{z}H^{(2)}_p(z) = H^{(2)}_{p-1}(z) + H^{(2)}_{p+1}(z).$$

It follows from the definition of $J_p(z)$ that $J_p(z)$ *and* $N_p(z)$ *are real while* $H^{(1)}_p(z)$ *and* $H^{(2)}_p(z)$ *are complex conjugates when* p *and* z *are real.*

149. The expansion of the Neumann functionn with an integer subscript. When the subscript is an integer the solutions $J_n(z)$ and $J_{-n}(z)$ will be linearly dependent and we can take $N_n(z)$ for the second linearly-independent solution. It is therefore interesting to deduce an expansion for this solution which will hold in the whole plane. According to the general theorem of Fuchs this expansion, apart from integral powers of z, will also contain terms in $\log z$.

To start with we shall explain certain formulae which refer to the function $\Gamma(z)$. We obtained for this function the following infinite Weierstrass product

$$\frac{1}{\Gamma(z)} = e^{Cz} z \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \qquad (C = 0.57\ldots),$$

where C is Euler's constant. We know from [68] that the logarithmic derivative of this product can be obtained in the same way as for a finite product. Hence

$$-\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} + C + \sum_{k=1}^{\infty}\left(\frac{1}{z+k} - \frac{1}{k}\right);$$

and putting $z = n$, where n is a positive number, we have

$$\frac{\Gamma'(n)}{\Gamma(n)} = -\frac{1}{n} - C - \sum_{k=1}^{\infty}\left(\frac{1}{n+k} - \frac{1}{k}\right) =$$

$$= -\frac{1}{n} - C + \left(\frac{1}{1} - \frac{1}{n+1}\right) + \left(\frac{1}{2} - \frac{1}{n+2}\right) + \left(\frac{1}{3} - \frac{1}{n+3}\right) + \cdots$$

or

$$\frac{\Gamma'(n)}{\Gamma(n)} = \frac{1}{n-1} + \frac{1}{n-2} + \ldots + 1 - C \qquad (n = 2, 3, \ldots),$$

We also have $\Gamma(n) = (n-1)!$ and therefore:

$$\frac{d}{dt}\frac{1}{\Gamma(t)} = -\frac{\Gamma'(t)}{\Gamma^2(t)} = -\frac{1}{(t-1)!}\left(\frac{1}{t-1} + \frac{1}{t-2} + \ldots + 1 - C\right) \quad (t = 2, 3. \ldots), \quad (67)$$

also, when $t = 1$, we have $\Gamma(1) = 1$ and $\Gamma'(1) = -C$, and therefore

$$\frac{d}{dt}\frac{1}{\Gamma(t)} = C \qquad (t = 1). \quad (68)$$

Let us now consider the case when t is a negative integer or zero. We know that $\Gamma(z)$ has a pole of order one with a residue $(-1)^n/n!$ at the point $z = -n$, i.e. in the neighbourhood of this point we have the expansion

$$\Gamma(z) = \frac{(-1)^n}{n!\,(z+n)} + \alpha_0 + \alpha_1 (z+n) + \ldots$$

or

$$\frac{1}{\Gamma(z)} = (-1)^n n! \frac{z+n}{1 + \beta_1 (z+n) + \beta_2 (z+n)^2 + \ldots}.$$

By performing a simple differentiation we obtain directly

$$\frac{d}{dt}\frac{1}{\Gamma(t)}\Big|_{t=-n} = (-1)^n n! \qquad (n = 0, 1, 2, \ldots). \quad (69)$$

Let us now try to find the expansion of the solution $N_n(z)$ which is given by the formula (60). We have

$$J_{\pm p}(z) = \left(\frac{z}{2}\right)^{\pm p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{z}{2}\right)^{2k} \frac{1}{\Gamma(\pm p + k + 1)}$$

and, differentiating with respect to the parameter p, we obtain:

$$\frac{\partial J_p(z)}{\partial p} = \log \frac{z}{2} J_p(z) + \left(\frac{z}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{z}{2}\right)^{2k} \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\Gamma(t)}\right)_{t=p+k+1},$$

$$\frac{\partial J_{-p}(z)}{\partial p} = -\log \frac{z}{2} J_{-p}(z) - \left(\frac{z}{2}\right)^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{z}{2}\right)^{2k} \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\Gamma(t)}\right)_{t=-p+k+1}.$$

We now put $p = n$ and we obtain

$$\left.\frac{\partial J_p(z)}{\partial p}\right|_{p=n} = \log \frac{z}{2} J_n(z) + \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{z}{2}\right)^{2k} \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\Gamma(t)}\right)_{t=n+k+1}$$

and

$$\left.\frac{\partial J_{-p}(z)}{\partial p}\right|_{p=n} = -\log \frac{z}{2} J_{-n}(z) - \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{z}{2}\right)^{2k} \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\Gamma(t)}\right)_{t=-n+k+1}.$$

Substituting in formula (60) and using the formulae (67) and (68) we obtain finally, when $n > 1$:

$$\begin{aligned} \pi N_n(z) = 2J_n(z)\left(\log \frac{z}{2} + C\right) - \left(\frac{z}{2}\right)^{-n} \sum_{k=1}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k} - \\ - \left(\frac{z}{2}\right)^n \frac{1}{n!} \left(\frac{1}{n} + \frac{1}{n-1} + \dots + 1\right) - \\ - \left(\frac{z}{2}\right)^n \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{z}{2}\right)^{2k} \left(\frac{2}{n+k} + \frac{1}{n+k-1} + \dots + 1 + \right. \\ \left. + \frac{1}{k} + \frac{1}{k-1} + \dots + 1\right), \end{aligned} \tag{70}$$

and when $n = 0$:

$$\pi N_0(z) = 2J_0(z)\left(\log \frac{z}{2} + C\right) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \left(\frac{1}{k} + \frac{1}{k-1} + \dots + 1\right). \tag{71}$$

150. The case of the purely imaginary argument. If $Z_p(z)$ is a solution of the Bessel equation then, as we know, $Z_p(kz)$ is a solution of the equation [II, 49]

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{1}{z} \frac{\mathrm{d}w}{\mathrm{d}z} + \left(k^2 - \frac{p^2}{z^2}\right) w = 0. \tag{72}$$

If we put that $k = i$ we can see that the function $Z_p(iz)$ is a solution of the equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \left(1 + \frac{p^2}{z^2}\right)w = 0. \tag{73}$$

Let us suppose, to start with, that $Z_p(z)$ is equal to $J_p(z)$:

$$J_p(iz) = \sum_{k=0}^{\infty} \frac{(-1)^k i^p i^{2k}}{k!\,\Gamma(p+k+1)} \left(\frac{z}{2}\right)^{p+2k} = i^p \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(p+k+1)} \left(\frac{z}{2}\right)^{p+2k}.$$

To obtain a solution of the equation (73) which is real when p and $z > 0$ we multiply the above solution by the constant $i^{-p} = e^{-(1/2)p\pi i}$. We then obtain the following solution of the equation (73):

$$I_p(z) = e^{-\frac{1}{2}p\pi i} J_p(iz) = \sum_{k=0}^{8} \frac{1}{k!\,\Gamma(p+k+1)} \left(\frac{z}{2}\right)^{p+2k}. \tag{74}$$

The function $I_{-p}(z)$ is also a solution of the equation (73) and, when p is not an integer, $I_p(z)$ and $I_{-p}(z)$ are two linearly independent solutions of the equation (73).

If we now take $Z_p(z)$ to be equal to the first Hankel function $H_p^{(1)}(z)$ then, by adding another constant factor, we arrive at the following solution of the equation (73):

$$K_p(z) = \frac{1}{2}\pi i e^{\frac{1}{2}p\pi i} H_p^{(1)}(iz). \tag{75}$$

From (62) we can rewrite this formula as follows:

$$K_p(z) = \frac{1}{2}\pi i e^{-\frac{1}{2}p\pi i} H_{-p}^{(1)}(iz). \tag{76}$$

Using the first of the formulae (61) we can express $K_p(z)$ in terms of $I_{+p}(z)$. In fact this formula gives:

$$K_p(z) = -\frac{1}{2}\pi e^{\frac{1}{2}p\pi i} \frac{J_p(iz)\,e^{-ip\pi} - J_{-p}(iz)}{\sin p\pi},$$

or, using (74):

$$K_p(z) = -\frac{1}{2}\pi e^{\frac{1}{2}p\pi i} \frac{I_p(z)\,e^{-\frac{1}{2}p\pi i} - I_{-p}(z)\,e^{-\frac{1}{2}p\pi i}}{\sin p\pi},$$

and finally:

$$K_p(z) = \frac{1}{2}\pi \frac{I_{-p}(z) - I_p(z)}{\sin p\pi}. \tag{77}$$

The functions $I_p(z)$ and $K_p(z)$ satisfy relationships analogous to the relationships deduced for $J_p(z)$ in [144].

Using (74) and the fact that $J_{-n}(z) = (-1)^n J_n(z)$ for Bessel functions with integral subscripts, it can readily be shown that

$$I_{-n}(z) = I_n(z). \tag{78}$$

An expression for the function $K_n(z)$ with an integral subscript can be obtained from (77) by taking the limit as $p \to n$ and eliminating the indefiniteness by taking the differentials:

$$K_n(z) = \frac{(-1)^n}{2}\left[\frac{\partial I_{-p}(z)}{\partial p} - \frac{\partial I_p(z)}{\partial p}\right]_{p=n}. \tag{79}$$

As we said in [112] the asymptotic formula:

$$H_p^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right)}\left[1 + O(|z|^{-1})\right]$$

is valid when $-\pi + \varepsilon < \arg z < \pi - \varepsilon$, and therefore we can replace z by iz, where z is real and positive and $\arg(iz) = \pi/2$. Using formula (75) we obtain an asymptotic expression for $K_p(z)$ when $z > 0$:

$$K_p(z) = \frac{1}{2}\pi i e^{\frac{1}{2}p\pi i}\sqrt{\frac{2}{\pi z}}\, e^{-\frac{\pi}{4}i}\, e^{i\left(iz - \frac{p\pi}{2} - \frac{\pi}{4}\right)}\left[1 + O(z^{-1})\right]$$

or

$$K_p(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1 + O(z^{-1})\right], \qquad (z > 0), \tag{80}$$

i.e. the function $K_p(z)$ decreases exponentially as $z \to +\infty$.

The equation (73) is often met in mathematical physics and therefore the solution $K_p(z)$ with its exponential decrease is of great importance in applications to problems in physics.

In some cases the symbol $K_p(z)$ denotes a function which, in our notation, is equal to $\cos p\pi\, K_p\pi(z)$.

If k is replaced by ik in the equation (72) then we find that the functions $I_p(kz)$ and $K_p(kz)$ are solutions of the equation:

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \left(k^2 + \frac{p^2}{z^2}\right)w = 0. \tag{81}$$

These solutions will be linearly independent in the same way as $J_p(z)$ and $H_p^{(1)}(z)$ are independent in the Bessel equation.

Numerous tables for Bessel functions are available. We mention, for example, the book by Prof. Kuzmin *Bessel Functions* in which tables are given.

151. Integral representation. To explain certain properties of Bessel functions it is convenient to use integral representations which differ from those we considered earlier. These representations can be obtained by superimposing two-dimensional waves (Frank and Mizes, *Equations of Mathematical Physics)* or by the method of integral transformations (Courant–Gilbert, *Methods of Mathematical Physics)*, or, finally by the method of linear transformation of Bessel functions which we introduced above. We shall pursue the third method. Replacing $1/\Gamma(p+k+1)$ in formula (7) by its expression in terms of a contour integral [74], i.e.

$$\frac{1}{\Gamma(p+k+1)} = \frac{1}{2\pi i}\int_{l'} e^{\tau}\tau^{-(p+k+1)}\,d\tau,$$

where l' is a contour which encircles the negative part of the imaginary axis, we obtain

$$J_p(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{l'} e^{\tau}\tau^{-(p+k+1)}\left(\frac{z}{2}\right)^{p+2k}d\tau =$$

$$= \frac{1}{2\pi i}\int_{l'} e^{\tau}\tau^{-(p+1)}\left(\frac{z}{2}\right)^{p}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\tau^{-k}\left(\frac{z}{2}\right)^{2k}d\tau.$$

As a result of the uniform convergence of this latter series the transposition of summation and integration is permissible. The summation gives:

$$J_p(z) = \frac{1}{2\pi i}\int_{l'}\left(\frac{z}{2}\right)^{p}\tau^{-(p+1)}e^{-\frac{z^2}{4\tau}+\tau}\,d\tau.$$

We assume that the complex number z satisfies the condition

$$|\arg z| < \frac{\pi}{2}, \tag{82}$$

and we replace the variables according to the formula $\tau = (1/2)\,zt$. We then have

$$J_p(z) = \frac{1}{2\pi i}\int_{l} t^{-p-1}e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)}\,dt, \tag{83}$$

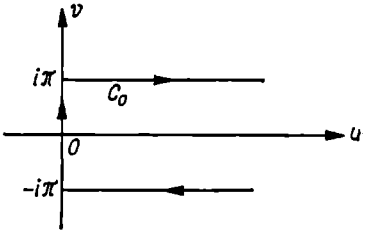

FIG. 74

where we can take the former looped contour l' as the contour of integration. Formula (83) was developed by N. Ia. Sonin (in 1870).

Take for l a contour consisting of the lower edge of the cut along the negative part of the real axis, the circle $|t| = 1$ and the upper edge of the same cut. If we take the new variable of integration given by the formula $t = e^{w}$, the contour of integration l will be transformed into the contour C_0 shown in Fig. 74 and the function $J_p(z)$ will be given by the following final expression:

$$J_p(z) = \frac{1}{2\pi i}\int_{C_0} e^{z\sinh w - pw}\,dw. \tag{84}$$

Notice that all parts of the contour at a finite distance from the origin can be deformed in any way we please. To obtain further results it is convenient to transform the integral (84). This can easily be done if we assume that C_0 has the form shown in Fig. 74 and that $w = \varphi - \pi i$.

Using the relationship $\sinh(\varphi + 2\pi i) = \sinh\varphi$ the following formula can readily be obtained (cf. [146]):

$$J_p(z) = \frac{1}{\pi}\int_0^{\pi} \cos(p\varphi - z\sin\varphi)\,d\varphi - \frac{\sin p\pi}{\pi}\int_0^{\infty} e^{-p\varphi - z\sinh\varphi}\,d\varphi. \tag{85}$$

Let us now construct an integral representation of the type (84), for the remaining cylindrical functions.

If we use formula (85) and also the relationship [148]

$$N_p(z) = \frac{J_p(z)\cos p\pi - J_{-p}(z)}{\sin p\pi},$$

we can obtain the equation

$$\pi N_p(z) = \cot p\pi \int_0^{\pi} \cos(p\varphi - z\sin\varphi)\,d\varphi -$$

$$- \frac{1}{\sin p\pi}\int_0^{\pi} \cos(p\varphi + z\sin\varphi)\,d\varphi - \int_0^{\infty} e^{p\varphi - z\sinh\varphi}\,d\varphi -$$

$$- \cos p\pi \int_0^{\infty} e^{-p\varphi - z\sinh\varphi}\,d\varphi$$

or

$$N_p(z) = \frac{1}{\pi}\int_0^{\pi} \sin(z\sin\varphi - p\varphi)\,d\varphi - \frac{1}{\pi}\int_0^{\infty} (e^{p\varphi} + e^{-p\varphi}\cos p\pi)\,e^{-z\sinh\varphi}\,d\varphi. \tag{86}$$

This formula, together with formula (85), enables us to find an integral representation for the Hankel functions:

$$H_p^{(1)}(z) = J_p(z) + iN_p(z); \quad H_p^{(2)}(z) = J_p(z) - iN_p(z).$$

We have:

$$\left.\begin{aligned} H_p^{(1)}(z) &= \frac{1}{\pi i}\int_{C_1} e^{z\sinh w - pw}\,dw, \\ H_p^{(2)}(z) &= -\frac{1}{\pi i}\int_{C_2} e^{z\sinh w - pw}\,dw, \end{aligned}\right\} \tag{87}$$

where C_1 and C_2 are infinite contours connecting $(-\infty)$ with the points $(\infty, +\pi i)$ and $(\infty, -\pi i)$ respectively. The extension of the formulae (85) and (87) when z is arbitrary can be performed by the method of analytic continuation

152. The asymptotic representation of Hankel functions. The integral representations (84) and (87) are convenient for finding the approximate expression for cylindrical functions when the values of $|z|$ and $|p|$ are large. Put

$$\frac{p}{z} = \xi \tag{88}$$

and consider the function

$$f(w) = \sinh w - \xi w. \tag{89}$$

The integrals in the formulae (84) and (89) will then take the following form

$$\int_{C_\nu} e^{zf(w)}\, dw. \tag{90}$$

We shall use the method of the steepest descent and assume that p and z are positive and real.

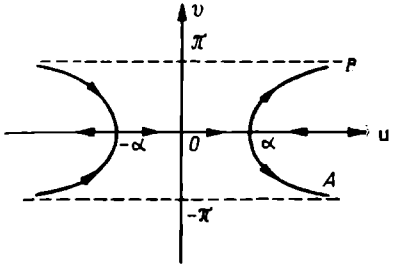

FIG. 75

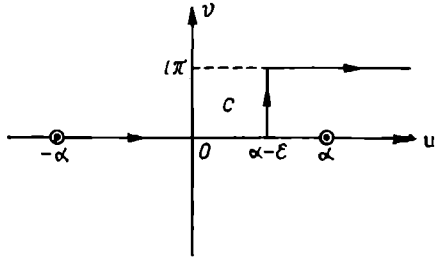

FIG. 76

Before using this method it is important to explain the position of the saddle points w_0 which are determined by the condition

$$f'(w_0) = \cosh w_0 - \xi = 0,$$

and to establish the position of the contours

$$I_m(\sinh w - \xi w) = I_m(\sinh w_0 - \xi w_0)$$

and, lastly, to convince ourselves that the contours C_1, C_2 and C_0 can be transformed to the lines of the steepest descent of the function (89).

We shall investigate all these cases and consider three separate instances depending on the value of $\xi = p/z$.

CASE 1. $\xi > 1$, $z \gg 1$. The saddle points are at $w_0 = \pm a$, where $a > 0$ is derived from the equation $\cosh a = \xi$. The equation of the stationary contours which pass through the saddle points are as follows:

$$v = 0 \text{ and } \sin v \cosh u = v \cosh a \qquad (w = u + vi). \tag{91}$$

These stationary contours, situated symmetrically with respect to the axes of coordinates, are shown in Fig. 75 where the directions in which the real part of $f(w)$ decreases are shown by arrows. Considering

$$\mathcal{R}[f(w)] = \sinh u \cos v - \xi u,$$

it can readily be seen that if we take for the contours of integration C_0, C_1 and C_2 in the formulae (84) and (87) the stationary contours $(-\infty, -a, a, B)$ $(-\infty, -a, a, A)$ and (A, a, B) respectively, then the cylindrical functions for large values of the argument z are determined by integration along small sections of the contours in the neighbourhood of the saddle points. We shall give details of these calculations by taking the function $H_p^{(1)}(z)$ as an example. Changing the path of integration we replace the stationary contour $(-\infty, -a, a, B)$ by the contour C shown in Fig. 76. We then have

$$H_p^{(1)}(z) = \frac{1}{\pi i}\int_C e^{zf(w)}\,dw. \tag{92}$$

Choose the value

$$\varepsilon = \left(\frac{12}{z\cosh a}\right)^{\frac{1}{3}} \tag{93}$$

and assume that z is such that

$$\frac{z\sinh a}{2}\varepsilon^2 = N \geqslant 8. \tag{94}$$

Notice that we had to make similar conditions in Example 2 in [80].

It follows from (93) and (94):

$$z\sinh a = \frac{N}{\sqrt[3]{18}}(z\cosh a)^{\frac{2}{3}} > 3\,(z\cosh a)^{\frac{2}{3}} \tag{95}$$

and

$$\varepsilon = \frac{6\sinh a}{N\cosh a} < 0.75\,\frac{\sinh a}{\cosh a}. \tag{96}$$

This will be useful in later results.

Break the integral (92) into the sum of the following five integrals:

$$\int_{-\infty}^{-a-\varepsilon} e^{zf(w)}\,dw + \int_{-a-\varepsilon}^{-a+\varepsilon} e^{zf(w)}\,dw + \int_{-a+\varepsilon}^{a-\varepsilon} e^{zf(w)}\,dw + \\ + \int_{a-\varepsilon}^{a-\varepsilon+\pi i} e^{zf(w)}\,dw + \int_{a-\varepsilon+\pi i}^{\infty+\pi i} e^{zf(w)}\,dw. \tag{97}$$

The second integral we considered in [80]. Let us find the upper bounds of all the remaining integrals in (97). To do so we consider

$$\Phi(w) = \mathcal{R}\left[f(w)\right] = \sinh u\cos v - u\cosh a. \tag{98}$$

In the interval $-\infty < u < -a-\varepsilon$ we have:

$$\Phi(w) = \Phi(-a-\varepsilon) + \left[\Phi(w) - \Phi(-a-\varepsilon)\right] = \Phi(-a-\varepsilon) - \\ - \left[\cosh(a+\varepsilon) - \cosh a\right]|u+a+\varepsilon| - \frac{\sinh(a+\varepsilon)}{2!}|u+a+\varepsilon|^2 - \ldots < \\ < \Phi(-a-\varepsilon) - \left[\cosh(a+\varepsilon) - \cosh a\right]|u+a+\varepsilon|.$$

But

$$\Phi(-a-\varepsilon) = f(-a-\varepsilon) = f(-a) - \frac{\sinh a}{2!}\varepsilon^2 - \frac{\cosh a}{3!}\varepsilon^3 - \ldots <$$

$$< f(-a) - \frac{\sinh a}{2}\varepsilon^2 = f(-a) - \frac{N}{z}$$

and

$$\cosh(a+\varepsilon) - \cosh a = \frac{\sinh a}{1}\varepsilon + \frac{\cosh a}{2}\varepsilon^2 + \ldots > \frac{\sinh a}{1}\varepsilon + \frac{\cosh a}{2}\varepsilon^2 >$$

$$> \frac{\varepsilon^2 \sinh a}{2}\frac{2}{\varepsilon} + \frac{\cosh a}{2}\varepsilon^2 > \frac{3N}{z}\frac{\cosh a}{\sinh a}.$$

Therefore finally

$$\Phi(w) < f(-a) - \frac{N}{z} - \frac{3N}{z}|u+a+\varepsilon|\frac{\cosh a}{\sinh a}.$$

Using this inequality we find

$$\left|\int_{-\infty}^{-a-\varepsilon} e^{zf(w)}\,dw\right| < e^{zf(-a)}\frac{e^{-N}\sinh a}{3N\cosh a}. \tag{99}$$

In the interval $-a+\varepsilon \leqslant u \leqslant a-\varepsilon$, $v=0$ we have:

$$\Phi(u) = f(u) = \sinh u - u\cosh a$$

$$f'(u) = -(\cosh a - \cosh u) \leqslant -[\cosh a - \cosh(a-\varepsilon)]$$

$$f(u) < f(-a+\varepsilon) - [\cosh a - \cosh(a-\varepsilon)](u+a-\varepsilon).$$

But from (96)

$$f(-a+\varepsilon) = f(-a) - \frac{\sinh a}{2!}\varepsilon^2 + \frac{\cosh a}{3!}\varepsilon^3 - \frac{\sinh a}{4!}\varepsilon^4 + \ldots <$$

$$< f(-a) - \frac{\sinh a}{2!}\varepsilon^2 + \frac{\cosh a}{3!}\varepsilon^3 =$$

$$= f(-a) - \frac{\sinh a}{2}\varepsilon^2\left(1 - \frac{\varepsilon\cosh a}{3\sinh a}\right) \leqslant f(-a) - 0.75\frac{N}{z},$$

and

$$\cosh a - \cosh(a-\varepsilon) = \frac{\sinh a}{1!}\varepsilon - \frac{\cosh a}{2!}\varepsilon^2 + \frac{\sinh a}{3!}\varepsilon^3 - \ldots >$$

$$> \varepsilon\sinh a\left(1 - \frac{\varepsilon\cosh a}{2\sinh a}\right) \geqslant \frac{5}{8}\frac{\sinh a}{2}\varepsilon^2\frac{2}{\varepsilon} > \frac{5N\cosh a}{3z\sinh a},$$

using (94) and (96).

We have finally

$$f(u) < f(-a) - 0.75\frac{N}{z} - \frac{N}{z}(u+a-\varepsilon)\frac{5\cosh a}{3\sinh a}.$$

Using this inequality we obtain:

$$\left| \int_{-a+\varepsilon}^{a-\varepsilon} e^{zf(w)}\, dw \right| < \frac{3 \sinh a}{5N \cosh a}\, e^{zf(-a)-0.75N}. \tag{100}$$

To obtain inequalities for the last two integrals in (97) we consider

$$f(-a) = -f(a) = (a - \tanh a) \cosh a.$$

Using the expansion

$$a = \operatorname{arc\,tanh} \eta = \eta + \frac{\eta^3}{3} + \frac{\eta^5}{5} + \dots,$$

we obtain

$$f(-a) > \cosh a \frac{\tanh^3 a}{3} = \frac{\sinh^3 a}{3 \cosh^2 a}.$$

If, however, we use (95) we can write

$$f(-a) < \frac{N^3}{54z}.$$

In this case, from the inequality $a > \varepsilon$, it is evident that

$$f(a-\varepsilon) < 0 < f(-a) - \frac{N^3}{54z}.$$

To find an upper bound for the fourth integral in (97) we notice that in the interval $u = a - \varepsilon$, $0 \leqslant v \leqslant \pi$ and we have

$$\Phi(w) = f(a-\varepsilon) - (1 - \cos v) \sinh(a-\varepsilon).$$

But

$$1 - \cos v > \frac{2v^2}{\pi^2},$$

and from (96),

$$\frac{\sinh a}{\sinh(a-\varepsilon)} = \frac{\sinh a}{\sinh a - \varepsilon \cosh a + \varepsilon^2 \dfrac{\sinh a}{2} - \dots} <$$

$$< \frac{\sinh a}{\sinh a - \varepsilon \cosh a} \leqslant \frac{100}{25} = 4,$$

i.e. $\sinh(a-\varepsilon) > (1/4) \sinh a$. Therefore

$$\Phi(w) < f(a-\varepsilon) - \frac{v^2}{2\pi^2} \sinh a < f(-a) - \frac{N^3}{54z} - \frac{v^2}{2\pi^2} \sinh a.$$

Using this inequality we obtain

$$\left| \int_{a-\varepsilon}^{a-\varepsilon+\pi i} e^{zf(w)}\, dw \right| < \sqrt{\frac{\pi^3}{2z \sinh a}}\, e^{zf(-a)-\frac{N^3}{54}}. \tag{101}$$

Finally in the interval $\alpha - \varepsilon \leqslant u < \infty$, $v = \pi$ and we have

$$\Phi(w) = -\sinh u - u\cosh\alpha < -u\cosh\alpha < f(-\alpha) - \frac{N^3}{54z} - u\cosh\alpha.$$

Therefore we obtain the following inequality for the last integral in (97):

$$\left| \int_{\alpha-\varepsilon+\pi i}^{\infty+\pi i} e^{zf(w)}\,dw \right| < e^{zf(-\alpha)} \frac{e^{-\frac{N}{54}}}{z\cosh\alpha}. \tag{102}$$

Notice that the inequalities (101) and (102) could easily be made more exact.

Let us now return to the expression (92). Using (97) and also the inequalities (99), (100), (101) and (102) we obtain the following expressions:

$$H_p^{(1)}(z) = \frac{1}{\pi i}\left[\int_{-\alpha-\varepsilon}^{-\alpha+\varepsilon} e^{zf(w)}\,dw + \omega\right], \tag{103}$$

in which

$$\omega < e^{zf(-\alpha)}\left[\frac{3\sinh\alpha}{5N\cosh\alpha}\,e^{-0.75N} + \frac{\sinh\alpha}{3N\cosh\alpha}\,e^{-N} + \right.$$
$$\left. + \left(\frac{4.4}{\sqrt{z\sinh\alpha}} + \frac{1}{z\cosh\alpha}\right) e^{-\frac{N^3}{54}}\right]. \tag{104}$$

The integral in (103) we investigated in [80] where we saw that it can be represented by the formula

$$\frac{1}{\pi i}\int_{-\alpha-\varepsilon}^{-\alpha+\varepsilon} e^{zf(w)}\,dw =$$

$$= -\frac{i}{\sqrt{\pi}}\,e^{zf(-\alpha)}\left(\frac{2}{z\sinh\alpha}\right)^{\frac{1}{2}}\left[1 - \frac{1}{8}\left(1 - \frac{5\cosh^2\alpha}{3\sinh^2\alpha}\right)\frac{1}{z\sinh\alpha} + \omega'\right], \tag{105}$$

where

$$|\omega'| \leqslant \frac{e^{-N}}{\sqrt{\pi}}\left(1 + \frac{N^{\frac{5}{2}}\cosh^2\alpha}{6z\sinh^3\alpha}\right) + \left(\frac{2}{z\sinh\alpha}\right)^2\left(\frac{1}{8} + \frac{\cosh^2\alpha}{25\sinh^2\alpha} + \frac{\cosh^4\alpha}{8\sinh^4\alpha}\right). \tag{106}$$

If the term ω in (103) is taken into account then the function $H_p^{(1)}(z)$ can be represented by the right-hand side of the formula (105) where instead of ω we have $\omega' + \omega''$ and where ω'' satisfies the following condition:

$$|\omega''| < \frac{1}{\sqrt{\pi}}\left(\frac{z\sinh\alpha}{2}\right)^{\frac{1}{2}}\left[\frac{3\sinh\alpha}{5N\cosh\alpha}\,e^{-0.75N} + \frac{\sinh\alpha}{3N\cosh\alpha}\,e^{-N} + \right.$$
$$\left. + \left(\frac{4.4}{\sqrt{z\sinh\alpha}} + \frac{1}{z\sinh\alpha}\right) e^{-\frac{N^3}{54}}\right]. \tag{107}$$

We can easily find an upper bound for the right-hand side. To do so it is sufficient to use the equation

$$\left(\frac{z \sinh a}{2}\right)^{\frac{1}{2}} \frac{\sinh a}{N \cosh a} = \frac{\sqrt{N}}{6},$$

which follows from (95). With the aid of this equation the values of the right-hand sides of (106) and (107) can be compared. It thus appears that when $N > 8$ the value of (106) exceeds the value of the right-hand side of (107). Therefore when $N > 8$ the error in the formula for $H_p^{(1)}(z)$ [of the type (105)] will be determined by the second term in (106). Notice that the condition $N > 8$ in our calculations is equivalent to the requirement:

$$z \sinh a > 3 (z \cosh a)^{\frac{2}{3}}, \tag{108}$$

i.e.

$$\sqrt{p^2 - z^2} > 3p^{\frac{2}{3}}. \tag{109}$$

Terms of an even smaller order can be calculated similarly. We then obtain the following formulae: (cf. Watson: *A Treatise on The Theory of Bessel Functions*)

$$\left.\begin{aligned} H_p^{(1)}(z) &\sim -i \sqrt{\frac{2}{\pi s}}\, e^{-s + p \operatorname{arc\,tanh} \frac{s}{p}} G(-s) \\ H_p^{(2)}(z) &\sim i \sqrt{\frac{2}{\pi s}}\, e^{-s + p \operatorname{arc\,tanh} \frac{s}{p}} G(-s) \end{aligned}\right\} \tag{110}$$

and

$$J_p(z) \sim \frac{1}{2} \sqrt{\frac{2}{\pi s}}\, e^{s - p \operatorname{arc\,tanh} \frac{s}{p}} G(s), \tag{111}$$

where

$$s^2 = p^2 - z^2,$$

and $$G(s) = 1 + \frac{1}{8}\left(\frac{1}{s} - \frac{5p^2}{3s^3}\right) + \frac{1 \cdot 3}{8^2}\left(\frac{3}{2s^2} - \frac{77p^2}{9s^4} + \frac{385p^4}{54s^6}\right) + \dots \tag{112}$$

This series does not converge for any s and p. However when s and p are sufficiently large then the terms decrease before beginning to increase again. The series (112) must always be terminated by terms which are still decreasing. It can be proved that if the series (112) is discontinued in the way described above and if the inequality below is satisfied

$$\sqrt{p^2 - z^2} = s > 2.5p^{\frac{2}{3}}, \tag{113}$$

(Note that this condition can only apply when $p > (2.5)^3 \sim 16$), then the right-hand side of (111) gives approximate values for Bessel functions to an accuracy greater than that of the last remaining term.

To obtain a clear picture of the behaviour of Bessel functions when $z < p$ the following expansion can be used:

$$\operatorname{arc\,tanh} \frac{s}{p} = \frac{s}{p} + \frac{s^3}{3p^3} + \dots$$

It is then apparent that the expression

$$-s + p \operatorname{arc\,tanh} \frac{s}{p} = \frac{(p^2 - z^2)^{\frac{3}{2}}}{3p^2} + \dots$$

increases as z tends to zero from values close to p. It can be seen from (110) and (111) that when the values of z vary in the way described above, the Hankel functions will grow exponentially and the Bessel functions will decrease exponentially. This latter fact is used, for example, for testing the convergence of series of the following type:

$$\sum_{n=0}^{\infty} c_n J_n(\varrho).$$

If $|c_n| < Mn^{\sigma}$ $(\sigma > 0)$, then the above series will, in any case, converge when $n > \varrho$.

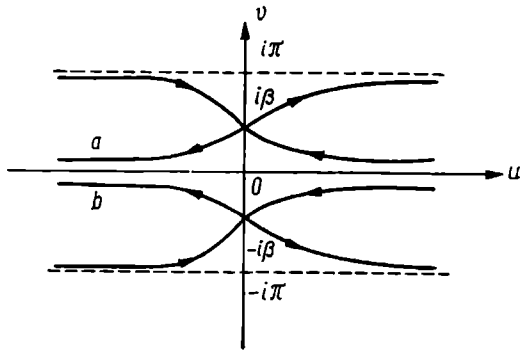

FIG. 77

CASE 2. $\xi < 1$, $z \gg 1$. The saddle-points now have coordinates $w_0 = \pm \beta i$ where $\cos \beta = \xi$ $(\beta > 0)$. The stationary contours, determined from the equations

$$\left.\begin{aligned} \sinh u \sin v &= (v - \beta) \cos \beta + \sin \beta, \\ \sinh u \sin v &= (v + \beta) \cos \beta - \sin \beta, \end{aligned}\right\} \tag{114}$$

are situated symmetrically with respect to the axes of coordinates and pass through the saddle-points $\pm i\beta$ and ∞ respectively. These stationary contours are represented in Fig. 77 where the directions in which $\mathcal{R}[f(w)]$ decreases are marked by arrows.

If we take the stationary curves (a) and (b) which connect $(-\infty)$ with the points $(\infty, +\pi i)$ and $(\infty, -\pi i)$ respectively, for the contours of integration C_1 and C_2 in the formulae (87), then the determination of the principal parts of the Hankel function will involve integration in the neighbourhood of the saddle-points $\pm \beta i$. It, is then apparent that the values of both Hankel functions are of the same order as their sum. Therefore to obtain the asymptotic representa-

tion for the function $J_p(z)$ it is not necessary to perform additional calculations but it is sufficient to use the formula:

$$J_p(z) = \frac{1}{2}\left[H_p^{(1)}(z) + H_p^{(2)}(z)\right].$$

The asymptotic formulae can be constructed by the usual method of the steepest descent. Without performing these calculations here we give the following final formulae:

$$\left.\begin{aligned} H_p^{(1)}(z) &\sim \sqrt{\frac{2}{\pi s}}\, G(si)\, e^{\varphi i}, \\ H_p^{(2)}(z) &\sim \sqrt{\frac{2}{\pi s}}\, G(-si)\, e^{-\varphi i}, \\ J_p(z) &\sim \sqrt{\frac{2}{\pi s}}\,(G_1 \cos\varphi + G_2 \sin\varphi), \end{aligned}\right\} \tag{115}$$

in which

$$z^2 = p^2 + s^2;\ G(si) = G_1 - G_2 i;\ \varphi = s - p \arctan\frac{s}{p} - \frac{\pi}{4}, \tag{116}$$

and where $G(s)$ is the series (112). It can be shown that when

$$\sqrt{z^2 - p^2} = s > 2.5p^{\frac{2}{3}} \quad \text{and} \quad s > 6 \tag{117}$$

and if in the expressions G, G_1 and G_2 only decreasing terms are retained, then the error in the formula (115) will not exceed the value of the last remaining term. It can readily be shown that the asymptotic formulae (115) become the Hankel formulae which we obtained in [112] when $z \gg p$.

We can obtain the corresponding formulae on the assumption that in the series G, G_1 and G_2 only the first terms are retained. Noting that when $z \gg p$ the following approximate equations are valid

$$s \sim z \quad \text{and} \quad \arctan\frac{s}{p} \sim \arctan\frac{z}{p} \sim \frac{\pi}{2},$$

we obtain from (115):

$$\left.\begin{aligned} H_p^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}}\, e^{i\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right)}, \\ H_p^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}}\, e^{-i\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right)}, \\ J_p(z) &\sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right). \end{aligned}\right\} \tag{118}$$

Case 3. $\xi \sim 1$, $p \gg 1$. The position of the stationary contours and of the saddle-points can, in this case, be established by taking the limit as $\xi \to 1$. It is then apparent that the saddle-points lie near the origin and that by moving along the stationary contours, the value of integrand changes very rapidly. But, in spite of this, the above calculations become devoid of meaning since the condition (113) is no longer satisfied.

The case of the asymptotic representation of Bessel functions in which we are interested corresponds to the conditions

$$\sqrt{|p^2 - z^2|} \sim |p|^{\frac{2}{3}} \; p \gg 1, \tag{119}$$

and was first systematically studied by Academician V. A. Fok (V. A. Fok, *A New Asymptotic Representation of Bessel Functions*, Reports of the Academy of Science, 1934, vol. 1, No. 3, pp. 97—99; V. A. Fok, *The Diffraction of Radio Waves Round The Surface of The Earth;* V. A. Fok, *Tables of Eiry functions).* We explained the result of his investigations in the example of the function $H_p^{(1)}(z)$.

If in the first of the formulae (87) w is replaced by $(-w)$ we obtain for $H_p^{(1)}(z)$ the expression

$$H_p^{(1)}(z) = \frac{1}{\pi i} \int_C e^{-z \sinh w + pw} \, dw, \tag{120}$$

in which the path of integration connects $(-\infty, -\pi i)$ with $(+\infty)$.

The saddle-point of the integrand function in (120) lies very close to the origin and the stationary contour can be modified to a contour along the straight line $I_m(w) = -\pi$, from $(-\infty, -\pi i)$ to the point $w_1 = -\pi/\sqrt{3} - \pi i$, then along the straight line from w_1 to the origin and, finally from the origin along the positive part of the real axis. When moving along this path away from the origin the integrand function decreases very rapidly. Therefore the value of the integral (120) is determined by integration along a small section of the contour near the origin. Denote this section of the contour by l_ε. We can then write:

$$H_p^{(1)}(z) = \frac{1}{\pi i}\left[\int_{l_\varepsilon} e^{-z \sinh w + pw} \, dw + w_\varepsilon(z, p) \, , \right] \tag{121}$$

where the upper bound of $w_\varepsilon(z, p)$ is approximated to in the same way as we did when $\xi > 1$. When the values of z are greater the value of w_ε appears to be negligibly small.

Let us suppose that

$$p = z + \left(\frac{z}{2}\right)^{\frac{1}{3}} t \tag{122}$$

and introduce a new variable of integration

$$\tau = \left(\frac{z}{2}\right)^{\frac{1}{3}} w. \tag{123}$$

We thus have

$$-z \sinh w + pw = t\tau - \frac{\tau^3}{3} - \frac{z}{120}\left(\tau \sqrt[3]{\frac{2}{z}}\right)^5 - \frac{z}{5040}\left(\tau \sqrt[3]{\frac{2}{z}}\right)^7 - \dots \tag{124}$$

and

$$e^{-z \sinh w + pw} = e^{t\tau - \frac{\tau^3}{3}}\left[1 - \frac{1}{60}\left(\frac{z}{2}\right)^{-\frac{2}{3}} \tau^5 + \dots\right]. \tag{125}$$

Notice that the expansion on the right-hand side converges rapidly on the contour l_ε.

To obtain an approximate expression for $H_p^{(1)}(z)$ the equation (125) can be substituted in (121). We then obtain the following expression for the integral along l_ε:

$$\left(\frac{z}{2}\right)^{-\frac{1}{3}}\left[\int_{L_\varepsilon} e^{t\tau-\frac{\tau^3}{3}}\,d\tau - \frac{1}{60}\left(\frac{z}{2}\right)^{-\frac{2}{3}}\int_{L_\varepsilon}\tau^5 e^{t\tau-\frac{\tau^3}{3}}\,d\tau+\dots\right], \tag{126}$$

in which small terms are omitted. Notice that the contour ε_ε in this formula consists of straight lines connecting the point $(z/2)(-\pi/\sqrt{3}-\pi i)\varepsilon/3$ with the origin and the origin with the point $(z/2)^{\frac{1}{2}}\varepsilon$.

Let us consider finally the contour Γ formed by the ray $\arg\tau = 4\pi/3$ with the positive part of the real axis. The first integral in formula (126) can then be written in the form:

$$\int_{L_\varepsilon} e^{t\tau-\frac{\tau^3}{3}}\,d\tau = \int_{\Gamma} e^{t\tau-\frac{\tau^3}{3}}\,d\tau - \int_{\Gamma-L_\varepsilon} e^{t\tau-\frac{\tau^3}{3}}\,d\tau,$$

and the same thing applies to the second integral.

The upper bound of the integral along that part of the contour Γ which does not belong to the contour L (this part we denoted by $\Gamma - L_\varepsilon$) can easily be calculated to be negligibly small when z is large. Therefore when $z \gg 1$ the contour of integration L_ε in formula (126) can be replaced by the contour Γ. As a result we obtain the following approximate formula:

$$H_p^{(1)}(z) = \frac{-i}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{-\frac{1}{3}}\left[w(t) - \frac{1}{60}\left(\frac{z}{2}\right)^{-\frac{2}{3}}\frac{d^5 w(t)}{dt^5}+\dots\right], \tag{127}$$

in which

$$w(t) = \frac{1}{\sqrt{\pi}}\int_{\Gamma} e^{t\tau-\frac{\tau^3}{3}}\,d\tau \tag{128}$$

is the Airy function investigated by V. A. Fok. Tables have been constructed for this function. We notice in conclusion that formulae for calculating the residue in formula (127) can be obtained without difficulty in the way described above.

The treatment of this subject and the examples in [80] are due to Prof. G. I. Petrashen.

153. Bessel functions and the Laplace equation. The Bessel equation occurs frequently in problems of mathematical physics. Owing to the lack of space we are unable to investigate fully the applications of Bessel functions and we shall only consider basic facts which connect the Bessel equation with the fundamental equations of mathematical physics.

Let us begin with the Laplace equation. Previously we investigated the Laplace equation in spherical coordinates and we thus arrived at spherical functions. Similarly, by writing the Laplace equation in cylindrical coordinates and separating the variables we arrive at Bessel functions.

The Laplace equation in cylindrical coordinates has the form:

$$\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial U}{\partial \varrho}\right) + \frac{1}{\varrho}\frac{\partial^2 U}{\partial \varphi^2} + \varrho \frac{\partial^2 U}{\mathrm{d}z^2} = 0.$$

We shall seek a solution of this equation in the form of a product of three functions, one of which is a function of ϱ, the second a function of φ and the third a function of z:

$$U = R(\varrho)\, \Phi(\varphi)\, Z(z).$$

Substituting in the Laplace equation and separating the variables we have:

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\varrho}\left[\varrho \frac{\mathrm{d}R(\varrho)}{\mathrm{d}\varrho}\right]}{R(\varrho)} + \frac{1}{\varrho}\frac{\frac{\mathrm{d}^2 \Phi(\varphi)}{\mathrm{d}\varphi^2}}{\Phi(\varphi)} + \varrho \frac{\frac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2}}{Z(z)} = 0.$$

Each of the above fractions will be equal to a constant since only the independent variable ϱ varies in the first fraction while z only varies in the third fraction. Equating the second fraction to a constant $(-p^2)$ and the third to a constant k^2 we obtain the following three equations:

$$\Phi''(\varphi) + p^2 \Phi(\varphi) = 0; \quad Z''(z) - k^2 Z(z) = 0;$$

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\left[\varrho R'(\varrho)\right] - \frac{p^2}{\varrho} R(\varrho) + k^2 \varrho R(\varrho) = 0,$$

or

$$R''(\varrho) + \frac{1}{\varrho} R'(\varrho) + \left(k^2 - \frac{p^2}{\varrho^2}\right) R(\varrho) = 0.$$

We shall, for the moment, assume that the constants ϱ and k are not zero. The first two equations give

$$\Phi(\varphi) = \mathrm{e}^{\pm ip\varphi} \text{ or } \Phi(\varphi) = \begin{matrix}\cos p\varphi \\ \sin p\varphi\end{matrix}$$

$$Z(z) = \mathrm{e}^{\pm kz}.$$

Finally, the third equation gives $Z_p(k\varrho)$, where $Z_p(z)$ is an arbitrary solution of the Bessel equation with a parameter p. If we require a single-valued solution then we must assume that the constant p is equal to an integer n.

We then obtain the solution of the Laplace equation in the following form:

$$e^{\pm kz} \begin{matrix} \cos n\varphi \\ \sin n\varphi \end{matrix} \left[C_1 J_n(k\varrho) + C_2 N_n(k\varrho)\right], \tag{129}$$

where n is any integer and the constant k is arbitrary.

If $k = 0$ then in place of $Z(z) = e^{\pm kz}$ we assume that $Z(z) = 1$ or $Z(z) = z$ and the equation for $R(\varrho)$ will give $R(\varrho) = \varrho^{\pm p}$. Finally when $p = 0$ we assume that $\Phi(\varphi) = A + B\varphi$, and when $p = k = 0$ that $R(\varrho) = C + D \log \varrho$. When $n = 0$ formula (129) gives the solution in the following form:

$$e^{\pm kz}\left[C_1 J_0(k\varrho) + C_2 N_0(k\varrho)\right], \tag{130}$$

which does not depend on the angle φ. These equations are of importance in connection with potentials of masses with an axial symmetry. If we require a finite solution when $\varrho = 0$ then we must assume in formula (130) that the constant C_2 is zero and we then obtain a solution in the form

$$e^{\pm kz} J_0(k\varrho). \tag{131}$$

When solving Laplace equations of this type it is possible to obtain the solution $1/r$ which is of fundamental importance in the theory of Newton's potentials, viz. the following formula will hold

$$\int_0^\infty e^{-kz} J_0(k\varrho)\, dk = \frac{1}{\sqrt{\varrho^2 + z^2}} = \frac{1}{r} \qquad (z > 0), \tag{132}$$

which has numerous applications in the theory of potentials. To prove this formula we consider formula (42) which gives

$$e^{-kz} J_0(k\varrho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-kz - ik\varrho \sin\varphi}\, d\varphi,$$

and integrating with respect to k we obtain:

$$\int_0^\infty e^{-kz} J_0(k\varrho)\, dk = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{e^{-kz - ik\varrho\sin\varphi}}{-z - i\varrho\sin\varphi}\right]_{k=0}^{k=\infty} d\varphi,$$

or, substituting the limits

$$\int_0^\infty e^{-kz} J_0(k\varrho)\, dk = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{z + i\varrho\sin\varphi}\, d\varphi.$$

This latter integral can easily be evaluated by the method given in [57] whence formula (132) follows directly.

If we replace the constant $(+k^2)$ by the constant $(-k^2)$ then $e^{\pm kz}$ becomes $\cos kz$ and $\sin kz$, while $J_p(k\varrho)$ and $N_p(k\varrho)$ can be replaced by $I_p(k\varrho)$ and $K_p(k\varrho)$.

154. The wave equation in cylindrical coordinates. Let us consider the wave-equation:

$$\frac{\partial^2 U}{\partial t^2} = a^2 \Delta U, \tag{133}$$

where

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2};$$

we shall seek its solution in the form of a product

$$U = e^{-i\omega t} V(x, y, z). \tag{134}$$

Substituting in the equation (133) we obtain for V a solution in the form

$$\Delta V + k^2 V = 0, \tag{135}$$

where

$$k^2 = \frac{\omega^2}{a^2}. \tag{136}$$

The equation (135) is sometimes called the *Helmholtz equation*. If we take any solution of this equation, substitute in formula (134) and separate the real parts then it will give us a real solution of the wave equation which, in relation to time, represents a harmonic vibration of frequency. In some cases this solution can represent a *stationary wave* and in other cases a *propagating wave*. We shall explain this with simple examples. If we examine, for example, the product $e^{-i\omega t} \sin kx$ then its real part $\cos \omega t \sin kx$ defines a stationary wave. Similarly, the product $e^{-i\omega t} \cos kx$ also defines a stationary wave. If, however, we examine the product $e^{-i\omega t} e^{ikx}$ then its real part $\cos(kx - \omega t)$ is the sine wave which moves in the direction of the X-axis with velocity ω/k. When Bessel functions are applied, $\cos kx$ and $\sin kx$ can be replaced by $J_p(k\varrho)$ and $N_p(k\varrho)$ while e^{ikx} and e^{-ikx} should be replaced by $H_p^{(1)}(k\varrho)$ and $H_p^{(2)}(k\varrho)$.

Let us now return to the equation (138) and write the Laplace operator in cylindrical coordinates assuming, for the moment, that V does not depend on z [II, 178]

$$\frac{\partial^2 V}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial V}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2 V}{\partial \varphi^2} + k^2 V = 0.$$

We have already solved such an equation by separating the variables and we know that it has solutions of the form $Z_p(k\varrho)\begin{matrix}\sin p\varphi\\ \cos p\varphi\end{matrix}$, where $Z_p(z)$ is any solution of the Bessel equation with parameter p.

By assuming that $p = n$ is an integer we obtain a single-valued solution. When taking the Bessel function we obtain the solutions

$$e^{-i\omega t} J_n(k\varrho) \begin{matrix}\cos n\varphi\\ \sin n\varphi\end{matrix},$$

the real part of which

$$\cos \omega t\, J_n(k\varrho) \begin{matrix}\cos n\varphi\\ \sin n\varphi\end{matrix}$$

determines a stationary wave. If we take the first Hankel function as a solution then, bearing in mind the asymptotic representation of Hankel functions when the value of the argument is large, we have, by taking the first terms only the following asymptotic representation:

$$e^{-i\omega t} H_p^{(1)}(k\varrho) = e^{i\left(k\varrho - \frac{n\pi}{2} - \frac{\pi}{4} - \omega t\right)} \sqrt{\frac{2}{\pi k\varrho}} \left[1 + O(\varrho^{-1})\right],$$

i.e. *we have a propagating wave at infinity the phase of which moves to infinity.* We say that these solutions satisfy the *radiation principle.* If, however, instead of the factor $e^{-i\omega t}$ we take the factor $e^{i\omega t}$, then in order to satisfy the principle of radiation we must take the second Hankel function as the second factor on the left-hand side since, according to the asymptotic expansion, we have the following asymptotic equation:

$$e^{i\omega t} H_n^{(2)}(k\varrho) = e^{i\left(\omega t - k\varrho + \frac{n\pi}{2} + \frac{\pi}{4}\right)} \sqrt{\frac{2}{\pi k\varrho}} \left[1 + O(\varrho^{-1})\right].$$

Consider now the general case when the function V depends on the coordinate z. Equation (135) then becomes [II, 119]:

$$\frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial V}{\partial \varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} + k^2 V = 0.$$

We will seek its solution in the form:

$$V = R(\varrho)\, \Phi(\varphi)\, Z(z).$$

Separating the variables as usual we obtain the solution of the equation:

$$Z_p\left(\sqrt{k^2 - h^2}\,\varrho\right) e^{\pm ihz} \begin{matrix}\cos p\varphi\\ \sin p\varphi\end{matrix}, \tag{137}$$

where $Z_p(z)$ is any solution of the Bessel equation. Putting $k^2 - h^2 = \lambda^2$ and considering single-valued solutions ($p = n$ which is a positive integer) we obtain the following solutions:

$$J_n(\lambda\varrho)\, e^{\sqrt{\lambda^2-k^2}\cdot z} \begin{matrix}\cos n\varphi \\ \sin n\varphi\end{matrix} \quad (138) \quad \text{and} \quad H_n^{(1)}(\lambda\varrho)\, e^{\sqrt{\lambda^2-k^2}\cdot z} \begin{matrix}\cos n\varphi \\ \sin n\varphi\end{matrix}. \quad (139)$$

The first of these remains finite when $\varrho = 0$ and describes a stationary wave. The second solution satisfies the radiation principle. The solutions of the first type are generally used when the inner part of a cylinder containing the axis $\varrho = 0$ is the domain in which the vibrations occur. Solutions of the second type are used for the space outside the cylinder. In diffraction problems many-valued solutions where p is not an integer are also be frequently used.

Let us consider one particular problem. The equation (135) has the obvious solution $e^{ikx} = e^{ik\varrho\cos\varphi}$. Multiplying this by $e^{-i\omega t}$ we obtain the solution $e^{i(kx-\omega t)}$ which represents an elementary flat wave propagated along the X-axis. Let us suppose that this flat wave exists not in the whole infinite space but only outside the cylinder $\varrho = a$, where it must satisfy the boundary condition:

$$V = 0 \text{ (when } \varrho = a).$$

To satisfy this boundary condition we must add to the solution e^{ikx} of the equation (135) a certain other solution of this equation (additional disturbance caused by diffraction) and this additional solution must be single-valued and satisfy the radiation principle. Bearing all this in mind as well as the independence of z of the fundamental solution we shall seek the additional solution by using exponential functions instead of trigonometric functions, in the form of a linear combination of solutions of the form (139) when $\lambda = k$:

$$\sum_{n=-\infty}^{+\infty}{}' a_n H_n^{(1)}(k\varrho)\, e^{in\varphi} \qquad (\varrho > a). \quad (140)$$

We only have to determine the coefficients a_n from the boundary condition. Remembering formula (37) and putting $t = ie^{i\varphi}$ and $z = k\varrho$ in this formula, we can write the given fundamental solution in the form:

$$e^{ikx} = e^{ik\varrho\cos\varphi} = \sum_{n=-\infty}^{+\infty}{}' i^n J_n(k\varrho)\, e^{in\varphi}. \quad (141)$$

The boundary condition gives:

$$\sum_{n=-\infty}^{\infty}{}' i^n J_n(ka)\, e^{in\varphi} + \sum_{n=-\infty}^{+\infty}{}' a_n H_n^{(1)}(ka)\, e^{in\varphi} = 0,$$

and we thus obtain the following expressions for the coefficients a_n:

$$a_n = -i^n \frac{J_n(ka)}{H_n^{(1)}(ka)}.$$

The final solution of the problem will thus have the following form:

$$V = e^{ikx} - \sum_{n=-\infty}^{+\infty} i^n \frac{J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(k\varrho)\, e^{in\varphi} \qquad (\varrho > a).$$

The above problem has applications in certain cases of diffraction of electromagnetic waves from an infinite conducting cylinder. The series obtained above are of practical use only in cases where the waves are comparatively long.

It is interesting to compare the diffraction of an elementary flat wave with the vibration of a round membrane [II, 178]. Notice, first of all, that in the first case the number k is given (it is determined by the frequency ω of the falling wave) whereas in the second case it was determined from the boundary conditions. In the diffraction problem the coefficients of the expansion are determined from the boundary condition, whereas in the second case they are determined from the initial condition, i.e. from the vibration picture when $t = 0$. In the diffraction problem we have no initial condition since we are not considering the general diffraction problem with an initial disturbance but only an established sinusoidal case with a given frequency ω with respect to time.

155. The wave equation in spherical coordinates. Consider now the equation (135) in spherical coordinates. It has the form:

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\Delta_1 V + k^2 V = 0.$$

We consider a solution in the usual form:

$$V = f(r)\, Y(\theta, \varphi). \tag{142}$$

Substituting in the equation and separating the variables we obtain

$$\frac{f''(r)}{f(r)} + \frac{2}{r}\frac{f'(r)}{f(r)} + \frac{1}{r^2}\frac{\Delta_1 Y(\theta, \varphi)}{Y(\theta, \varphi)} + k^2 = 0,$$

where $\Delta_1 Y$ is determined from the formula (71) in [135]. We thus obtain two equations of the following form:

$$\Delta_1 Y + \lambda Y = 0 \tag{143}$$

and

$$f''(r) + \frac{2}{r} f'(r) + \left(k^2 - \frac{\lambda}{r^2}\right) f(r) = 0. \tag{144}$$

The equation (143) is the same as the equation we obtained for spherical functions. If we suppose the solution is single-valued and

continuous we obtain the following possible values for the constant λ:

$$\lambda_n = n(n+1) \qquad (n = 0, 1, 2, \ldots)$$

and these will correspond to the solutions of the equation (143) which are the usual spherical functions $Y_n(\theta, \varphi)$. The equation (144) can be rewritten as follows

$$f_n''(r) + \frac{2}{r} f_n'(r) + \left(k^2 - \frac{n(n+1)}{r^2}\right) f_n(r) = 0. \tag{145}$$

We now replace $f(r)$ by a new unknown function $R(r)$ according to the formula

$$f_n(r) = \frac{1}{\sqrt{r}} R_n(r).$$

Substituting in the equation (145) we obtain for $R_n(r)$ an equation of the form

$$R_n''(r) + \frac{1}{r} R_n'(r) + \left(k^2 - \frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right) R_n(r) = 0,$$

and therefore $R_n(r)$ is equal to $Z_{n+1/2}(kr)$, where $Z_{n+1/2}(r)$ is the solution of the Bessel equation with the parameter $p = n + 1/2$, and, according to (142), we have

$$V = \frac{Z_{n+1/2}(kr)}{\sqrt{r}} Y_n(\theta, \varphi) \qquad (n = 0, 1, 2, \ldots). \tag{146}$$

Notice that here the solutions of the Bessel equation are expressed in finite form in terms of elementary functions. The choice of the solution $Z_{n+1/2}(kr)$ is determined, as in the previous section, from the physical conditions of the problem. The following three functions are usually considered:

$$\left.\begin{aligned} &\zeta_n^{(1)}(\varrho) = \sqrt{\frac{\pi}{2\varrho}} H^{(1)}_{n+\frac{1}{2}}(\varrho); \quad \zeta_n^{(2)}(\varrho) = \sqrt{\frac{\pi}{2\varrho}} H^{(2)}_{n+\frac{1}{2}}(\varrho); \\ &\psi_n(\varrho) = \sqrt{\frac{\pi}{2\varrho}} J_{n+\frac{1}{2}}(\varrho) = \frac{1}{2}\left[\zeta_n^{(1)}(\varrho) + \zeta_n^{(2)}(\varrho)\right], \end{aligned}\right\} \tag{147}$$

where the constant term $\sqrt{\pi/2}$ is added to make calculations more convenient. In particular when $n = 0$ we obtain, from [148]:

$$\zeta_0^{(1)}(\varrho) = -i\frac{e^{i\varrho}}{\varrho}; \qquad \zeta_0^{(2)} = i\frac{e^{-i\varrho}}{\varrho}; \qquad \psi_0(\varrho) = \frac{\sin\varrho}{\varrho}.$$

Solutions which are independent of φ have the form:

$$\frac{Z_{n+\frac{1}{2}}(kr)}{\sqrt{r}} P_n(\cos\theta),$$

and when $n = 0$ we have

$$\frac{Z_{\frac{1}{2}}(kr)}{\sqrt{r}}.$$

To obtain solutions of the fundamental equation (133) we must multiply the solutions (146) by $e^{\pm i\omega t}$ or, which comes, to the same thing, by $\cos\omega t$ and $\sin\omega t$, where ω and k are connected by the relationship (136). If we separate the variables in the equation (133), as usual, assuming that $U = T(t)\,V(x, y, z)$, we obtain the equation (135) for V, whereas for $T(t)$ we have

$$T''(t) + a^2 k^2 T(t) = 0 \qquad (a^2 k^2 = \omega^2),$$

which has the above functions of t as solutions. But we have assumed until now that k (or ω) is not zero. When $k = 0$ then we must take $T(t) = A + Bt$ and we simply obtain for V the Laplace equation $\Delta V = 0$. Thus we also obtain a solution in the form:

$$(A + Bt)\, r^n\, Y_n(\theta, \varphi), \tag{148}$$

which must be linked with the solutions (146).

Here, as in the above case with cylindrical coordinates, we can pursue to the end the solution of the problem of vibrations inside a sphere with given initial and boundary conditions, as well as the problem of the diffraction of a flat wave from a sphere.

To start with let us suppose that we want to find the solution of the wave equation

$$\frac{\partial^2 U}{\partial t^2} = a^2\,\Delta U, \tag{149}$$

which satisfies the initial conditions:

$$U|_{t=0} = f_1(r, \theta, \varphi); \qquad \left.\frac{\partial U}{\partial t}\right|_{t=0} = f_2(r, \theta, \varphi) \qquad (r < a) \tag{150}$$

and the boundary condition

$$\left.\frac{\partial U}{\partial r}\right|_{r=a} = 0. \tag{151}$$

On referring to the solutions (146) and bearing in mind that our solution is required to be finite when $r=0$, we take $Z_{n+1/2}(kr)$ equal to $J_{n+1/2}(kr)$ and we

determine the values of k for a given n from the boundary condition:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left.\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\right|_{r=a}=0 \quad \text{or} \quad J_{n+\frac{1}{2}}(ka)-2kaJ'_{n+\frac{1}{2}}(ka)=0. \quad (152)$$

In future we shall denote the positive zeros of this equation by

$$k_m^{(n)}(m=0,1,2,\ldots).$$

Also the solutions (148) satisfy the boundary condition (151) when $n=0$. According to Fourier's theorem we must seek the solution of our problem in the form:

$$U=A+Bt+$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[Y_n^{(1)}(\theta,\varphi)\cos ak_m^{(n)}t+Y_n^{(2)}(\theta,\varphi)\sin ak_m^{(n)}t)]\frac{J_{n+\frac{1}{2}}(k_m^{(n)}r)}{\sqrt{r}}. \quad (153)$$

The spherical functions $Y_n^{(1)}(\theta,\varphi)$ and $Y_n^{(2)}(\theta,\varphi)$ of order n remain to be determined from the initial conditions (150). Notice that the equation (152) is the form which we considered in [150], and we can determine the above spherical functions by using the property of orthogonality of Bessel functions. We shall not explain this in greater detail.

Let us now consider the diffraction of a flat wave from a sphere $r=a$ given by the solution $e^{i(kz-\omega t)}$ of the equation (149), when the boundary condition is

$$U|_{r=a}=0.$$

In this case we have taken a wave propagated along the Z-axis. Instead of the formula (141) the following formula given in spherical coordinates, applies:

$$e^{ikz}=e^{ikr\cos\theta}=\sum_{n=0}^{\infty}(2n+1)\,i^n\psi_n(kr)\,P_n(\cos\theta), \quad (154)$$

where $P_n(x)$ are the usual Legendre polynomials. We shall not give the proof of this formula. Bearing in mind the radiation principle we shall seek the additional disturbance in the form:

$$\sum_{n=0}^{\infty}a_n\zeta_n^{(1)}(kr)\,P_n(\cos\theta). \quad (155)$$

The coefficients a_n are determined from the condition that the sum of the solutions (154) and (155) must vanish when $r=a$, and this gives

$$a_n=-\frac{(2n+1)\,i^n\psi_n(ka)}{\zeta_n^{(1)}(ka)}.$$

§ 3. The Hermitian and Laguerre polynomials

156. The linear oscillator and the Hermitian polynomial. The Schrödinger equation, as we know, has the form:

$$\frac{h^2}{2m}\Delta\psi + (E - V)\psi = 0.$$

We assume that the function ψ is a function of x only and that the potential V is determined by the formula $V = kx^2/2$ which refers to the case of an elastic force $f = -kx$. We thus obtain the equation:

$$\frac{h^2}{2m}\frac{d^2\psi}{dx^2} + \left(E - \frac{k}{2}x^2\right)\psi = 0,$$

where the value of the parameter E is determined by the condition that the solution of the equation must remain finite in the whole interval $-\infty < x < +\infty$. Let us introduce two new constants:

$$a^2 = \frac{mk}{h^2}; \quad \lambda = \frac{2mE}{h^2} \qquad (a < 0). \tag{1}$$

Here a^2 is given and λ is a parameter which replaces E. The equation can then be rewritten in the form:

$$\frac{d^2\psi}{dx^2} + (\lambda - a^2 x^2)\psi = 0. \tag{2}$$

$x = \infty$ is an irregular singularity of this linear equation. We shall treat it in the same way as we did in [105], viz. suppose that:

$$\psi = e^{\omega(x)} u(x)$$

and determine the function $\omega(x)$ from the condition that in the coefficient of the unknown function $u(x)$ in the differential equation should be there no term containing x^2. Differentiating and substituting in the equation (2) we obtain the following equation for $u(x)$

$$u''(x) + 2\omega'(x)u'(x) + \left[\omega''(x) + \omega'^2(x) + \lambda - a^2x^2\right]u(x) = 0,$$

and to eliminate the term a^2x^2 we take:

$$\omega(x) = -\frac{a}{2}x^2,$$

where the minus sign is chosen so as to obtain a decrease as $x \to \pm\infty$. We thus obtain

$$\psi(x) = e^{-\frac{a}{2}x^2} u(x), \tag{3}$$

where $u(x)$ satisfies the following equation:

$$\frac{d^2u}{dx^2} - 2ax\frac{du}{dx} + (\lambda - a)\,u = 0. \tag{4}$$

If for a certain parameter λ this equation has a solution in the form of a polynomial then the function $\varphi(x)$ will decrease at infinity and, consequently, it will satisfy the necessary boundary conditions. We shall therefore seek the solution of the equation (4) in the form of a polynomial. Replacing x by the new independent variable

$$\xi = \sqrt{a}\,x,$$

whence

$$\frac{du}{dx} = \frac{du}{d\xi}\sqrt{a}; \qquad \frac{d^2u}{dx^2} = \frac{d^2u}{d\xi^2}\,a,$$

and substituting in the equation (4) we obtain the following equation for u:

$$\frac{d^2u}{d\xi^2} - 2\xi\frac{du}{d\xi} + \left(\frac{\lambda}{a} - 1\right)u = 0. \tag{5}$$

For this equation the origin is not a singularity and the solution can be found in the form of an ordinary power series:

$$u = \sum_{k=0}^{\infty} a_k \xi^k$$

in which the first two coefficients a_0 and a_1 are arbitrary. Substituting in the equation (5) we obtain a reduction formula for the coefficients:

$$(k+2)(k+1)\,a_{k+2} - 2ka_k + \left(\frac{\lambda}{a} - 1\right)a_k = 0,$$

whence

$$a_{k+2} = \frac{2k - \left(\frac{\lambda}{a} - 1\right)}{(k+2)(k+1)}\,a_k \qquad (k = 0, 1, 2, \ldots). \tag{6}$$

We shall now demonstrate a method for obtaining the solution of the equation as a polynomial of the nth degree. We assume that the parameter λ is chosen from the condition

$$\frac{\lambda}{a} - 1 = 2n,$$

i.e.

$$\lambda_n = (2n+1)\,a. \tag{7}$$

Formula (6) then gives successively

$$a_{r+2} = a_{n+4} = a_{n+6} = \ldots = 0. \tag{8}$$

When n is even we must also assume that $a_1 = 0$ and $a_0 \neq 0$. From (6) we have $a_1 = a_3 = a_5 = \ldots = 0$ and also all a_k's with even subscripts to $k = n$ inclusively will not be zero while, according to (8), the remaining ones will be zero. If, however, n is odd we must assume conversely that $a_0 = 0$ and $a_1 \neq 0$. We then obtain the solution in the form of polynomials and formula (7) gives the corresponding individual values of the parameter λ. Substituting these in the equation (5) and denoting by $H_n(\xi)$ the introduced polynomials we obtain for them the following differential equation:

$$H_n''(\xi) - 2\xi H_n'(\xi) + 2nH_n(\xi) = 0. \tag{9}$$

From (3) we have for the function $\psi_n(\xi)$

$$\psi_n(\xi) = e^{-\frac{1}{2}\xi^2} H_n(\xi). \tag{10}$$

The polynomials $H_n(\xi)$ are usually known as *Hermitian polynomials* and the functions (10) as *Hermitian functions*.

We have equation (2) for the Hermitian functions in which x must be replaced by ξ. After substitution the equation takes the form:

$$\frac{d^2\psi_n(\xi)}{d\xi^2} + \left(\frac{\lambda_n}{a} - \xi^2\right)\psi_n(\xi) = 0 \qquad \left(\frac{\lambda_n}{a} = 2n + 1\right). \tag{11}$$

We shall now introduce a simple formula for Hermitian polynomials. Assume that $v = e^{-\xi^2}$ whence $v' = -2\xi v$. Differentiating this equation $(n + 1)$ times by applying the Leibniz formula to the derivative of the product we have

$$v^{(n+2)} = -2\xi v^{(n+1)} - (n + 1)\, 2v^{(n)}$$

or

$$v^{(n+2)} + 2\xi v^{(n+1)} + 2(n + 1)\, v^{(n)} = 0. \tag{12}$$

Let us introduce a new function $K_n(\xi) = e^{\xi^2} v^{(n)}$ and show that the equation (9) is satisfied by it. The function $K_n(\xi)$ must be a polynomial of the nth degree in ξ:

$$K_n(\xi) = e^{\xi^2} \frac{d^n}{d\xi^n}(e^{-\xi^2}). \tag{13}$$

Substituting the expression

$$v^{(n)} = e^{-\xi^2} K_n(\xi)$$

in the equation (12) we do, in fact, obtain the equation (9) for $K_n(\xi)$.

Hence the Hermitian polynomials which have so far only been determined as far as the constant term are the same as the functions (13). Notice that the second solution of the equation (9) cannot be a polynomial since the point $\xi = \infty$ is an irregular singularity of this equation. To obtain a positive coefficient for the first term we ascribe the constant factor $(-1)^n$ to the expression (13) and determine the Hermitian polynomial from the following formula:

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} (e^{-\xi^2}). \tag{14}$$

Let us write the first three Hermitian polynomials:

$$H_0(\xi) = 1; \quad H_1(\xi) = 2\xi; \quad H_2(\xi) = 4\xi^2 - 2.$$

$H_n(\xi)$ generally contains only even powers of ξ when n is even and odd powers of ξ when n is odd. This follows from the method described above by which the coefficients a_k were determined. It follows from formula (14) that the first coefficient ξ^n in the polynomial $H_n(\xi)$ is equal to 2^n. This is due to the fact that the differentiation of the index $(-\xi^2)$ gives (-2ξ).

It can be shown, but we shall not do so here, that Hermitian functions represent the complete set of solutions of the equation (2) which satisfy the boundary conditions.

157. Orthogonality. Consider two different Hermitian functions $\psi_n(\xi)$ and $\psi_m(\xi)$. The following equations are satisfied by these functions:

$$\frac{d^2\psi_n(\xi)}{d\xi^2} + \left(\frac{\lambda_n}{a} - \xi^2\right)\psi_n(\xi) = 0,$$

$$\frac{d^2\psi_m(\xi)}{d\xi^2} + \left(\frac{\lambda_m}{a} - \xi^2\right)\psi_m(\xi) = 0.$$

Multiplying the first function by $\psi_m(\xi)$ and the second by $\psi_n(\xi)$, subtracting and integrating over the interval $(-\infty, +\infty)$ we obtain, as always, the equation proving the orthogonality of Hermitian functions:

$$\int_{-\infty}^{+\infty} \psi_n(\xi)\,\psi_m(\xi)\,d\xi = 0 \qquad (n \neq m), \tag{15}$$

or, from (10),

$$\int_{-\infty}^{+\infty} e^{-\xi^2} H_n(\xi) H_m(\xi) \, d\xi = 0 \qquad (n \neq m), \tag{16}$$

i.e. we can say that *Hermitian polynomials are orthogonal with* $e^{-\xi^2}$ *in the interval* $(-\infty, +\infty)$. Let us now evaluate the integral (16) when $n = m$. According to formula (14) we have:

$$I_n = \int_{-\infty}^{+\infty} e^{-\xi^2} H_n^2(\xi) \, d\xi = (-1)^n \int_{-\infty}^{+\infty} H_n(\xi) \frac{d^n(e^{-\xi^2})}{d\xi^n} d\xi,$$

or, integrating by parts,

$$I_n = (-1)^n H_n(\xi) \frac{d^{n-1}(e^{-\xi^2})}{d\xi^{n-1}} \Bigg|_{\xi=-\infty}^{\xi=+\infty} +$$

$$+ (-1)^{n+1} \int_{-\infty}^{+\infty} H_n'(\xi) \frac{d^{n-1}(e^{-\xi^2})}{d\xi^{n-1}} \, d\xi.$$

The first term on the right-hand side is a product of $e^{-\xi^2}$ and a polynomial and therefore it vanishes when $\xi = \pm\infty$. Continuing the integration by parts we obtain

$$I_n = \int_{-\infty}^{+\infty} H_n^{(n)}(\xi) e^{-\xi^2} d\xi,$$

or, bearing in mind the fact that the first coefficient of the polynomial $H_n(\xi)$ is equal to 2^n

$$I_n = 2^n n! \int_{-\infty}^{+\infty} e^{-\xi^2} d\xi,$$

we obtain finally [II, 78]:

$$I_n = \int_{-\infty}^{+\infty} e^{-\xi^2} H_n^2(\xi) \, d\xi = 2^n n! \sqrt{\pi}. \tag{17}$$

Series containing Hermitian polynomials and analogous with Fourier series can be constructed similarly to those containing Legendre polynomials [132]. In this case instead of the finite interval $(-1, +1)$ we have an infinite interval $(-\infty, +\infty)$. In this interval we obtain the following expansion

$$f(\xi) = \sum_{n=0}^{\infty} a_n H_n(\xi), \tag{18}$$

where the coefficients a_n, as a result of the above orthogonality and from formula (17), are determined as follows:

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{+\infty} f(\xi)\, e^{-\xi^2} H_n(\xi)\, d\xi. \tag{19}$$

For the expansion (18) to become valid it is, of course, essential that the function $f(\xi)$ should satisfy certain additional conditions.

158. The conversion function. Using (14) and Cauchy's formula for the derivative of the function e^{-z^2} in the form of a contour integral, we can write

$$e^{-\xi^2} H_n(\xi) = (-1)^n \frac{n!}{2\pi i} \int_{l_\xi} \frac{e^{-z^2}}{(z-\xi)^{n+1}}\, dz,$$

where l_ξ is an arbitrary closed contour which encircles the point $z = \xi$. Replace z by a new variable of integration according to the formula

$$z = \xi - t.$$

Replacing the variables in the integral and dividing both sides by $e^{-\xi}$ we obtain

$$\frac{1}{n!} H_n(\xi) = \frac{1}{2\pi i} \int_{l_0'} \frac{e^{-t^2+2t\xi}}{t^{n+1}}\, dt,$$

where l_0' is a simple contour which encircles the origin. It follows from this formula that $H_n(\xi)/n!$ is the coefficient of t^n in the expansion of the function

$$e^{-t^2+2t\xi} \tag{20}$$

into a McLaurin's series, i.e. *the function* (20) *is the conversion function for Hermitian polynomials multiplied by the constant* $1/n!$:

$$e^{-t^2+2t\xi} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(\xi)\, t^n. \tag{21}$$

From this formula the fundamental relationships for Hermitian polynomials are readily obtainable. Differentiating the identity (21) with respect to ξ we have

$$e^{-t^2+2t\xi} \cdot 2t = \sum_{n=0}^{\infty} \frac{1}{n!} H_n'(\xi)\, t^n$$

or

$$\sum_{n=0}^{\infty} \frac{2}{n!} H_n(\xi)\, t^{n+1} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n'(\xi)\, t^n$$

and comparing the coefficients of equal powers of t we obtain the relationship

$$H_n'(\xi) = 2nH_{n-1}(\xi). \tag{22}$$

On differentiating the identity (21) with respect to t we obtain:

$$e^{-t^2+2t\xi} \cdot (2\xi - 2t) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_n(\xi)\, t^{n-1}$$

or

$$\sum_{n=0}^{\infty} \frac{2\xi}{n!} H_n(\xi)\, t^n - \sum_{n=0}^{\infty} \frac{2}{n!} H_n(\xi)\, t^{n+1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_n(\xi)\, t^{n-1},$$

and again comparing the coefficients we also obtain the following relationship:

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi). \tag{23}$$

Finally let us determine the constant term in the Hermitian polynomial, i.e. $H_n(0)$. When n is odd this term must be zero since an odd Hermitian polynomial contains only odd powers of ξ. When n is even we have, first of all, $H_0(0) = 1$. Subsequently formula (23), with $n = 1$ and $\xi = 0$, gives

$$H_2(0) = -2H_0(0) = -2.$$

The same formula when $n = 3$ and $\xi = 0$ gives

$$H_4(0) = -2 \cdot 3H_2(0) = 2^2 \cdot 1 \cdot 3.$$

Further, when $n = 5$ and $\xi = 0$ we have

$$H_6(0) = -2^3 \cdot 1 \cdot 3 \cdot 5$$

and in general

$$H_{2n}(0) = (-1)^n \cdot 2^n \cdot 1 \cdot 3 \cdot 5 \ldots \cdot (2n-1). \tag{24}$$

Notice that if Rolle's theorem is applied several times to the equation (14) then it can be shown that all the zeros of $H_n(\xi)$ are real and different. We used similar arguments in [102] to show that all the zeros of $P_n(x)$ are different and lie in the interval $(-1, +1)$.

Sometimes Hermitian polynomials are introduced in a slightly different form, viz. instead of formula (14) the Hermitian polynomials are determined from the formula

$$\widetilde{H}_n(\xi) = \frac{1}{n!} e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n} e^{-\frac{\xi^2}{2}}.$$

The only difference is in the constant terms one of which stands outside the polynomial while the other refers to the argument ξ.

159. Parabolic coordinates and Hermitian functions. We shall now give one particular case of substitution of the variables in the wave equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 U = 0. \tag{25}$$

Replace x and y by two other variables ξ and η and let us suppose that this substitution satisfies the formula

$$x + iy = f(\zeta) = \varphi(\xi, \eta) + i\psi(\xi, \eta) \qquad (\zeta = \xi + i\eta),$$

where $f(\zeta)$ is a regular function of the complex variable ζ. Differentiating in accordance with the law for differentiating complicated functions we have

$$\frac{\partial U}{\partial \xi} = \frac{\partial U}{\partial x}\frac{\partial \varphi}{\partial \xi} + \frac{\partial U}{\partial y}\frac{\partial \psi}{\partial \xi}; \qquad \frac{\partial U}{\partial \eta} = \frac{\partial U}{\partial x}\frac{\partial \varphi}{\partial \eta} + \frac{\partial U}{\partial y}\frac{\partial \psi}{\partial \eta}$$

and further

$$\frac{\partial^2 U}{\partial \xi^2} = \frac{\partial^2 U}{\partial x^2}\left(\frac{\partial \varphi}{\partial \xi}\right)^2 + 2\frac{\partial^2 U}{\partial x \partial y}\frac{\partial \varphi}{\partial \xi}\frac{\partial \psi}{\partial \xi} + \frac{\partial^2 U}{\partial y^2}\left(\frac{\partial \psi}{\partial \xi}\right)^2 + \frac{\partial U}{\partial x}\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial U}{\partial y}\frac{\partial^2 \psi}{\partial \xi^2},$$

$$\frac{\partial^2 U}{\partial \eta^2} = \frac{\partial^2 U}{\partial x^2}\left(\frac{\partial \varphi}{\partial \eta}\right)^2 + 2\frac{\partial^2 U}{\partial x \partial y}\frac{\partial \varphi}{\partial \eta}\frac{\partial \psi}{\partial \eta} + \frac{\partial^2 U}{\partial y^2}\left(\frac{\partial \psi}{\partial \eta}\right)^2 + \frac{\partial U}{\partial x}\frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial U}{\partial y}\frac{\partial^2 \psi}{\partial \eta^2}.$$

Using the Cauchy-Riemann equations

$$\frac{\partial \varphi}{\partial \xi} = \frac{\partial \psi}{\partial \eta}; \qquad \frac{\partial \varphi}{\partial \eta} = -\frac{\partial \psi}{\partial \xi},$$

and also the fact that $\varphi(\xi, \eta)$ and $\psi(\xi, \eta)$ satisfy the Laplace equation we can readily prove the following formula:

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)\left[\left(\frac{\partial \varphi}{\partial \xi}\right)^2 + \left(\frac{\partial \psi}{\partial \xi}\right)^2\right]$$

or

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)|f'(\zeta)|^2.$$

Consider the particular case when

$$f(\zeta) = \frac{1}{2}(\xi + i\eta)^2; \quad f'(\zeta) = \xi + i\eta$$

or

$$\varphi(\xi, \eta) = \frac{1}{2}(\xi^2 - \eta^2); \quad \psi(\xi, \eta) = \xi\eta.$$

The coordinate lines $\xi = C_1$ and $\eta = C_2$ represent parabolae [32] in the (x, y)-plane and therefore the new coordinates ξ and η

are called parabolic. Transforming the wave-equation in the way described above we have

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + k^2 |f'(\zeta)|^2 U = 0,$$

and therefore equation (25) with the new coordinates will, in this case, have the following form:

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + k^2 (\xi^2 + \eta^2) U = 0. \tag{26}$$

We shall seek its solution in the form of a product of two factors, one of which depends on ξ and the other on η:

$$U = X(\xi) Y(\eta).$$

Substituting in the equation (26) and separating the variables in the usua way we obtain

$$\frac{X''(\xi)}{X(\xi)} + k^2 \xi^2 = -\frac{Y''(\eta)}{Y(\eta)} - k^2 \eta^2.$$

Both sides of the above identity will be equal to the same constant which we denote by $(-\beta^2)$. We thus arrive at the following two equations:

$$X''(\xi) + (k^2 \xi^2 + \beta^2) X(\xi) = 0; \qquad Y''(\eta) + (k^2 \eta^2 - \beta^2) Y(\eta) = 0. \tag{27}$$

Let us recall the differential equation (11) which is satisfied by Hermitian functions:

$$\psi_n''(\xi) + (2n + 1 - \xi^2) \psi_n(\xi) = 0, \tag{28}$$

where we have the following formula for the Hermitian function:

$$\psi_n(\xi) = e^{-\frac{\xi^2}{2}} H_n(\xi) = (-1)^n e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n} (e^{-\xi^2}). \tag{29}$$

Consider the first of the equations (27) and replace ξ by a new variable ξ_1 according to the formula

$$\xi_1 = \sqrt{ik}\, \xi.$$

This gives

$$\frac{d}{d\xi} = \sqrt{ik} \frac{d}{d\xi_1}; \quad \frac{d^2}{d\xi^2} = ik \frac{d^2}{d\xi_1^2}$$

and substituting in (27) we obtain the equation:

$$\frac{d^2 X}{d\xi_1^2} + \left(\frac{\beta^2}{ik} - \xi_1^2\right) X = 0. \tag{30}$$

If we determine the constant β^2 from the equation

$$\beta_n^2 = (2n + 1) ik,$$

where n is a positive integer or zero, we can obtain the equation (30) in the form (28). Hence by using this new variable ξ_1 we can take the following Hermitian function for X

$$X_n = C_n \psi_n(\xi_1) = C_n e^{-\frac{\xi_1^2}{2}} H_n(\xi_1)$$

or returning to the old variable we obtain

$$X_n = C_n \psi_n(\sqrt{ik}\,\xi) = C_n e^{-\frac{ik\xi^2}{2}} H_n(\sqrt{ik}\,\xi),$$

where C_n is an arbitrary constant.

Similarly considering the second of the equations (27) and replacing η by the new variable

$$\eta_1 = i\sqrt{ik}\,\eta,$$

we can also obtain the second solution in the form (28) for the same value of the parameter β_n. Returning to the old variable we have

$$Y_n = D_n \psi_n(\eta_1) = D_n e^{\frac{ik\eta^2}{2}} H_n(i\sqrt{ik}\,\eta).$$

We thus obtain an infinite number of solutions of the wave-equations (25) in the following form:

$$U_n = A_n \psi_n(\sqrt{ik}\,\xi)\,\psi_n(i\sqrt{ik}\,\eta) \qquad (n = 0, 1, 2, \ldots). \tag{31}$$

These solutions comprise a full system of functions and are analogous with Bessel functions expressed in cylindrical coordinates. Here, as before, functions analogous with Hankel functions can be constructed and this makes it possible to solve diffraction problems with reference to a parabolic cylinder.

160. The Laguerre polynomials. We obtained the Laguerre polynomials in a generalized form on solving an equation of the type [115]

$$x\frac{d^2y}{dx^2} + (s + 1 - x)\frac{dy}{dx} + \mu y = 0. \tag{32}$$

Bearing in mind the formulae (218), (219) and (222) from [115] we can say that the solution of the equation (32) will be obtained in the form of a polynomial of the nth degree if the parameter μ is equal to $\mu_n = n$; in this case, the solution of the equation will be expressed by Laguerre polynomials for which we obtained the following expressions:

$$Q_n^{(s)}(x) = x^{-s} e^x \frac{d^n}{dx^n}(x^{s+n} e^{-x}). \tag{33}$$

Hence these polynomials are the solution of the equation

$$x\frac{d^2 y_n}{dx^2} + (s + 1 - x)\frac{dy_n}{dx} + n y_n = 0. \tag{34}$$

We are assume throughout that the number s is real and > -1.

In (32) notice that the independent variable x only differs by a constant term from the radius-vector and therefore the principal interval of variation of this variable is the interval $(0, +\infty)$. The Laguerre polynomials are analogous with Hermitian polynomials but for them the principal interval is not the interval $(-\infty, +\infty)$ but the interval $(0, +\infty)$. Formula (216) from [115] gives the Laguerre functions which are analogous with Hermitian functions:

$$\omega_n^{(s)}(x) = e^{-\frac{x}{2}} x^{\frac{s}{2}} Q_n^{(s)}(x) = x^{-\frac{s}{2}} e^{\frac{x}{2}} \frac{d^n}{dx^n}(x^{s+n} e^{-x}). \tag{35}$$

According to (213) and (222) from [115] these functions are solutions of the equation

$$\frac{d}{dx}\left[x \frac{dw}{dx}\right] + \left(\lambda_n - \frac{x}{4} - \frac{s^2}{4x}\right) w = 0, \tag{36}$$

where

$$\lambda_n = \frac{s+1}{2} + n. \tag{37}$$

As before the orthogonality of these functions can easily be deduced:

$$\int_0^\infty \omega_m^{(s)}(x)\, \omega_n^{(s)}(x)\, dx = 0 \qquad (m \neq n) \tag{38}$$

and from (35)

$$\int_0^\infty x^s e^{-x} Q_m^{(s)}(x)\, Q_n^{(s)}(x)\, dx = 0 \qquad (m \neq n). \tag{39}$$

We shall now try to evaluate the integral (39) when $m = n$. According to the definition of the Laguerre polynomial we have

$$I_n = \int_0^\infty x^s e^{-x} \left[Q_n^{(s)}(x)\right]^2 dx = \int_0^\infty Q_n^{(s)}(x) \frac{d^n}{dx^n}(x^{s+n} e^{-x})\, dx.$$

Integrating by parts we have

$$I_n = Q_n^{(s)}(x) \frac{d^{n-1}}{dx^{n-1}}(x^{s+n} e^{-x}) \Bigg|_{x=0}^{x=\infty} - \int_0^\infty \frac{dQ_n^{(s)}(x)}{dx} \frac{d^{n-1}(x^{s+n} e^{-x})}{dx^{n-1}}\, dx,$$

where as for Hermitian polynomials, the first term on the right-hand side vanishes. Integrating by parts several times we finally obtain the integral

$$I_n = (-1)^n \int_0^\infty x^{s+n} e^{-x} \frac{d^n Q_n^{(s)}(x)}{dx^n}\, dx.$$

But $\mathrm{d}^n Q_n^{(s)}(x)/\mathrm{d}x^n$ is a product of $n!$ and the first coefficient of $Q_n^{(s)}(x)$. Applying the Leibniz formula to the derivative in formula (33) we can see that this first coefficient is equal to $(-1)^n$ and we can therefore write

$$I_n = n! \int_0^\infty x^{s+n} \mathrm{e}^{-x} \mathrm{d}x,$$

or, remembering the definition of the function $\Gamma(z)$, we finally obtain

$$\int_0^\infty x^s \mathrm{e}^{-x} \left[Q_n^{(s)}(x)\right]^2 \mathrm{d}x = n!\ \Gamma(s+n+1). \tag{40}$$

In the same way as we did with Hermitian functions we can consider the expansion of an arbitrary function $f(x)$ into a series of Laguerre polynomials in the interval $(0, +\infty)$.

We shall now construct the conversion function for Laguerre polynomials. According to formula (33) and Cauchy's theorem which gives the derivative of order n of the function $z^{s+n}\mathrm{e}^{-z}$, when $z = x$, we can write

$$x^s \mathrm{e}^{-x} Q_n^{(s)}(x) = \frac{n!}{2\pi i} \int_{l_x} \frac{z^{s+n}\mathrm{e}^{-z}}{(z-x)^{n+1}} \mathrm{d}z,$$

where l_x is a small closed contour encircling the point $z = x$. Notice that the function $z^{s+n}\mathrm{e}^{-z}$ is regular in the whole plane except at the point $z = 0$ where it has a branch-point, provided s is not an integer. Replacing z by a new variable of integration

$$t = \frac{z-x}{z}, \qquad z = \frac{x}{1-t} = \frac{xt}{1-t} + x,$$

substituting in the integral and dividing through by $x^s \mathrm{e}^{-x}$ we obtain

$$\frac{1}{n!} Q_n^{(s)}(x) = \frac{1}{2\pi i} \int_{l_0} \mathrm{e}^{-\frac{xt}{1-t}} \frac{1}{(1-t)^{s+1}} \frac{\mathrm{d}t}{t^{n+1}},$$

where l_0 is a small closed contour which encircles the point $t = 0$.

This shows that $Q_n^{(s)}(x)/n!$ are the coefficients in the expansion of the function

$$\mathrm{e}^{-\frac{xt}{1-t}} \frac{1}{(1-t)^{s+1}}$$

into a McLaurin's series in powers of t, i.e.

$$\mathrm{e}^{-\frac{xt}{1-t}} \frac{1}{(1-t)^{s+1}} = \sum_{n=0}^{\infty} \frac{1}{n!} Q_n^{(s)}(x)\, t^n. \tag{41}$$

From this formula several simple relationships for Laguerre polynomials can be deduced. Differentiating both sides of (41) with respect to x we obtain

$$-e^{-\frac{xt}{1-t}}\frac{t}{(1-t)^{s+2}}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{dQ_n^{(s)}(x)}{dx}t^n$$

or

$$-\sum_{n=0}^{\infty}\frac{1}{n!}Q_n^{(s+1)}(x)\,t^{n+1}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{dQ_n^{(s)}(x)}{dx}t^n,$$

which, by comparing the coefficients of t^n, gives:

$$\frac{dQ_n^{(s)}(x)}{dx}=-nQ_{n-1}^{(s+1)}(x). \tag{42}$$

Similarly, differentiating both sides of (41) with respect to t, we obtain the relationship

$$xQ_n^{(s)}(x)=(n+s)\,Q_n^{(s-1)}(x)-Q_{n+1}^{(s-1)}(x). \tag{43}$$

Finally if we multiply both sides of (41) by $(1-t)$ we also obtain the following relationship:

$$Q_n^{(s-1)}(x)=Q_n^{(s)}(x)-nQ_{n-1}^{(s)}(x). \tag{44}$$

Frequently instead of the polynomials $Q_n^{(s)}(x)$ the polynomials $(1/n!)\,Q_n^{(s)}(x)$ are considered.

By applying Rolle's theorem several times to formula (33) it can be shown that all the zeros of $Q_n^{(s)}(x)$ are real and different and lie in the interval $(0, +\infty)$.

161. The connection between Hermitian and Laguerre polynomials. The Hermitian polynomials can be simply expressed in terms of the Laguerre polynomials $Q_n^{(s)}(x)$. These latter are, as we know, the solutions of the equation (34) when $s=-1/2$, i.e.

$$x\frac{d^2y_n}{dx^2}+\left(\frac{1}{2}-x\right)\frac{dy_n}{dx}+ny_n=0. \tag{45}$$

Replace x by a new variable ξ according to the formula

$$\frac{d}{dx}=\frac{1}{2\xi}\frac{d}{d\xi};\quad \frac{d^2}{dx^2}=\frac{1}{2\xi}\frac{d}{d\xi}\left(\frac{1}{2\xi}\frac{d}{d\xi}\right)=\frac{1}{4\xi^2}\frac{d^2}{d\xi^2}-\frac{1}{4\xi^3}\frac{d}{d\xi}.$$

Substituting in the equation (45) we obtain the equation

$$\frac{d^2y_n}{d\xi^2}-2\xi\frac{dy_n}{d\xi}+4ny_n=0, \tag{46}$$

which is the same as the equation (9) if we write $2n$ instead of n in the latter equation. As we mentioned before the second solution of the equation (9) is no longer a polynomial and we can therefore say that $Q_n^{(-1/2)}(\xi^2)$ is the same as $H_{2n}(\xi)$ except for a constant factor, i.e.

$$H_{2n}(\xi) = C_n Q_n^{\left(-\frac{1}{2}\right)}(\xi^2).$$

To determine the constant C_n we compare the coefficients on both sides of the above equation. On the left-hand side the first coefficient, as we know from [156] is equal to 2^{2n} and on the right-hand side to $(-1)^n C_n$ [160], so that $C_n = (-1)^n 2^{2n}$, and therefore

$$H_{2n}(\xi) = (-1)^n 2^{2n} Q_n^{\left(-\frac{1}{2}\right)}(\xi^2). \tag{47}$$

Let us now deduce an analogous formula for $H_{2n+1}(\xi)$. The function $Q_n^{(1/2)}(x)$ satisfies the equation:

$$x \frac{d^2 y_n}{dx^2} + \left(\frac{3}{2} - x\right) \frac{dy_n}{dx} + n y_n = 0,$$

which can be transformed by the formula $x = \xi^2$ into the following equation:

$$\frac{d^2 y_n}{d\xi^2} + \left(\frac{2}{\xi} - 2\xi\right) \frac{dy_n}{d\xi} + 4n y_n = 0.$$

Replace y_n by a new function z_n according to the formula

$$y_n = \frac{1}{\xi} z_n.$$

On differentiating with respect to ξ and substituting in the equation we obtain an equation for z_n:

$$\frac{d^2 z_n}{d\xi^2} - 2\xi \frac{dz_n}{d\xi} + (4n + 2) z_n = 0.$$

This equation is the same as (9) if we write $(2n + 1)$ for n in the latter equation. The above transformations give us directly:

$$H_{2n+1}(\xi) = D_n \xi Q_n^{\left(\frac{1}{2}\right)}(\xi^2).$$

The comparison of the first coefficients gives $D_n = (-1)^n 2^{2n+1}$, and therefore:

$$H_{2n+1}(\xi) = (-1)^n 2^{2n+1} Q_n^{\left(\frac{1}{2}\right)}(\xi^2). \tag{48}$$

162. The asymptotic expression for Hermitian polynomials. The Hermitian function:

$$\psi_n(x) = e^{-\frac{1}{2}x^2} H_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \tag{49}$$

satisfies the equation (11)

$$\psi_n''(x) + (2n + 1 - x^2)\psi_n(x) = 0. \tag{50}$$

Consider the case when n is even. We then have

$$\psi_{2n}''(x) + (4n + 1 - x^2)\psi_{2n}(x) = 0. \tag{51}$$

Also from (24) and the fact that $H_{2n}(x)$ is a polynomial in x^2, we obtain the following initial conditions

$$\psi_{2n}(0) = (-1)^n 2^n 1 \cdot 3 \cdot 5 \ldots (2n-1); \quad \psi'_{2n}(0) = 0. \tag{52}$$

The equation (51) together with the initial condition (52) makes it possible to obtain the asymptotic expression for Hermitian polynomials when n is large. Let us recall that the solution of the equation

$$y'' + k^2 y = f(x), \tag{53}$$

which satisfies zero initial conditions $y(0) = y'(0) = 0$ has the form [II, 28]

$$y = \frac{1}{k} \int_0^x f(u) \sin k(x-u)\, du. \tag{54}$$

If instead of the zero initial conditions we have the following initial conditions

$$y(0) = a; \quad y'(0) = b, \tag{55}$$

then we must add to the solution (54) a solution of a homogeneous equation which satisfies the initial conditions (55); the final solution of the equation (53) which satisfies the initial conditions (55) will then be

$$y = a \cos kx + \frac{b}{k} \sin kx + \frac{1}{k} \int_0^x f(u) \sin k(x-u)\, du. \tag{56}$$

Take the equation (51) and rewrite it as follows:

$$\psi''_{2n}(x) + (4n+1)\psi_n(x) = x^2 \psi_{2n}(x).$$

Let us suppose that in this case $k^2 = 4n+1$ and $f(x) = x^2 \psi_{2n}(x)$. We obtain from (56):

$$\psi_{2n}(x) = \psi_{2n}(0) \cos \sqrt{4n+1}\, x + \frac{1}{\sqrt{4n+1}} \int_0^x u^2 \psi_{2n}(u) \sin \sqrt{4n+1}\,(x-u)\, du. \tag{57}$$

It can be shown that when the values of n are large the first term on the right-hand side gives the principal value of the function $\psi_{2n}(x)$. To prove this we determine the upper bound of the integral term on the right-hand side when $x > 0$. Applying Buniakowski's inequalityw we obtain from (17):

$$\left| \int_0^x u^2 \psi_{2n}(u) \sin \sqrt{4n+1}\,(x-u)\, du \right| <$$

$$< \sqrt{\int_0^x \psi_{2n}^2(u)\, du} \sqrt{\int_0^x u^4 \sin^2 \sqrt{4n+1}\,(x-u)\, du} <$$

$$< \sqrt{\int_{-\infty}^{+\infty} \psi_{2n}^2(u)\, du} \sqrt{\int_0^x u^4\, du} = \sqrt{2^{2n}(2n)!\sqrt{\pi}\, \frac{x^5}{5}}$$

and substituting in (57) we have

$$\psi_{2n}(x) = \psi_{2n}(0)\cos\sqrt{4n+1}\,x + \frac{2^n\sqrt{(2n)!}\sqrt[4]{\pi}x^{\frac{5}{2}}}{\sqrt{5}\sqrt{4n+1}}\theta_n(x),$$

where $\theta_n(x)$ is a function of x which satisfies the condition

$$-1 < \theta_n(x) < 1.$$

Taking $\psi_{2n}(0)$ outside the bracket and remembering its value from (52) we have

$$\psi_{2n}(x) = \psi_{2n}(0)\left[\cos\sqrt{4n+1}\,x + \frac{(-1)^n\sqrt{(2n)!}\sqrt[4]{\pi}x^{\frac{5}{2}}}{\sqrt{5}\sqrt{4n+1}\cdot 1\cdot 3\ldots(2n-1)}\theta_n(x)\right]. \tag{58}$$

Let us consider in greater detail the coefficient of $\theta_n(x)$:

$$\frac{\sqrt[4]{\pi}x^{\frac{5}{2}}}{\sqrt{5}}\cdot\frac{\sqrt{1\cdot 2\cdot 3\ldots 2n}}{1\cdot 3\cdot 5\ldots(2n-1)} = \frac{\sqrt[4]{\pi}x^{\frac{5}{2}}}{\sqrt{5}}\sqrt{\frac{2\cdot 4\cdot 6\ldots 2n}{1\cdot 3\cdot 5\ldots(2n-1)}}.$$

If we put

$$I_k = \int_0^{\frac{\pi}{2}} \sin^k x\,dx,$$

then, as we know [I, 100],

$$I_{2n} = \frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}\frac{\pi}{2}, \quad I_{2n+1} = \frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3},$$

where $I_{2n+1} < I_{2n}$, i.e.

$$\frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3} < \frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}\frac{\pi}{2},$$

or

$$\left(\frac{2n(2n-2)\ldots 2}{(2n-1)(2n-3)\ldots 1}\right)^2 < (2n+1)\frac{\pi}{2}.$$

Whence

$$\sqrt{\frac{2\cdot 4\cdot 6\ldots 2n}{1\cdot 3\cdot 5\ldots(2n-1)}} < \frac{\sqrt[4]{\pi}}{\sqrt[4]{2}}\sqrt[4]{2n+1},$$

and, finally, the coefficient of $\theta_n(x)$ will be

$$\frac{\sqrt[4]{\pi}}{\sqrt[4]{50}}\sqrt[4]{\frac{2n+1}{4n+1}}x^{\frac{5}{2}}\frac{1}{\sqrt[4]{4n+1}}\theta_n',$$

where $0 < \theta_n' < 1$. Rejecting the factor which is less than unity we can write the above expression in the form

$$x^{\frac{5}{2}}\frac{1}{\sqrt[4]{4n+1}}\theta_n'',$$

where $0 < \theta_n'' < 1$. Substituting this expression in formula (58) we obtain the following asymptotic representation for the Hermitian function with an even subscript:

$$\psi_{2n}(x) = \psi_{2n}(0)\left[\cos\sqrt{4n+1}\,x + x^{\frac{5}{2}}\frac{1}{\sqrt[4]{4n+1}}\,\theta_n'''(x)\right],$$

where $-1 < \theta_n'''(x) < +1$. Hence for a given x the second term in the square brackets tends to zero as the subscript n increases. The assumption that $x > 0$, as can easily be shown, is not essential. By adding the factor $e^{(1/2)x^2}$ we obtain the asymptotic expression for Hermitian polynomials with an even subscript:

$$H_{2n}(x) = (-1)^n 2^n 1\cdot 3\cdot 5\ldots(2n-1)\,e^{\frac{1}{2}x^2}\left[\cos\sqrt{4n+1}\,x + O\left(\frac{1}{\sqrt[4]{n}}\right)\right].$$

When the subscript is odd we can obtain similarly

$$H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} \times \\ \times 1\cdot 3\cdot 5\ldots(2n-1)\sqrt{2n+1}\,e^{\frac{1}{2}x^2}\left[\sin\sqrt{4n+3}\,x + O\left(\frac{1}{\sqrt[4]{n}}\right)\right].$$

In these formulae $O(1/\sqrt[4]{n})$ denotes a value such that $4\sqrt[4]{n}\,O(1/\sqrt[4]{n})$ $(1/4dn)$ remains bounded as n increases provided x varies in an arbitrary finite interval. Notice that we can take any argument $\sqrt{4n+a}\,x$ for the trigonometric function, where a is a given real number. In fact we have, for example:

$$\cos\sqrt{4n+1}\,x - \cos\sqrt{4n+a}\,x = \\ = 2\sin\frac{\sqrt{4n+1}+\sqrt{4n+a}}{2}\,x\,\sin\frac{\sqrt{4n+a}-\sqrt{4n+1}}{2}\,x = \\ = 2\sin\frac{\sqrt{4n+1}+\sqrt{4n+a}}{2}\,x\,\sin\frac{a-1}{2\left(\sqrt{4n+a}+\sqrt{4n+1}\right)}\,x.$$

When x lies in a finite interval the above product will be equal to $O(1/\sqrt[4]{n})$ and therefore $\cos\sqrt{4n+1}x$ can be replaced by $\cos\sqrt{4n+a}x$ with an accuracy equal to this value. Using the above calculations it possible to obtain even more accurate results for the additional terms $O(1/\sqrt{n})$.

163. The asymptotic expression for Legendre polynomials. By using the same method the asymptotic expressions for the Legendre polynomials $P_n(x)$ can be deduced when n is large. We have the differential equation:

$$(1-x^2)\,P_n''(x) - 2xP_n'(x) + n(n+1)\,P_n(x) = 0.$$

Let us replace x by the new variable t according to the formula $x = \cos t$ and replace $P_n(x)$ by the new function:

$$v_n(t) = \sqrt{\sin t}\,P_n(\cos t) \text{ or } P_n(\cos t) = \frac{v_n(t)}{\sqrt{\sin t}}. \tag{59}$$

Substituting all this in the equation we obtain after simple operations the following expression for $v_n(t)$:

$$v_n''(t) + \left[n(n+1) + \frac{\frac{1}{2} - \frac{1}{4}\cos^2 t}{\sin^2 t} \right] v_n(t) = 0,$$

which we can rewrite in the form:

$$v_n''(t) + \left(n + \frac{1}{2}\right)^2 v_n(t) = -\frac{1}{4\sin^2 t} v_n(t).$$

x varies in the interval $-1 \leqslant x \leqslant +1$ which corresponds to $0 \leqslant t \leqslant \pi$. Take $t = \pi/2$ for the initial value corresponding to $x = 0$. Consider the case when the subscript is even:

$$v_{2n}''(t) + \left(2n + \frac{1}{2}\right)^2 v_{2n}(t) = -\frac{1}{4\sin^2 t} v_{2n}(t). \tag{60}$$

Bearing in mind formula (59) and also the fact that

$$P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \quad \text{and} \quad P_{2n}'(0) = 0,$$

we have the following initial conditions for $v_{2n}(t)$:

$$v_{2n}\left(\frac{\pi}{2}\right) = (-1)^n \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n}; \quad v_{2n}'\left(\frac{\pi}{2}\right) = 0. \tag{61}$$

If in the equation (60) we reject the right-hand side then the homogeneous equation so obtained will have the general solution:

$$C_1 \cos\left(2n + \frac{1}{2}\right) t + C_2 \sin\left(2n + \frac{1}{2}\right) t. \tag{62}$$

If we select C_1 and C_2 so that the conditions (61) are satisfied:

$$C_1 \cos\left(2n + \frac{1}{2}\right)\frac{\pi}{2} + C_2 \sin\left(2n + \frac{1}{2}\right)\frac{\pi}{2} = v_{2n}\left(\frac{\pi}{2}\right),$$

$$-C_1 \sin\left(2n + \frac{1}{2}\right)\frac{\pi}{2} + C_2 \cos\left(2n + \frac{1}{2}\right)\frac{\pi}{2} = 0,$$

or, using the formulae $\cos(n\pi + \varphi) = (-1)^n \cos\varphi$ and $\sin(n\pi + \varphi) = (-1)^n \sin\varphi$, we have:

$$C_1 \cos\frac{\pi}{4} + C_2 \sin\frac{\pi}{4} = (-1)^n v_{2n}\left(\frac{\pi}{2}\right); \qquad -C_1 \sin\frac{\pi}{4} + C_2 \cos\frac{\pi}{4} = 0,$$

whence

$$C_1 = C_2 = \frac{(-1)^n}{\sqrt{2}} v_{2n}\left(\frac{\pi}{2}\right) = (-1)^n v_{2n}\left(\frac{\pi}{2}\right) \sin\frac{\pi}{4},$$

and substituting in (62) we obtain the expression:

$$(-1)^n v_{2n}\left(\frac{\pi}{2}\right) \cos\left[\left(2n + \frac{1}{2}\right) t - \frac{\pi}{4}\right].$$

Hence the solution of the equation (60) with the initial conditions (61) will be:

$$v_{2n}(t) = (-1)^n v_{2n}\left(\frac{\pi}{2}\right)\cos\left[\left(2n+\frac{1}{2}\right)t-\frac{\pi}{4}\right]- \\ -\frac{1}{\left(2n+\frac{1}{2}\right)}\int_{\frac{\pi}{2}}^{t}\frac{1}{4\sin^2 u}\,v_{2n}(u)\sin\left(2n+\frac{1}{2}\right)(t-u)\,du, \quad (63)$$

where we assume that $0 \leqslant x < 1$ and $0 < t \leqslant \pi/2$. Notice also that the solution of the equation (53), which satisfies the initial conditions $y(a) = y'(a) = 0$, has the form (54) where the lower limit of the integral is not zero but a [II, 28].

If we consider the integral on the right-hand side and use formula (59):

$$K_{2n} = \int_{\frac{\pi}{2}}^{t}\frac{1}{4\sin^2 u}\sin\left(2n+\frac{1}{2}\right)(t-u)\,P_{2n}(\cos u)\sqrt{\sin u}\,du,$$

then, applying Buniakowski's inequality, we obtain:

$$K_{2n}^2 \leqslant \int_{t}^{\frac{\pi}{2}}\frac{\sin^2\left(2n+\frac{1}{2}\right)(t-u)}{16\sin^4 u}\,du\int_{t}^{\frac{\pi}{2}}P_{2n}^2(\cos u)\sin u\,du.$$

The first of the factors on the right-hand side will be less than

$$\int_{t}^{\frac{\pi}{2}}\frac{du}{16\sin^4 u} = \beta(t),$$

where $\beta(t)$ has a finite definite value for a given t and remains bounded when $0 < \varepsilon_1 < t \leqslant \pi/2$, where ε_1 is a given positive number. The second factor is less than

$$\int_{0}^{\frac{\pi}{2}}P_{2n}^2(\cos u)\sin u\,du = \int_{0}^{1}P_{2n}^2(x)\,dx = \frac{1}{4n+1}.$$

We finally obtain the result:

$$|K_{2n}| < \frac{\alpha(x)}{\sqrt{4n+1}},$$

where $\alpha(x)$ is independent of n and remains bounded when $0 \leqslant x < 1-\varepsilon$, where ε is any given positive number. Substituting in formula (63) we have:

$$v_{2n}(t) = (-1)^n v_{2n}\left(\frac{\pi}{2}\right)\cos\left[\left(2n+\frac{1}{2}\right)t-\frac{\pi}{4}\right]+\frac{\gamma(x)}{(4n+1)^{\frac{3}{2}}},$$

where $\gamma(x)$ remains bounded in the interval $0 \leqslant x < 1 - \varepsilon$ as n increases. Bearing in mind the expression (61) we obtain:

$$v_{2n}(t) = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \left\{ \cos\left[\left(2n + \frac{1}{2}\right)t - \frac{\pi}{4}\right] + \right.$$

$$\left. + (-1)^n \cdot \frac{2 \cdot 4 \ldots 2n}{1 \cdot 3 \ldots (2n-1)\,(4n+1)^{\frac{3}{2}}}\, \gamma(x) \right\}.$$

We shall use the inequality:

$$\frac{2 \cdot 4 \cdot 6 \ldots 2n}{1 \cdot 3 \cdot 5 \ldots (2n-1)} < \frac{\sqrt{\pi}}{\sqrt{2}} \sqrt{2n+1},$$

which we have proved in the previous section. It gives:

$$v_{2n}(t) = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \left\{ \cos\left[\left(2n + \frac{1}{2}\right)t - \frac{\pi}{4}\right] + \sqrt{\frac{2n+1}{4n+1}}\, \frac{\delta(x)}{4n+1} \right\}$$

or

$$v_{2n}(t) = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \left\{ \cos\left[\left(2n + \frac{1}{2}\right)t - \frac{\pi}{4}\right] + \frac{\eta(x)}{4n+1} \right\},$$

where $\delta(x)$ and $\eta(x)$ are functions of x which remain bounded when $0 \leqslant x > < 1 - \varepsilon$ and as n increases. These functions can be determined even more accurately by using the above calculations.

Finally, using formula (59) we obtain:

$$P_{2n}(\cos t) = \frac{1}{\sqrt{\sin t}}\, \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \left\{ \cos\left[\left(2n + \frac{1}{2}\right)t - \frac{\pi}{4}\right] + O\left(\frac{1}{n}\right) \right\}. \quad (64)$$

Similarly for an odd subscript

$$P_{2n-1}(\cos t) = \frac{1}{\sqrt{\sin t}}\, \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \left\{ \cos\left[\left(2n + \frac{3}{2}\right)t - \frac{\pi}{4}\right] + O\left(\frac{1}{n}\right) \right\}. \quad (65)$$

The result obtained is also valid for negative values of $x = \cos t$ and the symbol $O(1/n)$ in the above formulae denotes a number which is such that the product $n0(1/n)$ remains bounded as n increases independently of x, provided x lies anywhere in the interval $-1 + \varepsilon < x < 1 - \varepsilon$, where ε is any fixed small positive number.

We can write the above formula in a simpler form. To do so we use the Wallis formula [75]

$$\frac{\pi}{2} = \lim_{n \to \infty} \frac{2^2 \cdot 4^2 \ldots (2n-2)^2 \cdot 2n}{1^2 \cdot 3^2 \ldots (2n-1)^2}.$$

It gives:

$$\lim_{n \to \infty} \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \sqrt{2n} = \sqrt{\frac{2}{\pi}}$$

or

$$\frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \sqrt{2n} = \sqrt{\frac{2}{\pi}} + \eta_n.$$

where $\eta_n \to 0$ as $n \to \infty$, i.e.

$$\frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} = \sqrt{\frac{1}{n\pi}} + \frac{\eta_n}{\sqrt{2n}}.$$

After this, formula (64) can be rewritten as follows:

$$P_{2n}(\cos t) = \sqrt{\frac{1}{n\pi \sin t}} \left\{\cos\left[\left(2n + \frac{1}{2}\right)t - \frac{\pi}{4}\right] + \eta''_{2n}\right\}.$$

Applying the same procedure to formula (65) we can see that the following formula applies to any subscript

$$P_n(\cos t) = \sqrt{\frac{2}{n\pi \sin t}} \left\{\cos\left[\left(n + \frac{1}{2}\right)t - \frac{\pi}{4}\right] + \eta''_n\right\}, \tag{66}$$

where $\eta''_n \to 0$ as $n \to \infty$ uniformly with respect to t provided that $\varepsilon < t < \pi - \varepsilon$ where $\varepsilon > 0$.

We shall also give below the asymptotic expressions for Laguerre polynomials without going into the proof. When x varies in the interval $0 < a \leqslant x \leqslant b$, where a and b are small arbitrary finite numbers, then the following asymptotic formula holds:

$$Q_n^{(s)}(x) = \pi^{-\frac{1}{2}} n^{\frac{s}{2} - \frac{1}{4}} \cdot n!\, x^{-\frac{s}{2} - \frac{1}{4}} e^{\frac{x}{2}} \left\{\cos\left(2\sqrt{n}\, x - \frac{s\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{\sqrt{n}}\right)\right\}. \tag{67}$$

§ 4. Elliptic integrals and elliptic functions

164. The transformation of elliptic integrals into normal form. In this section we shall deal with certain functions of a complex variable which are not connected with linear differential equations and have a slightly different origin, viz. these functions are connected with certain integrals which cannot be expressed in finite form, i.e. with the so called *elliptic integrals*. We have already mentioned these integrals earlier [I, 199]. Here we shall consider them in detail.

Previously we considered an integral of the type

$$\int R(x, \sqrt{P(x)})\, dx, \tag{1}$$

where $R(x, y)$ is a rational function of its arguments, and $P(x)$ is a polynomial of the second degree. We saw that such integrals can be expressed in terms of elementary functions. *If, however, $P(x)$ is a polynomial of the third or fourth degree then an integral of the type* (1) *is known as an elliptic integral* and it cannot, in general, be ex-

pressed in finite form. In exceptional cases it is possible to express it in terms of elementary functions. Thus, for example, if we take the integral

$$\int \frac{x^{2n+1}\,dx}{\sqrt{x^4 + bx^2 + c}},$$

where n is an integer, then by introducing the new variable $t = x^2$ we obtain the following integral

$$\frac{1}{2}\int \frac{t^n\,dt}{\sqrt{t^2 + bt + c}},$$

which, as we know, can be expressed in terms of elementary functions. If an integral of the type (1), where $P(x)$ is a polynomial of the third or the fourth degree, can be expressed in terms of elementary functions then such an integral is known as *pseudoelliptic*.

Let us now consider elliptic integrals. We notice, first of all, the case when $P(x)$ is a polynomial of the third degree and does not differ fundamentally from a polynomial of the fourth degree. One can be transformed into the other by a simple replacement of the variable of integration. In fact, suppose that $P(x)$ is a polynomial of the fourth degree

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e, \tag{2}$$

and let $x = x_1$ be one of the zeros of this polynomial. We replace x by the new variable t given by the formula

$$x = x_1 + \frac{1}{t}. \tag{3}$$

Substituting in the expression (2) we obtain

$$P(x) = a\left(x_1 + \frac{1}{t}\right)^4 + b\left(x_1 + \frac{1}{t}\right)^3 + c\left(x_1 + \frac{1}{t}\right)^2 + d\left(x_1 + \frac{1}{t}\right) + e.$$

Removing the brackets and remembering that $x = x_1$ is a zero of the polynomial (2), we have

$$P(x) = \frac{P_1(t)}{t^4},$$

where $P_1(t)$ is a polynomial of the third degree. Hence we can go from a polynomial of the fourth degree to a polynomial of the third degree. Notice that the transformation (3) must involve the fact that one of the zeros of the polynomial (2), viz. $x = x_1$ is, with the new variable, transformed into the zero $t = \infty$.

Conversely, if $P(x)$ is a polynomial of the third degree, then, as a result of the bilinear transformation

$$x = \frac{\alpha t + \beta}{\gamma t + \delta}$$

we obtain

$$P(x) = \frac{P_2(t)}{(\gamma t + \delta)^4},$$

where $P_2(t)$ is a polynomial which is, in general, of the fourth degree.

Arguing in the same way as in [I, 199] we can show that the elliptic integral (1) can be transformed into integrals of the following types:

$$\int \frac{\varphi(x)}{\sqrt{P(x)}}\,dx \tag{4}$$

and

$$\int \frac{dx}{(x-\alpha)^k \sqrt{P(x)}}, \tag{5}$$

where $\varphi(x)$ is a certain polynomial. If we suppose that $P(x)$ is a polynomial of the third degree we can show that the above integrals can be transformed into integrals of one of three types. To do so consider an integral of the type

$$I_k = \int \frac{x^k}{\sqrt{P(x)}}\,dx, \tag{6}$$

where k is an integer, either positive or negative. Let us suppose that

$$P(x) = ax^3 + bx^2 + cx + d.$$

By differentiating we obtain:

$$(x^m \sqrt{P(x)})' = mx^{m-1}\sqrt{P(x)} + x^m \frac{3ax^2 + 2bx + c}{2\sqrt{P(x)}} =$$

$$= \frac{mx^{m-1}(ax^3 + bx^2 + cx + d)}{\sqrt{P(x)}} + \frac{x^m(3ax^2 + 2bx + c)}{2\sqrt{P(x)}},$$

which after integration, and by bearing in mind the notation (6), gives

$$x^m \sqrt{P(x)} + C = maI_{m+2} + mbI_{m+1} + mcI_m + mdI_{m-1} +$$

$$+ \frac{3a}{2} I_{m+2} + bI_{m+1} + \frac{c}{2} I_m$$

(C being an arbitrary constant) or

$$a\left(m + \frac{3}{2}\right) I_{m+2} + b(m+1) I_{m+1} + c\left(m + \frac{1}{2}\right) I_m + \tag{7}$$

$$+ dmI_{m-1} = x^m \sqrt{P(x)} + C.$$

When $m = 0$ and $m = 1$ we have

$$\frac{3}{2}aI_2 + bI_1 + \frac{c}{2}I_0 = \sqrt{P(x)} + C,$$

$$\frac{5}{2}aI_3 + 2bI_2 + \frac{3}{2}cI_1 + dI_0 = x\sqrt{P(x)} + C.$$

The above formulae make it possible to express I_2 and I_3 successively in terms of I_0 and I_1. Putting $m = 2$ in formula (7) we have:

$$\frac{7}{2}aI_4 + 3bI_3 + \frac{5}{2}cI_2 + 2dI_1 = x^2\sqrt{P(x)} + C,$$

whence we can determine I_4 etc. Hence all integrals of the type (6) can be expressed in terms of I_0 and I_1 when k is a positive integer. The integral (4) obviously has the same property.

Let us now consider the integral (5). Replacing x by the new variable $x - a = t$ we obtain the following integral

$$I'_k = \int \frac{t^k}{\sqrt{P_1(t)}}\,dt \qquad (k = -1, -2, \ldots), \tag{8}$$

where $P_1(t)$ is a polynomial of the third degree and k is a negative integer. Putting $m = -1$ in formula (7) we obtain

$$\frac{1}{2}a' I'_1 - \frac{1}{2}c' I'_{-1} - d' I'_{-2} = t^{-1}\sqrt{P_1(t)} + C,$$

where a', b', c' and d' are coefficients of $P_1(t)$.

If we now put $m = -2$ we obtain

$$-\frac{1}{2}a' I'_0 - b' I'_{-1} - \frac{3}{2}c' I'_{-2} - 2d' I'_{-3} = t^{-2}\sqrt{P_1(t)} + C$$

etc. This shows clearly that all integrals of the type (8) can be expressed in terms of I'_1, I'_0 and I'_{-1}, i.e. by using our former notation they can be expressed by the integrals

$$\int \frac{x-a}{\sqrt{P(x)}}\,dx; \quad \int \frac{dx}{\sqrt{P(x)}}; \quad \int \frac{dx}{(x-a)\sqrt{P(x)}}.$$

Hence we can finally say that when $P(x)$ is a polynomial of the third degree, then any elliptic integral can be transformed into an integral of one of the following three types:

$$\int \frac{dx}{\sqrt{P(x)}}; \quad \int \frac{x\,dx}{\sqrt{P(x)}}; \quad \int \frac{dx}{(x-a)\sqrt{P(x)}}. \tag{9}$$

The first of these integrals is known as *an elliptic integral of the first kind*, the second as *an elliptic integral of the second kind* and the third as *an elliptic integral of the third kind*.

Notice that when the initial integral is real then the above calculations might result in formulae which could contain complex numbers. Thus when using formula (3) the number x_1 might be complex if all four zeros of the polynomial $P(x)$ are complex. Similarly during the expansion of the rational fraction into partial fractions and the transformation of the integral into the form (5) it is possible to obtain complex values for a. We shall not give a detailed explanation of the way in which these calculations must be carried out so that real quantities only should result. In future we shall to a certain extent take this circumstance into account.

165. The conversion of the integrals into a trigonometric form. We shall for the moment only consider elliptic integrals of the first and second kinds and show that they can be obtained in a new form in which the integrand is expressed in terms of trigonometric functions. Let us begin with integrals of the first kind. We can, of course, assume that the first coefficient of $P(x)$ is equal to ± 1,

$$P(x) = \pm x^3 + bx^2 + cx + d.$$

Let us suppose, to begin with, that this polynomial with real coeffecients has three real roots: α, β and γ. We must, of course, assumi that there are no equal roots among them since otherwise the polynomial $P(x)$ would contain the square factor $(x - \alpha)^2$ which could be taken outside the square radical and we would then have under the radical a polynomial of the first degree. We now suppose that α is the smallest zreo when x^3 has the $+$ sign, and the greatest zero when x^3 has the $-$ sign. Let us suppose further that β denotes the middle zero. Replace x by the new variable φ according to the formula

$$x = \alpha + (\beta - \alpha)\sin^2\varphi. \tag{10}$$

Substituting this expression into our polynomial

$$P(x) = \pm x^3 + bx^2 + cx + d = \pm (x - \alpha)(x - \beta)(x - \gamma),$$

we obtain after simple calculations:

$$P(x) = |\gamma - \alpha|(\beta - \alpha)^2(1 - k^2\sin^2\varphi)\sin^2\varphi\cos^2\varphi,$$

where

$$k^2 = \frac{\beta - \alpha}{\gamma - \alpha}, \tag{11}$$

and we always assume that $k > 0$. We have from (10):

$$dx = 2(\beta - \alpha)\sin\varphi\cos\varphi\, d\varphi,$$

and therefore the expression

$$\frac{dx}{\sqrt{P(x)}} \tag{12}$$

differs only by a constant term from the expression

$$\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}. \tag{13}$$

We can show that it is possible to arrive at the same result when the polynomial $P(x)$ with real coefficients has only one real zero $x = a$. In this case this polynomial can be represented as follows

$$P(x) = \pm (x-a)(x^2+px+q),$$

where the trinomial (x^2+px+q) with real coefficients has no real zeros and therefore always remains positive when the values of x are real. Replace x by a new variable φ according to the formula

$$x = a \pm \sqrt{a^2+pa+q}\tan^2\frac{\varphi}{2}. \tag{14}$$

After substitution we obtain

$$\pm (x-a)(x^2+px+q) = (a^2+pa+q)^{\frac{3}{2}}(1-k^2\sin^2\varphi)\frac{\tan^2\frac{\varphi}{2}}{\cos^4\frac{\varphi}{2}},$$

where

$$k^2 = \frac{1}{2}\left(1 \pm \frac{a+\frac{p}{2}}{\sqrt{a^2+pa+q}}\right). \tag{15}$$

We shall show that the value of k^2 lies between 0 and 1. To do so it is sufficient to show that the modulus of the second term inside the large bracket in the expression (15) is less than unity, i.e. it is sufficient to show that the square of the denominator is greater than the square of the numerator. We obviously have:

$$a^2+pa+q = \left(a+\frac{p}{2}\right)^2 + \left(q-\frac{1}{4}p^2\right).$$

But the second term on the right-hand side is bound to be positive since it is given that the trinomial x^2+px+q has imaginary zeros, and we therefore have

$$a^2+pa+q > \left(a+\frac{p}{2}\right)^2.$$

Furthermore, it follows from the expression (14) that

$$dx = \pm \sqrt{a^2 + pa + q}\, \frac{\tan\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}\, d\varphi,$$

and, therefore, as a result of the substitution the expression (12) only differs by a constant term from the expression (13).

We thus see that *every real integral of the first kind can, by means of a real substitution, be obtained in the form*

$$\int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \qquad (0 < k^2 < 1). \tag{16}$$

Let us now consider integrals of the second kind

$$\int \frac{x}{\sqrt{P(x)}}\, dx.$$

Applying one of the above transformations we can obtain this integral in the form (16) together with one of the following two integrals:

$$\int \frac{\sin^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}\, d\varphi \text{ and } \int \frac{\tan^2 \frac{\varphi}{2}}{\sqrt{1 - k^2 \sin^2 \varphi}}\, d\varphi. \tag{17}$$

Adding to the first of these the constant k^2 we can write it in the form

$$k^2 \int \frac{\sin^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}\, d\varphi = \int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} - \int \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi,$$

and therefore can be converted into the integral (16) and an integral of the following type:

$$\int \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi. \tag{18}$$

Let us show that the second of the integrals (17) can be converted into the first of the integrals (17). To do so we shall use the following formula which can easily be proved by simple differentiation

$$2d\left(\tan\frac{\varphi}{2}\sqrt{1 - k^2 \sin^2 \varphi}\right) = \left[\left(1 + \tan^2\frac{\varphi}{2}\right) - 2k^2 \sin^2 \varphi\right] \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}.$$

Integrating this identity we obtain the required result:

$$\int \frac{\tan^2 \frac{\varphi}{2}}{\sqrt{1 - k^2 \sin^2 \varphi}}\, d\varphi = 2 \tan\frac{\varphi}{2} \sqrt{1 - k^2 \sin^2 \varphi} + \\ + 2k^2 \int \frac{\sin^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}\, d\varphi - \int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}.$$

We thus *an* see that *elliptic integral of the second kind can be converted into integrals of the following two types:*

$$\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}; \quad \int \sqrt{1-k^2\sin^2\varphi}\, d\varphi. \tag{19}$$

These integrals are sometimes known as *elliptic integrals of the first and second kind in the Legendre form.*

The integrals (19) can be written in a slightly different form if we replace φ by a new variable t according to the formula:

$$t = \sin\varphi.$$

In this case

$$d\varphi = \frac{dt}{\sqrt{1-t^2}},$$

and the first of the integrals (19) can be obtained in the following form:

$$\int \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Here we have under the radical a polynomial of the fourth degree of a special kind. We could, by taking an elliptic integral of the general kind, obtain such an integral if we substitute the independent variable by performing the general bilinear transformation

$$x = \frac{at+\beta}{\gamma t+\delta}.$$

Let us write the integrals (19) with a zero lower limit and a variable upper limit by introducing the special notation:

$$F(k,\varphi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}; \quad E(k,\varphi) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\varphi}\, d\varphi \tag{20}$$

If the upper limit is equal to $\varphi = \pi/2$ then these integrals will be functions of k alone:

$$F(k) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}; \quad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\, d\varphi, \tag{21}$$

and they are usually known as *full elliptic integrals of the first and second kind.*

Tables are in existence which give the values of the integrals (20) and (21). The Legendre tables, published in 1826, were the first of this kind. These tables contain among other things the logarithms

of the values of (21) for different values of k; it is assumed that $k = \sin\theta$ and values of θ are given in tenths of every degree. The fundamental table with a dual approach gives the values of the integral (20) where, as before, it is assumed that $k = \sin\theta$. This table contains values for integrals of φ and θ for every degree from 0 to 90 degrees. The values of the integrals are given with a nine tenths suffix. We can also mention the book by Janke and Emde: *Tables of Functions with Formulae And Curves*, Which Also Contains Tables of Elliptic Integrals.

166. Examples. 1. The time required for the full vibration of a simple pendulum of length l and amplitude of vibration a is expressed by the formula

$$T = \sqrt{\frac{2l}{g}} \int_0^a \frac{d\tau}{\sqrt{\cos\tau - \cos a}}, \tag{22}$$

where g is the acceleration due to gravity. It is not difficult to express the integral (22) as a full elliptic integral of the first kind. To do so we introduce the constant $k = \sin a/2$ and replace τ by a new variable determined by the formula $\sin\tau/2 = k\sin\varphi$. We then obtain

$$\cos\tau - \cos a = 2\left(\sin^2\frac{a}{2} - \sin^2\frac{\tau}{2}\right) = 2k^2\cos^2\varphi$$

and also

$$\cos\frac{\tau}{2}\,d\tau = 2k\cos\varphi\,d\varphi \text{ or } d\tau = \frac{2k\cos\varphi\,d\varphi}{\sqrt{1 - k^2\sin^2\varphi}},$$

bearing in mind that because $\sin\tau/2 = \sin a/2\sin\varphi$, the variable φ must vary between 0 and $\pi/2$, this gives us finally

$$T = 2\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = 2\sqrt{\frac{l}{g}}\,F(k).$$

2. Consider the elliptic integral

$$\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}},$$

where $k^2 > 1$ and assume that the upper limit φ_0 lies in the interval $(0, a)$, where a is determined by the equation $\sin a = 1/k$. Replace φ by the new variable ψ according to the formula $\sin\psi = k\sin\varphi$. ψ can only vary in the interval $(0, \psi_0)$ where $\sin\psi_0 = k\sin\varphi_0$, and after elementary transformation we have:

$$\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = \frac{1}{k}\,\frac{d\varphi}{\sqrt{1 - \frac{1}{k^2}\sin^2\psi}},$$

whence $$\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{1}{k} F\left(\frac{1}{k},\ \psi_0\right).$$

If the upper limit $\varphi_0 = a$ then $\psi_0 = \pi/2$ and we obtain, according to the notation (21):

$$\int_0^{a} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{1}{k} F\left(\frac{1}{k}\right).$$

Similarly we can consider the integral

$$\int_0^{\varphi_0} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

where $k^2 > 1$. In this case we obtain

$$\int_0^{a} \sqrt{1-k^2\sin^2\varphi}\, d\varphi = \frac{1}{k} F\left(\frac{1}{k}\right) + kE\left(\frac{1}{k}\right) - kF\left(\frac{1}{k}\right).$$

3. Consider the integral

$$\int \frac{dx}{\sqrt{1-x^4}}.$$

Putting $x = \cos\varphi$ we can obtain the integral in the normal form:

$$\int \frac{dx}{\sqrt{1-x^4}} = -\frac{1}{\sqrt{2}} \int \frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}}.$$

Similarly by using the same substitution we obtain

$$\int \frac{x^2\,dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2}} \int \frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}} - \sqrt{2} \int \sqrt{1-\frac{1}{2}\sin^2\varphi}\, d\varphi.$$

4. Everything that was said in [165] applies to the integral:

$$\int \frac{dx}{\sqrt{x^3+1}}$$

and replacing x by a new variable φ, according to the formula:

$$x = -1 + \sqrt{3}\tan^2\frac{\varphi}{2},$$

we obtain:

$$\int \frac{dx}{\sqrt{x^3+1}} = \frac{1}{\sqrt[4]{3}} \int \frac{d\varphi}{\sqrt{1-\left(\frac{1+\sqrt{3}}{2}\right)^2\sin^2\varphi}}.$$

5. It is more difficult to obtain the integral

$$\int \frac{dx}{\sqrt{x^4+1}} \tag{23}$$

in a simpler form. In this case the polynomial under the radical can be written as a product of two real polynomial of the second degree

$$x^4 + 1 = (x^2 + 1)^2 - (\sqrt{2}x)^2 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1).$$

In general, when a polynomial of the fourth degree under a radical has only imaginary zeros and factorizes into real polynomial of the second degree

$$P(x) = (x^2 + px + q)(x^2 + p'x + q'),$$

then the substitution of the variable must be performed in accordance with the formula

$$x = \frac{\lambda + \mu m \tan \varphi}{1 + m \tan \varphi},$$

where the numbers λ and μ must be determined from the formulae

$$(p - p')\lambda = q - q' - \sqrt{(q - q')^2 + (p - p')(pq' - qp')},$$

$$(p - p')\mu = q - q' + \sqrt{(q - q')^2 + (p - p')(pq' - qp')},$$

and m is the smaller of the two numbers

$$\sqrt{\frac{\lambda^2 - p\lambda + q}{\mu^2 - p\mu + q}} \quad \text{and} \quad \sqrt{\frac{\lambda^2 - p'\lambda + q'}{\mu^2 - p'\mu + q'}}.$$

In this case the substitution of the variables takes the form

$$x = \frac{\tan \varphi - (1 + \sqrt{2})}{\tan \varphi + (1 + \sqrt{2})},$$

and as a result of this transformation the integral (23) becomes

$$\int \frac{dx}{\sqrt{x^4 + 1}} = \int \frac{(2 + \sqrt{2})\, d\varphi}{\sqrt{\sin^4 \varphi + 6(3 + 2\sqrt{2}) \sin^2 \varphi \cos^2 \varphi + (3 + 2\sqrt{2})^2 \cos^4 \varphi}}.$$

It can readily be shown that the expression under the radical is a product of

$$\sin^2 \varphi + (3 + 2\sqrt{2})^2 \cos^2 \varphi = (3 + 2\sqrt{2})^2 \left(1 - \frac{4\sqrt{2}}{3 + 2\sqrt{2}} \sin^2 \varphi\right)$$

and $\sin^2 \varphi + \cos^2 \varphi$, and we finally obtain the following formula in which the integral (23) appears in the normal form:

$$\int \frac{dx}{\sqrt{x^4 + 1}} = \int \frac{(2 - \sqrt{2})\, d\varphi}{\sqrt{1 - \frac{4\sqrt{2}}{3 + 2\sqrt{2}} \sin^2 \varphi}}.$$

167. The conversion of elliptic integrals. Having explained the concept of an elliptic integral we shall now proceed to explain elliptic functions. In some respects these functions are similar to the known trigonometric functions and are so to speak their generalized form. We shall explain, first of all, the fact that the fundamental trigono-

metric functions, for example $x = \sin u$, can be obtained by transforming integrals. Consider the elementary integral

$$u = \int_0^x \frac{dx}{\sqrt{1-x^2}} = \arcsin x. \tag{24}$$

The value of the integral u is a function of the upper limit of integration x. Let us consider the inverse function, i.e. let us consider the upper limit x as a function of the value of the integral u. We thus obtain a single-valued, regular periodic function $x = \sin u$. It is said that this function is obtained by converting the integral (24). Similarly, if we take an elliptic integral of the first kind

$$u = \int_0^x \frac{dx}{\sqrt{P(x)}},$$

then we obtain an analytic single-valued function $x = f(u)$ as a result of its conversion. This function will no longer be an integral function. It will be a fractional function and will have not one but two essentially different periods. We shall explain this problem in greater detail later. At the moment we shall consider an elliptic integral of the first kind of the Legendre form

$$u = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \tag{25}$$

and assume that k is real and satisfies the inequality $0 < k < 1$.

We have already met the integral (25) in connection with the conformal transformation of the upper half-plane z into a rectangle in the u-plane [37]. We will recall the relevant results obtained at the time, but we shall slightly modify our notation. Formula (25) gives the conformal transformation of the upper half-plane z into a rectangle $ABCD$ in the u-plane. Its side AB lies on the real axis and the coordinates of A and B are as follows

$$\pm K = \pm \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \tag{26}$$

and

$$\text{length } AB = 2\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = 2K. \tag{27}$$

The length of the side BC is determined by the formula:

$$\text{length } BC = \int_1^{\frac{1}{k}} \frac{dz}{\sqrt{(z^2-1)(1-k^2z^2)}}.$$

In this formula if we replace z by a new variable of integration $z = 1/\sqrt{1-k'^2x^2}$, where $k'^2 = 1-k^2$, then after substitution we obtain a new expression for the length of the side BC which, like the length AB, will be expressed by a full elliptic integral of the first kind:

$$\text{length } BC = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}} = K', \tag{28}$$

where the number k'^2 is determined by the formula

$$k^2 + k'^2 = 1. \tag{29}$$

The number k is usually known as the *modulus of the integral* (25), the number k' as the *additional modulus*, and the two are connected by formula (29).

We shall now analytically continue the function (25). If, for example, we perform the analytic continuation from the upper half-plane into the lower half-plane across the line $(1, 1/k)$ on the real axis then the resulting function will transform the lower half-plane into another rectangle, obtained from the original rectangle $ABCD$ by reflection in the side BC, which is obtained from the above line on the real axis. Similarly other analytic continuations of one half-plane into the other give values of u in the form of a rectangle in the u-plane, obtained from the initial rectangle by reflection in that side of the rectangle which corresponds to the section of the real axis across which the analytic continuation is performed. Hence all the possible analytic continuations of the function (25) in the z-plane result in a net of similar rectangles in the u-plane which fill the entire u-plane without overlapping. Each of these rectangles corresponds either to the lower or to the upper half-plane z. This net is shown in Fig. 78 where the clear rectangles correspond to the upper half-plane and the shaded rectangles to the lower half-plane. Conversely, by performing the analytic continuation of the function $z = f(u)$, obtained as a result of the conversion (25), across a line l, we must only pay attention to those sides of the rectangle which are intersected by this line when we obtain in the z-plane transitions from one half-plane into the other across the corresponding sections of the real axis. If, for example, we encircle one apex of our net of rectangles

in the u-plane we obtain the former values of z in the z-plane. We can thus see that the function $f(u)$ is a single-valued analytic function in the whole u-plane.

The point $u = iK'$, which lies in the middle of the side CD of the initial rectangle corresponds to the value $z = \infty$ [37], where the

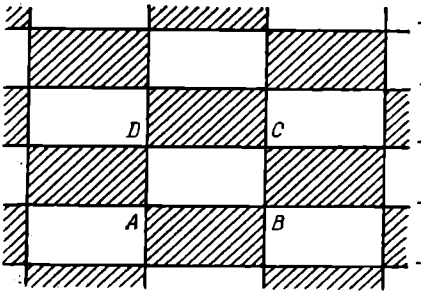

FIG. 78

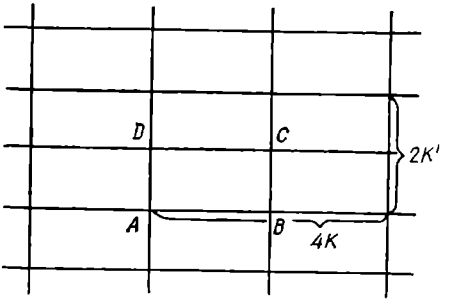

FIG. 79

one-sheeted neighbourhood of the point $u = iK'$ is transformed into the one-sheeted neighbourhood of the point $z = \infty$, and this shows [23] that our function $f(u)$ has a simple pole at the point iK'. There are analogous points in every rectangle of our net, i.e. $f(u)$ is a fractional function.

We shall show lastly that the function $f(u)$ has a real period equal to $4K$ and a purely imaginary period equal to $i2K'$. Let us take our net of rectangles and construct from it another net of larger rectangles,

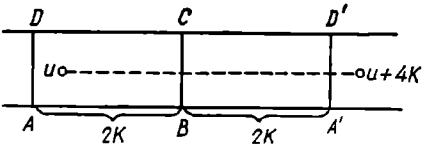

FIG. 80

by linking together four rectangles sharing a common apex (Fig. 79). This large rectangle has a side $4K$ in length parallel to the real axis and a side $2K'$ in length parallel to the imaginary axis.

The transition from u to $u + 4K$ or from u to $u + i\,2K'$ is equivalent to the geometric transition into the neighbouring rectangle, and the value of $f(u)$ does not alter during this transition. For example (Fig. 80) the transition from u to $u + 4K$ is equivalent to successive reflections in the straight lines BC and $A'D'$ which give two reflections in the real axis in the z-plane as well as former value of z. Hence the

function $f(u)$ does, in fact, have a dual periodicity which is expressed by the following formulae:

$$f(u + 4K) = f(u);$$
$$f(u + i2K') = f(u).$$

The single-valued function obtained is usually denoted as follows owing to its analogy with $\sin u$:

$$z = \sin(u).$$

We shall meet this function again later. As a result of the conversion of other elliptic integrals of the first kind we shall obtain other fractional functions with dual periodicity. We shall now deal with the general theory of such functions and of connected functions, and for this purpose we shall somewhat modify our earlier notation.

168. General properties of elliptic functions. Let ω_1 and ω_2 be two arbitrary complex numbers, the ratio of which is not equal to a real number. *The function $f(u)$ is known as an elliptic function when $f(u)$ is a fractional function with two periods ω_1 and ω_2, i.e.*

$$f(u + \omega_1) = f(u);$$
$$f(u + \omega_2) = f(u) \qquad (30)$$

are identical for any u. In other words the addition of ω_1 or ω_2 to the argument does not alter the value of the function. From formula (30) follows the more general formula

$$f(u + m_1\omega_1 + m_2\omega_2) = f(u), \qquad (31)$$

where m_1 and m_2 are any integers, positive or negative.

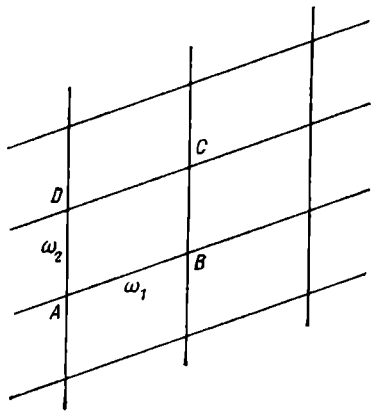

FIG. 81

Let us now explain the geometric aspect of dual periodicity. Draw from a point A in the u-plane two vectors AB and AD to correspond to the complex numbers ω_1 and ω_2. It is given that the relationship $\omega_2 : \omega_1$ is not real and therefore these vectors lie on different straight lines and therefore we can construct the parallelogram $ABCD$; by making the parallel transition of this parallelogram onto the vectors ω_1 and ω_2 we can cover the whole plane with a net of similar parallelograms (Fig. 81). The transition from any parallelogram to the neighbouring

parallelogram is equivalent to the transition from u to $u \pm \omega_1$ or $u \pm \omega_2$ and, as a result of dual periodicity, the values of $f(u)$ will be equal at corresponding points of the constructed parallelograms. Each parallelogram is known as a parallelogram of periods of the function $f(u)$. Notice that the choice of the fundamental apex A can be quite arbitrary. If, for example, we take the origin 0 as the fundamental apex then the apexes of our net of parallelograms will have

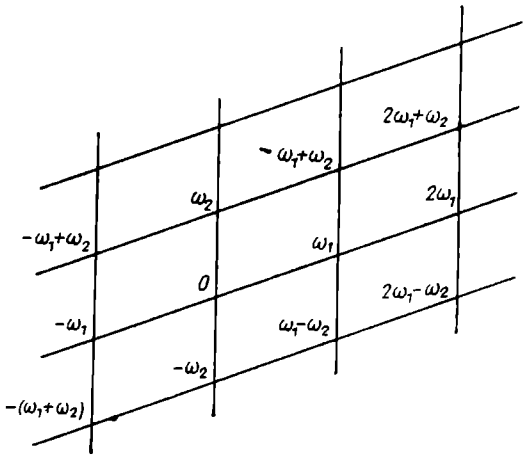

FIG. 82

complex coordinates $m_1\omega_1 + m_2\omega_2$, i.e. these apexes give the set of periods of the function $f(u)$ as shown by formula (31) (Fig. 82). If we take any point M in the u-plane and draw straight lines through it parallel to the vectors ω_1 and ω_2, then the radius vector from 0 to M gives the geometric sum of two vectors, one of which is parallel to ω_1 and the other to ω_2; hence any complex number can be represented in a unique way in the following form

$$u = k\omega_1 + l\omega_2$$

where k and l are real numbers. These numbers are the curvilinear coordinates of the point u if we adopt the vectors corresponding to the complex numbers ω_1 and ω_2 as the axes of coordinates. Above we used the phrase *the corresponding points of two parallelograms of the net*. These are points, the difference of the complex coordinates of which is equal to a period, i.e. they are expressed by the numbers $m_1\omega_1 + m_2\omega_2$, where m_1 and m_2 are integers. In this sense any point in the u-plane corresponds to a point of the fundamental parallelogram of the net.

If we take, for example, the net shown in Fig. 82 in which the origin is the fundamental apex, we can give the coordinates of any point u in the form

$$u = (k_1 \omega_1 + k_2 \omega_2) + m_1 \omega_1 + m_2 \omega_2,$$

where k_1 and k_2 are real numbers satisfying the conditions $0 \leqslant k_1 < 1$ and $0 \leqslant k_2 < 1$, and m_1 and m_2 are integers. Notice that we ascribe to each parallelogram one apex, and two sides which originate at this apex. The remaining sides and apexes are obtained by the addition of a period as mentioned above.

We shall now explain the fundamental properties of elliptic functions. Differentiating the identity (31) n times we obtain:

$$f^{(n)}(u + m_1 \omega_1 + m_2 \omega_2) = f^{(n)}(u),$$

i.e. *the derivatives of an elliptic function are also elliptic functions with the same periods*. Let us suppose that $f(u)$ has no poles, i.e. it is essentially not a fractional but an integral function. Its parallelogram of periods is a bounded domain of the plane and in this parallelogram, including its contour, it is regular and therefore also continuous and bounded, i.e. a positive number N exists such that in the fundamental parallelogram of periods the inequality $|f(u)| < N$ is satisfied. In other parallelograms of the net the values of $f(u)$ are repeated, and therefore the above inequality is satisfied in the whole plane, i.e. $f(u)$ is an integral function bounded in the whole plane. According to the theorem of Liouville we can say that such a function must be constant, i.e. we have the following theorem.

THEOREM I. *If $f(u)$ is an integral function of dual periodicity then $f(u)$ is constant.*

This theorem is very important owing to its two lemmas which we shall now establish. Let $f_1(u)$ and $f_2(u)$ be two elliptic functions with equal periods ω_1 and ω_2. Assume that they have equal poles in the parallelogram of periods, with equal infinite parts. The difference $f_2(u) - f_1(u)$ will then be a function of dual periodicity without poles, i.e. this difference will be an integral function of dual periodicity and it follows from the theorem that this difference must be constant. We therefore have:

LEMMA 1. *If two elliptic functions $f_1(u)$ and $f_2(u)$ with equal periods have the same poles in the parallelogram of periods and their infinite parts are also equal, then these functions differ only by a constant term.*

Let us now suppose that $f_1(u)$ and $f_2(u)$ have the same poles and zeros of the same order in the parallelogram of periods. In this case

the relationship $f_2(u) : f_1(u)$ will have neither poles nor zeros in the parallelogram of periods and it should therefore be constant, i.e. we have:

LEMMA 2. *If two elliptic functions $f_1(u)$ and $f_2(u)$ with equal periods have equal poles and zeros of the same order in the parallelogram of periods then these functions differ only by a constant term.*

Place the parallelograms of periods of the function $f(u)$ in such a way that the poles of this function do not lie on the sides and consider the integral of this function over the contour of the parallelogram

$$\int_{ABCD} f(u)\,du = \int_{AB} f(u)\,du + \int_{BC} f(u)\,du + \int_{CD} f(u)\,du + \int_{DA} f(u)\,du. \quad (32)$$

Let us consider the integral along the side CD and replace u by the new variable of integration $u = v + \omega_2$. In this case the side CD becomes the side BA in the v-plane and, as a result of the periodicity of the function, we have

$$\int_{CD} f(u)\,du = \int_{BA} f(v + \omega_2)\,dv = \int_{BA} f(v)\,dv = -\int_{AB} f(v)\,dv,$$

i.e. on the right-hand side of formula (32) the sum of the first and third term is zero. The same can be said of the sum of the second and fourth terms and we therefore have:

$$\int_{ABCD} f(u)\,du = 0, \quad (33)$$

i.e. if the poles of the elliptic function $f(u)$ do not lie on the contour of the parallelogram, then the integral of this function round the contour of the parallelogram is zero.

Consider the complex number a which is such that the equation $f(u) - a = 0$ has no zeros on the contour of the parallelogram. Applying the above result to the elliptic function

$$\varphi(u) = \frac{f'(u)}{f(u) - a},$$

we obtain

$$\int_{ABCD} \frac{f'(u)}{f(u) - a}\,du = 0.$$

The above integral expresses, as we know, the difference between the number of zeros and poles of the function $f(u) - a$ [22] and we can therefore say that the number of zeros of the equation $f(u) = a$

is equal to the number of poles of $f(u) - a$, or, which comes to the same thing, to the number of poles of $f(u)$, i.e. the function $f(u)$ takes the value of a and infinity an equal number of times inside the parallelogram.

This was proved for an arbitrary a on the assumption that the equation $f(u) = a$ had no zeros on the contour of the parallelogram. If this is not so then we can slightly displace our parallelogram so that the zeros of the above equation do fall inside the parallelogram and the poles remain inside as before. The above result will be valid for the displaced parallelogram. It can easily be seen that it will also be valid for the initial parallelogram, provided that when counting the number of zeros of the equation we ascribe one apex and two sides, originating at this apex, to the parallelogram. Notice, that when the equation $f(u) = a$ has the zero $u = u_0$ and the following expansion applies in this neighbourhood

$$f(u) = a + c_k (u - u_0)^k + c_{k+1} (u - u_0)^{k+1} + \dots \qquad (c_k \neq 0),$$

then this zero must be regarded as a zero of order k of the function $f(u) - a$ or of the above equation. In all such cases we obtain, from the above considerations, the following theorem.

THEOREM II. *The elliptic function takes any value (finite or infinite) an equal number of times in the parallelogram of periods.*

If $f(u)$ takes any value m times in the parallelogram of periods, then it is known as *an elliptic function of order m*. Such a function transforms the parallelogram of periods into an m-sheeted Riemann surface. The conformity of the transformation may be affected only at points where $f'(u)$ vanishes or where $f(u)$ has multiple poles. These values of u correspond to branch-points on the above Riemann surface.

We shall now show that the positive integer m cannot be unity. In fact, it follows directly from formula (33) that *the sum of the residues of an elliptic function at its poles inside the parallelogram of periods should be zero*. If we had $m = 1$ then the function $f(u)$ would have one simple pole inside the parallelogram of periods which contradicts the above result. We thus see that *an elliptic function of the first order does not exist*. Later we shall, in fact, construct elliptic functions of the second order. It can be shown that elliptic functions of the second order are obtained by converting elliptic integrals of the first kind. There are, of course, elliptic functions of higher orders.

169. Fundamental lemma. Consider the elementary function $\sin u$. This is an integral function with simple zeros at the points $u = k\pi$ ($k = 0, \pm 1, \ldots$) on the real axis, which are at a distance π from each other. Two other fundamental functions

$$\cot u = \frac{(\sin u)'}{\sin u} = \frac{\cos u}{\sin u} \quad \text{and} \quad -(\cot u)' = \frac{1}{\sin^2 u}, \tag{34}$$

have simple and double poles at these points. The function $\sin u$ can be represented by an infinite product and we had previously an expansion into partial fractions for the functions (34). In the next section we shall by an analogous process construct an integral function with simple zeros at the points

$$m_1\omega_1 + m_2\omega_2, \tag{35}$$

where ω_1 and ω_2 are complex numbers, the ratio between which is not equal to a real number, and m_1 and m_2 are any integers. The points (35) are the apexes of the net of parallelograms shown in Fig. 82. To construct this integral function we shall use the Weierstrass formula which gives an integral function in the form of an infinite product. To be able to apply this formula we have to find a number p which is such that the series

$$\sum_{m_1, m_2}{}' \frac{1}{|m_1\omega_1 + m_2\omega_2|^p} \tag{36}$$

converges. The apostrophe near the symbol of summation shows that summation includes all integral values of m_1 and m_2 except $m_1 = m_2 = 0$. Similar conditions will apply in future in all cases where there is an apostrophe near the symbol of a sum or product. We must rewrite the sum (36) in the form

$$\sum_{m_1, m_2}{}' \frac{1}{\delta^p_{m_1, m_2}} \tag{37}$$

where δ_{m_1, m_2} is the distance from the origin to that apex of the net in Fig. 82 which corresponds to the complex coordinate $m_1\omega_1 + m_2\omega_2$. Let 2δ be the shortest distance between any apex of the net, other than the origin, to the origin. This number 2δ will obviously also be the shortest distance between two apexes of the net. Imagine that two circles are drawn in the plane in Fig. 82, centres the origin and radii n and $(n+1)$ respectively, where n is an integer which satisfies the condition $n > \delta$. Let K_n be the annulus between these two circles. Let us find the approximate number of apexes of the net in the annulus K_n.

Let this number be t_n. Draw circles of radius δ with centres at the apexes of the net which lie inside the annulus K_n. It follows from the definition of δ that these circles will not overlap and their common surface area $\pi\delta^2 t_n$ will be less than the surface area of the annulus with inside radius $(n - \delta)$ and outside radius $(n + 1 + \delta)$, i.e.

$$\pi(n + 1 + \delta)^2 - \pi(n - \delta)^2 > \pi\delta^2 t_n$$

or, after elementary calculations,

$$t_n < A_1 n + A_2 \qquad \left(A_1 = \frac{4\delta + 2}{\delta^2}; \quad A_2 = \frac{2\delta + 1}{\delta^2}\right).$$

For any apex inside the annulus K_n the distance δ_{m_1,m_2} will not be less than n and therefore the sum of the terms of the series (37), which corresponds to the apexes in the annulus K_n will, in any case, be less than

$$\frac{A_1 n + A_2}{n^p} = \frac{A_1}{n^{p-1}} \frac{A_2}{n^p}.$$

This result will be valid for terms of the sum (37) for which δ_{m_1,m_2} is sufficiently large with respect to δ_1, for example $\delta_{m_1,m_2} > \delta + 1$. In this case the corresponding points are bound to lie inside the annulus K_n when $n > \delta$. Rejecting a finite number of terms of the series (37) and replacing the remaining terms by greater terms, we obtain a domineering series

$$\sum_n \left(\frac{A_1}{n^{p-1}} + \frac{A_2}{n^p}\right).$$

As we know from [I, 122], this series converges when $p > 2$ and, in particular, when $p = 3$ and we can therefore draw the conclusion:

FUNDAMENTAL LEMMA. *The series* (36) *converges when* $p > 2$ *and, in particular, when* $p = 3$.

170. The Weierstrass function. To simplify the notation let us put

$$w = m_1\omega_1 + m_2\omega_2. \tag{38}$$

Bearing in mind the fundamental lemma we can construct directly an integral function with simple zeros at the points (38). This function will be given by the following formula [69]:

$$\sigma(u) = u \prod_{m_1,m_2}{}' \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\left(\frac{u}{w}\right)^2}, \tag{39}$$

where the infinite product includes all pairs of integral values of m_1 and m_2, both positive and negative, except $m_1 = m_2 = 0$.

As we know from [68] we can evaluate the logarithmic derivative of this product in the same way as of a finite product; the logarithmic derivative of an individual factor of this product will be

$$\frac{1}{w} + \frac{u}{w^2} - \frac{1}{w}\frac{1}{1 - \frac{u}{w}} = \frac{1}{u - w} + \frac{1}{w} + \frac{u}{w^2}.$$

We thus obtain a second function

$$\zeta(u) = \frac{\sigma'(u)}{\sigma(u)} = \frac{1}{u} + \sum_{m_1, m_2}{}' \left(\frac{1}{u - w} + \frac{1}{w} + \frac{u}{w^2}\right), \tag{40}$$

which has simple poles at the points (38). This function is obtained from $\sigma(u)$ in the same way as $\cot u$ is obtained from $\sin u$. Bearing in mind the convergence of the series

$$\sum_{m_1, m_2}{}' \frac{1}{|w|^3},$$

it can readily be shown that the series (40) converges uniformly in any finite domain if a finite number of terms with poles in this domain are rejected. Differentiating the function (40) and changing the sign we obtain a new function

$$\wp(u) = -\zeta'(u) = \frac{1}{u^2} + \sum_{m_1, m_2}{}' \left[\frac{1}{(u - w)^2} - \frac{1}{w^2}\right]. \tag{41}$$

This new function is obtained from $\zeta(u)$ in the same way as $(1/\sin^2 u)$ is obtained from $\cot u$. It has double poles at the points w. The series (41) also converges uniformly in domains of the kind mentioned above [12].

We shall now explain some fundamental properties of the introduced functions. Let us write the expression for $\sigma(-u)$:

$$\sigma(-u) = -u \prod_{m_1, m_2}{}' \left(1 + \frac{u}{w}\right) e^{-\frac{u}{w} + \frac{1}{2}\left(\frac{u}{w}\right)^2}.$$

Owing to the fact that the product includes all pairs of integers m_1 and m_2, except $m_1 = m_2 = 0$, we can interchange the signs of m_1 and m_2, i.e. change the sign of w, whence we obtain

$$\sigma(-u) = -u \prod_{m_1, m_2} \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\left(\frac{u}{w}\right)^2} = -\sigma(u),$$

i.e. $\sigma(u)$ *is an odd function.* It can be shown similarly that $\zeta(u)$ *is also an odd function and that* $\wp(u)$ *is an even function.* This can, in fact, be obtained directly from the formulae

$$\zeta(u) = \frac{\sigma'(u)}{\sigma(u)}; \quad \wp(u) = -\zeta'(u), \tag{42}$$

since the differentiation of an odd function gives an even function and vice versa. It also follows directly from the formulae which define the introduced functions that

$$\frac{\sigma(u)}{u}\Big|_{u=0} = 1; \quad u\zeta(u)\Big|_{u=0} = 1; \quad u^2\wp(u)\Big|_{u=0} = 1. \tag{43}$$

The functions $\sigma(u)$ and $\zeta(u)$ cannot have periods ω_1 and ω_2 since the first is an integral function and the second has one simple pole in the parallelogram. We shall show that the function $\wp(u)$ has periods ω_1 and ω_2. To do so construct, to begin with,

$$\wp'(u) = -\frac{2}{u^3} - \sum_{m_1,m_2}{}' \frac{1}{(u-w)^3}$$

or

$$\wp'(u) = -2\sum_{m_1,m_2} \frac{1}{(u-w)^3} = -2\sum_{m_1,m_2} \frac{1}{(u-m_1\omega_1-m_2\omega_2)^3},$$

where summation includes all integer values for m_1 and m_2 without exception. Hence

$$\wp'(u+\omega_1) = -2\sum_{m_1,m_2} \frac{1}{(u_1+\omega_1-m_1\omega_1-m_2\omega_2)^3} =$$

$$= -2\sum_{m_1,m_2} \frac{1}{[u-(m_1-1)\omega_1-m_2\omega_2]^3}.$$

If m_1 runs through all integral values then the same can be said of $(m_1 - 1)$. We thus have

$$\wp'(u+\omega_1) = \wp'(u),$$

and it can be shown similarly that $\wp'(u+\omega_2) = \wp'(u)$. Hence

$$\wp'(u+\omega_k) = \wp'(u) \qquad (k=1,\ 2). \tag{44}$$

Let us now investigate the variation of the function $\wp(u)$ when ω_1 and ω_2 are added to the argument. Integrating (44) we have

$$\wp(u+\omega_k) = \wp(u) + C_k,$$

where C_k is a constant. Let us suppose in this identity that $u = -\omega_k/2$, where $\omega_k/2$ is not a pole of $\wp(u)$:

$$\wp\left(\frac{\omega_k}{2}\right) = \wp\left(-\frac{\omega_k}{2}\right) + C_k.$$

Owing to the fact that the function $\wp(u)$ is even we have $\wp(-\omega_k/2) = \wp(\omega_k/2)$ and therefore $C_k = 0$, i.e.

$$\wp(u + \omega_k) = \wp(u) \qquad (k = 1, 2). \tag{45}$$

Let us now investigate the variation of the function $\zeta(u)$ when ω_k is added to the argument. We have from (45) and (42)

$$\zeta'(u + \omega_k) = \zeta'(u).$$

Integrating this we obtain

$$\zeta(u + \omega_k) = \zeta(u) + \eta_k \qquad (k = 1, 2), \tag{46}$$

where η_k is a constant, i.e. *the function $\zeta(u)$ acquires a constant term η_k when the number ω_k is added to the argument.* A more general formula also follows from formula (46):

$$\zeta(u + m_1\omega_1 + m_2\omega_2) = \zeta(u) + m_1\eta_1 + m_2\eta_2, \tag{47}$$

where m_1 and m_2 are arbitrary integers.

The number η_k can be defined as a particular value of the function $\zeta(u)$, viz. assuming in formula (46) that $u = -\omega_k/2$ and remembering that the function $\xi(u)$ is odd, we obtain

$$\eta_k = 2\zeta\left(\frac{\omega_k}{2}\right) \qquad (k = 1, 2). \tag{48}$$

Let us now turn to the function $\sigma(u)$. As a result of (46) and (42) we can write

$$\frac{\sigma'(u + \omega_k)}{\sigma(u + \omega_k)} = \frac{\sigma'(u)}{\sigma(u)} + \eta_k.$$

Integrating this we obtain

$$\log \sigma(u + \omega_k) = \log \sigma(u) + \eta_k u + D_k$$

or

$$\sigma(u + \omega_k) = C_k e^{\eta_k u}\, \sigma(u),$$

where $C_k = e^D$ is a constant. To determine this constant we suppose in the above identity that $u = -\omega_k/2$:

$$\sigma\left(\frac{\omega_k}{2}\right) = C_k e^{-\frac{\eta_k \omega_k}{2}} \sigma\left(-\frac{\omega_k}{2}\right).$$

Remembering that the function $\sigma(u)$ is odd and simplifying by the factor $\sigma(\omega_k/2)$ which is not zero we obtain

$$C_k = -e^{\frac{\eta_k \omega_k}{2}},$$

and finally

$$\sigma(u + \omega_k) = -e^{\eta_k\left(u + \frac{\omega_k}{2}\right)} \sigma(u) \qquad (k = 1, 2). \tag{49}$$

Conclusion: *the function $\sigma(u)$ is multiplied by an exponential factor when ω_k is added to the argument.* Instead of formula (49) a more general formula can be obtained, which is similar to formula (47), viz.:

$$\sigma(u + w) = \varepsilon e^{\eta\left(u + \frac{w}{2}\right)} \sigma(u), \tag{50}$$

where

$$w = m_1\omega_1 + m_2\omega_2; \quad \eta = m_1\eta_1 + m_2\eta_2$$

and $\varepsilon = +1$ or $\varepsilon = -1$, depending on whether both integers m_1 and m_2 are even or not. In the last case the relationship (50) follows directly from (47) in the same way as (40) follows from (46). We shall only use this case in future when $m_1 = m_2 = 1$.

In conclusion of this section, we shall introduce a relationship connecting the constants ω_k and η_k. To begin with we shall introduce a rule for the notation of the periods ω_1 and ω_2. Consider the fundamental parallelogram $ABCD$ (Fig. 81). One of its sides AD makes a positive angle smaller than π with its other side AB. We shall always assume that ω_1 corresponds to that side AB, from which the angle is measured and ω_2 to that side AD of the parallelogram to which the positive angle, smaller than π is measured; the amplitude of the fraction ω_2/ω_1 will then be equal to an angle between 0 and π, i.e. the imaginary part of this fraction must be positive. The imaginary part of the reciprocal fraction ω_1/ω_2 will evidently be negative. Hence *we shall always denote the numbers ω_k in such a way that the relationship ω_2/ω_1 has a positive imaginary part.* Let us now construct a parallelogram with the principal apex at $A(u = u_0)$ so that the pole $\zeta(u)$ lies inside it. This single pole has, from (40), a residue of unity and, as a result of the fundamental theorem of residues, the

integral of the function $\zeta(u)$ round the contour of the parallelogram will be equal to $2\pi i$, i.e.

$$\int_{u_0}^{u_0+\omega_1} \zeta(u)\,du + \int_{u_0+\omega_1}^{u_0+\omega_1+\omega_2} \zeta(u)\,du + \int_{u_0+\omega_1+\omega_2}^{u_0+\omega_2} \zeta(u)\,du + \int_{u_0+\omega_2}^{u_0} \zeta(u)\,du = 2\pi i.$$

Replacing the variable u in the second of the above integrals by $u = v_1 + \omega_1$ and the variable u in the third integral by the new variable $u = v_2 + \omega_2$, we obtain

$$\int_{u_0}^{u_0+\omega_1} \zeta(u)\,du + \int_{u_0}^{u_0+\omega_2} \zeta(v_1+\omega_1)\,dv_1 + \int_{u_0+\omega_1}^{u_0} \zeta(v_2+w_2)\,dv_2 + \int_{u_0+\omega_2}^{u_0} \zeta(u)\,du = 2\pi i,$$

where we integrate along straight lines or by changing the sign of the variable of integration

$$\int_{u_0}^{u_0+\omega_2} [\zeta(u+\omega_1) - \zeta(u)]\,du - \int_{u_0}^{u_0+\omega_1} [\zeta(u+\omega_2) - \zeta(u)]\,du = 2\pi i.$$

This, from (46), gives the required relationship between ω_k and η_k:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i. \tag{51}$$

This relationship is usually known as *the Legendre relationship.*

The functions $\sigma(u)$, $\zeta(u)$ and $\wp(u)$ were first introduced by Weierstrass. It is evident from their definition that any two complex numbers ω_1 and ω_2 can be used for their construction, the only necessary condition being that their ratio is not be real; these Weierstrass functions are not only functions of the argument u but also of the complex parameters ω_1 and ω_2 which we just mentioned. For this reason they are sometimes denoted as follows:

$$\sigma(u;\ \omega_1,\ \omega_2);\quad \zeta(u;\ \omega_1,\ \omega_2);\quad \wp(u;\ \omega_1,\ \omega_2). \tag{52}$$

171. The differential equation for $\wp(u)$. Having established the fundamental properties of the Weierstrass functions we shall now study in greater detail the function $\wp(u)$ and, in particular, deduce a differential equation of the first order for this function. First of all we shall derive the expansion of the function $\wp(u)$ near the point

$u = 0$ which is a pole of the second order of this function. To do so let us turn to the fundamental formula (41). We have near $u = 0$:

$$\frac{1}{w-u} = \frac{1}{w} + \frac{u}{w^2} + \ldots + \frac{u^n}{w^{n+1}} + \ldots$$

and differentiating with respect to u we obtain

$$\frac{1}{(w-u)^2} = \frac{1}{w^2} + \frac{2u}{w^3} + \ldots + \frac{(n+1)u^n}{w^{n+2}} + \ldots$$

This, from (41), gives the following expansion for $\wp(u)$ near the origin

$$\wp(u) = \frac{1}{u^2} + \sum_{n=1}^{\infty} (n+1)\, u^n \sum_{m_1, m_2}{}' \frac{1}{w^{n+2}}.$$

When n is odd, the summation in terms of m_1 and m_2 contains terms which are equal in value and opposite in sign in pairs so that we have

$$\wp(u) = \frac{1}{u^2} + c_2 u^2 + c_3 u^4 + \ldots + c_n u^{2n-2} + \ldots, \tag{53}$$

where

$$c_n = (2n-1) \sum_{m_1, m_2}{}' \frac{1}{w^{2n}} \qquad (n = 2,\ 3, \ldots). \tag{54}$$

Let us now establish the form of the expansion for $\wp'^2(u)$ and $\wp^3(u)$. We evidently have

$$\wp'(u) = -\frac{2}{u^3} + 2c_2 u + 4c_3 u^3 + \ldots$$

$$\wp'^2(u) = \frac{4}{u^6} - \frac{8c_2}{u^2} - 16c_3 + \ldots$$

$$\wp^3(u) = \frac{1}{u^6} + \frac{3c_2}{u^2} + 3c_3 + \ldots,$$

where in the last two expressions terms in positive powers of u are omitted. Hence

$$\wp'^2(u) - 4\wp^3(u) + 20c_2\wp(u) = -28c_3 + \ldots, \tag{55}$$

where terms in positive powers of u are again omitted. Therefore the point $u = 0$ will no longer be a pole of the expression on the left-hand side. This expression will be an elliptic function without poles in the parallelogram of periods since the only poles of the function $\wp(u)$ are at the point $u = 0$ and at the corresponding apexes of other parallelograms. For this reason the expression (55) (an elliptic

function without poles) will be constant. But the right-hand side becomes $-28c_3$, i.e.

$$\wp'^2(u) = 4\wp^3(u) - 20c_2\wp(u) - 28c_3.$$

Let us introduce the notation:

$$g_2 = 20c_2 = 60 \sum_{m_1, m_2}{}' \frac{1}{w^4}; \quad g_3 = 28c_3 = 140 \sum_{m_1, m_2}{}' \frac{1}{w^6}. \tag{56}$$

The above calculations give rise to the following theorem.

THEOREM. *The function $\wp(u)$ satisfies the differential equation*

$$\wp'^2(u) = 4\wp^3(u) - g_2\wp(u) - g_3. \tag{57}$$

The numbers g_2 and g_3 are known as *invariants of the function $\wp(u)$*.

The function $\wp(u)$ has only one double pole in the parallelogram of periods with its main apex at the point $u = 0$. The remaining apexes should no longer be ascribed to this parallelogram and therefore $\wp(u)$ is an elliptic function of the second order; any equation $\wp(u) = a$ for any given value of the complex number a will have two zeros in the parallelogram of periods.

When $\wp(u_0) = a$ and $\wp'(u_0) = 0$ then the zero $u = u_0$ must, in any case, be a double zero. It can not, however, be of an order greater than a double zero since $\wp(u)$ is a function of the second order and therefore it is equal to a only at one point $u = u_0$ in the parallelogram. If $\wp'(u_0) \neq 0$ then the equation $\wp(u) = a$ will have two different simple zeros in the parallelogram. Let us now see where those values of u for which $\wp'(u) = 0$ lie. Assuming in the identities

$$\wp'(u + \omega_k) = \wp'(u) \quad \text{or} \quad \wp'(u + \omega_1 + \omega_2) = \wp'(u)$$

that $u = -\omega_k/2$ or $u = -(\omega_1 + \omega_2)2$ we obtain, owing to the fact that $\wp'(u)$ is odd

$$\wp'\left(\frac{\omega_k}{2}\right) = 0 \quad (k = 1, 2) \quad \text{and} \quad \wp'\left(\frac{\omega_1 + \omega_2}{2}\right) = 0, \tag{58}$$

i.e. $\wp'(u)$ vanishes in the middle of the sides and in the centre of the diagonal of the parallelogram with the main apex at $u = 0$. Consider the values of the function $\wp(u)$ at these points

$$\wp\left(\frac{\omega_1}{2}\right) = e_1; \quad \wp\left(\frac{\omega_1 + \omega_2}{2}\right) = e_2; \quad \wp\left(\frac{\omega_2}{2}\right) = e_3. \tag{59}$$

Each equation $\wp(u) = e_k$ has a double zero at the corresponding point. Bearing in mind that $\wp(u)$ is a function of the second order we can say that the numbers e_k are different.

Let us now turn to formula (57). The right-hand side of this formula is a polynomial of the third degree in $\wp(u)$. Assuming that either $u=\omega_k/2$ or $u=(\omega_1+\omega)/2$ we can see that this polynomial vanishes when $\wp(u)=e_k$ since for the given substitution the left-hand side of the equation (57) vanishes and $\wp(u)$ becomes e_k. Factorizing the polynomial we can rewrite formula (57) as follows:

$$\wp'^2(u) = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3). \tag{60}$$

Comparing the right-hand sides of the formulae (57) and (60) we obtain formulae connecting the numbers e_k and the invariants g_2 and g_3:

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2; \quad e_1e_2e_3 = \frac{1}{4}g_3. \tag{61}$$

If we put $x = \wp(u)$ then the equation (57) can be rewritten in the form

$$\left(\frac{dx}{du}\right)^2 = 4x^3 - g_2x - g_3.$$

Bearing in mind that when $u=0$ we have $x=\infty$, we separate the variables and integrate to obtain

$$u = \int_{\infty}^{x} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} \tag{62}$$

i.e. *the function $\wp(u)$ is obtained as a result of the conversion of an elliptic integral of the first kind* (62). It can be shown conversely that by choosing the constants g_2 and g_3 in such a way that the polynomial under the radical has no square zero, the conversion of the integral (62) leads to the Weierstrass function $\wp(u)$.

It can be shown further that any elliptic function with periods ω_1 and ω_2 is a rational function of $\wp(u)$ and $\wp'(u)$ so that the set of rational functions of $\wp'(u)$ and $\wp(u)$ represents the full set of elliptic functions with periods ω_1 and ω_2.

172. The functions $\sigma_k(u)$. It follows directly from formula (60) that the product on the right-hand side is the square of a single-valued analytic function $\wp'(u)$. It appears that the same can be said about each of the factors $\wp(u) - e_k$. The same thing also applies to trigonometric functions

$$(\cos u)'^2 = \sin^2 u = (1 - \cos u)(1 + \cos u),$$

for each factor on the right-hand side is the square of a single-valued analytic function

$$1 - \cos u = 2 \sin^2 \frac{u}{2} \quad \text{and} \quad 1 + \cos u = 2 \cos^2 \frac{u}{2} .$$

To prove our hypothesis for the difference $\wp(u) - e_k$ we shall deduce an auxiliary formula. Consider the difference

$$\wp(u) - \wp(v) \tag{63}$$

as being a function of the argument u. In the parallelogram with the main apex at $u = 0$ it has a double pole $u = 0$. Owing to the fact that the function $\wp(u)$ is even, the zeros of the function (63) lie at the two points of the parallelogram which correspond to the complex numbers $u = \pm v$, i.e. strictly speaking, the zeros lie at points of the parallelogram which differ by a period from $\pm v$. If such a point of the parellelogram proves to be a period then the above two points coincide and form a double point in the way explained above. Together with the function (63) let us also consider the function

$$f(u) = \frac{\sigma(u - v)\,\sigma(u + v)}{\sigma^2(u)} . \tag{64}$$

Let us prove, first of all, that this latter function also has the periods ω_1 and ω_2. We have from (49)

$$f(u + \omega_k) = \frac{\varrho(u - v + \omega_k)\,\sigma(u + v + \omega_k)}{\sigma^2(u + \omega_k)} =$$

$$= \frac{e^{\eta_k\left(u - v + \frac{\omega_k}{2}\right)}\,\sigma(u - v)\,e^{\eta_k\left(u + v + \frac{\omega_k}{2}\right)}\,\sigma(u + v)}{e^{2\eta_k\left(u + \frac{\omega_k}{2}\right)}\,\sigma^2(u)} = \frac{\sigma(u - v)\,\sigma(u + v)}{\sigma^2(u)} = f(u)$$

Therefore the function (64) does, in fact, have periods ω_1 and ω_2. It follows directly from formula (64) that it has a double zero at $u = 0$ in the fundamental parallelogram and two zeros at points of the parallelogram which differ by a period from $\pm v$. All this follows from the position of the zeros of the function $\sigma(u)$, viz. it is due to the fact that these zeros are simple zeros equal to $w = m_1 \omega_1 + m_2 \omega_2$. Therefore the functions (63) and (64) both with periods ω_1 and ω_2 have the same poles and the same zeros of the same order in the fundamental parallelogram. We can therefore say that these functions differ only by a constant term [168]:

$$\wp(u) - \wp(v) = C \frac{\sigma(u - v)\,\sigma(u + v)}{\sigma^2(u)} .$$

To determine the constant C we multiply both sides by u^2 and subsequently put $u = 0$

$$u^2\wp(u) - u^2\wp(u)\Big|_{u=0} = \frac{C\sigma(u-v)\sigma(u+v)}{\left[\frac{\sigma(u)}{u}\right]^2}\Bigg|_{u=0}.$$

We have from (43)

$$1 = C\sigma(-v)\sigma(v) = -C\sigma^2(v),$$

and we finally obtain the required formula

$$\wp(u) - \wp(v) = -\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(v)\,\sigma^2(u)}. \tag{65}$$

To investigate the difference $\wp(u) - e_k$ we only have to assume in formula (65) that

$$v = \frac{\omega_1}{2}; \quad v = \frac{\omega_1+\omega_2}{2} \quad \text{and} \quad v = \frac{\omega_2}{2}.$$

Thus, for example,

$$\wp(u) - e_1 = \wp(u) - \wp\left(\frac{\omega_1}{2}\right) = -\frac{\sigma\left(u-\frac{\omega_1}{2}\right)\sigma\left(u+\frac{\omega_1}{2}\right)}{\sigma^2\left(\frac{\omega_1}{2}\right)\sigma^2(u)}, \tag{66}$$

or, bearing in mind that we have from (49)

$$\sigma\left(u+\frac{\omega_1}{2}\right) = \sigma\left(u-\frac{\omega_1}{2}+\omega_1\right) = -\mathrm{e}^{\eta_1\left(u-\frac{\omega_1}{2}+\frac{\omega_1}{2}\right)}\sigma\left(u-\frac{\omega_1}{2}\right),$$

i.e.

$$\sigma\left(u+\frac{\omega_1}{2}\right) = -\mathrm{e}^{\eta_1 u}\sigma\left(u-\frac{\omega_1}{2}\right), \tag{67}$$

instead of (66) we can write

$$\wp(u) - e_1 = \mathrm{e}^{\eta_1 u}\frac{\sigma^2\left(u-\frac{\omega_1}{2}\right)}{\sigma_2\left(\frac{\omega_1}{2}\right)\sigma^2(u)}$$

or

$$\wp(u) - e_1 = \left[\frac{\mathrm{e}^{\frac{1}{2}\eta_1 u}\,\sigma\left(\frac{\omega_1}{2}-u\right)}{\varrho\left(\frac{\omega_1}{2}\right)\varrho(u)}\right]^2.$$

Two other differences can be analysed similarly. We therefore obtain the following representations for $\wp(u) - e_k$ as the square of the quotient

of two integral functions:

$$\wp(u) - e_k = \left[\frac{\sigma_k(u)}{\sigma(u)}\right]^2, \tag{68}$$

where we have used the following notation:

$$\left.\begin{aligned} &\sigma_1(u) = e^{\frac{1}{2}\eta_1 u}\,\frac{\sigma\left(\frac{\omega_1}{2} - u\right)}{\sigma\left(\frac{\omega_1}{2}\right)}; \\ &\sigma_2(u) = e^{\frac{1}{2}(\eta_1+\eta_3)u}\,\frac{\sigma\left(\frac{\omega_1+\omega_2}{2} - u\right)}{\sigma\left(\frac{\omega_1+\omega_2}{2}\right)}; \quad \sigma_3(u) = e^{\frac{1}{2}\eta_3 u}\,\frac{\sigma\left(\frac{\omega_2}{2} - u\right)}{\sigma\left(\frac{\omega_2}{2}\right)}. \end{aligned}\right\} \tag{69}$$

Let us now establish certain properties of the function $\sigma(u)$. These functions are obviously, *integral functions* and, putting $u = 0$, we have

$$\sigma_k(0) = 1 \qquad (k = 1,\ 2,\ 3). \tag{70}$$

Rewriting the relationship (67) in the form

$$\sigma\left(\frac{\omega_1}{2} - u\right) = e^{-\eta_1 u}\,\sigma\left(\frac{\omega_1}{2} + u\right),$$

we obtain

$$\sigma_1(u) = e^{-\frac{1}{2}\eta_1 u}\,\frac{\sigma\left(\frac{\omega_1}{2} + u\right)}{\varrho\left(\frac{\omega_1}{2}\right)} = \sigma_1(-u)$$

and the same applies to the two other functions $\sigma_k(u)$, i.e. *the functions $\sigma(u)$ are even integral functions.*

Substituting the expressions (68) in the right-hand side of formula (60) and extracting the zero we have

$$\wp'(u) = \pm\, 2\,\frac{\sigma_1(u)\,\sigma_2(u)\,\sigma_3(u)}{\sigma^3(u)}.$$

To determine the sign we multiply both sides by u^3 and subsequently put $u = 0$. Bearing in mind the expansion

$$\wp'(u) = -\frac{2}{u^3} + 2c_2 u + 4c_3 u^3 + \ldots,$$

and also the formulae (70) and (43) we can see that we must take the $-$ sign in the above formula, i.e.

$$\wp'(u) = -\,2\,\frac{\sigma_1(u)\,\sigma_2(u)\,\sigma_3(u)}{\sigma^3(u)}. \tag{71}$$

173. The expansion of a periodic integral function. The integral function $\sigma(u)$ has no periods at all. We shall show later that by multiplying it by an exponential factor we can obtain a periodic integral function. At present we shall consider a periodic integral function and deduce an expansion for such a function in the form of a power series or a Fourier series [cf. 119].

Let us suppose that the integral function $\varphi(u)$ has a period ω, i.e. for any complex u

$$\varphi(u+\omega)=\varphi(u). \tag{72}$$

We draw a vector ω through the origin and two lines perpendicular to it through its ends (Fig. 83). The latter lines form the period band of the function $\varphi(u)$. The line CD is obtained from the line AB by the transformation $u'=u+\omega$. Let us apply the transformation $\tau=u\cdot 2\pi i/\omega$ to the u-plane. In the plane $\tau=\tau_1+i\tau_2$ our band will be transformed into a band bounded by the lines $\tau_2=0$ and $\tau_2=2\pi$. If we subsequently perform the transformation

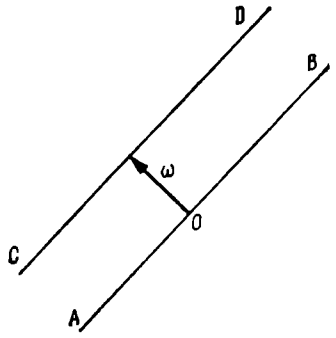

FIG. 83

$$\zeta=e^{\tau}=e^{\frac{2\pi iu}{\omega}},$$

then in the new ζ-plane our band will be represented by the whole plane cut along the positive part of the real axis ζ with the exception of the point $\zeta=0$ [19]. The edges of the cut correspond to the two lines which bounded the original band in the u-plane and the corresponding points on the two edges refer to values of u connected by the relationship $u'=u+\omega$. As a result of (72) our function has equal values on both edges of the cut and therefore the values of the derivatives of all orders will also be equal, i.e. our function will be single-valued and regular not only in the cut ζ-plane but in the whole ζ-plane except at the point $\zeta=0$. We can therefore say that it can be expanded in this plane into a Laurent's series

$$\varphi(u)=\sum_{n=-\infty}^{+\infty}{}' a_n\zeta^n=\sum_{n=-\infty}^{+\infty} a_n e^{\frac{2\pi iu}{\omega}n}. \tag{73}$$

We thus derive the following theorem.

THEOREM. *Any integral function $\varphi(u)$ with a period ω can be represented in the whole u-plane by the following series*

$$\varphi(u) = \sum_{n=-\infty}^{+\infty} a_n e^{\frac{2\pi i u}{\omega} n}. \tag{74}$$

The above series must converge uniformly in any bounded part of the plane. If we use Euler's formula and group together terms with corresponding values of n, equal in magnitude but opposite in sign, we obtain an expression for the function $\varphi(u)$ in the form of the trigonometric series

$$\varphi(u) = a_0 + \sum_{n=1}^{\infty} \left(a'_n \cos \frac{2\pi u n}{\omega} + b'_n \sin \frac{2\pi u n}{\omega} \right), \tag{75}$$

where

$$a'_n = a_n + a_{-n}; \quad b'_n = i(a_n - a_{-n}) \quad (n = 1, 2, 3, \ldots). \tag{76}$$

174. The new notation. When the theory of elliptic functions is studied in detail it contains an extensive formal apparatus which is used in the applications of these functions. Unfortunately not all authors keep to the same notation. We are giving here the mere basis of the theory and therefore we shall not give numerous and frequently evry useful formulae which appear in the theory of elliptic functions. However below we shall deal with a more complicated formal apparatus than we have used until now. We shall therefore give the notation due to Jacobi which is systematically explained in the book by Hurwitz *The General Theory of Functions and of Elliptic Functions*. We shall follow the treatment described in this book in the next few sections.

In future we shall frequently deal with halves of the numbers ω_1 and ω_2 and therefore, to abolish the use of fractions, we introduce the following symbols for ω_1 and ω_2:

$$\omega_1 = 2\omega; \quad \omega_2 = 2\omega'. \tag{77}$$

In relation to this we also put

$$\eta_1 = 2\eta; \quad \eta_2 = 2\eta'. \tag{78}$$

In future the fundamental element in the construction of the functions will not be the numbers ω_1 and ω_2 themselves, as for example in the case of the function $\wp(u)$, but their relationship

$$\tau = \frac{\omega_2}{\omega_1} = \frac{\omega'}{\omega} \tag{79}$$

or another number directly dependent on this relationship:

$$h = e^{i\pi\tau}. \tag{80}$$

We also replace the argument u by two other arguments

$$v = \frac{u}{2\omega}; \quad z = e^{i\pi v} = e^{\frac{i\pi u}{2\omega}}. \tag{81}$$

The above notation affects the symmetry of the numbers ω and ω', i.e. these numbers play different parts when the above notation is used. We always assume, as before, that in the relationship ω'/ω the coefficient of i is positive, i.e. if it is assumed that $\omega'/\omega = r + is$, then $s > 0$, and therefore

$$|h| = e^{-\pi s} < 1. \tag{82}$$

For these values of ω_1 and ω_2 we had Legendre's relationship (51) which, by using the above notation, can be rewritten in the form:

$$\eta\omega' - \eta'\omega = \frac{1}{2}\pi i. \tag{83}$$

Notice certain consequences due to the use of the above notation. We have from (81):

$$\frac{u+\omega}{2\omega} = v + \frac{1}{2}; \quad \frac{u+\omega}{2\omega} = v + 1; \quad e^{i\pi\left(r+\frac{1}{2}\right)} = iz; \quad e^{i\pi(v+1)} = -z,$$

and similarly

$$\frac{u+\omega'}{2\omega} = v + \frac{\tau}{2}; \quad \frac{u+2\omega'}{2\omega} = v + \tau; \quad e^{i\pi\left(v+\frac{\tau}{2}\right)} = h^{\frac{1}{2}}z; \quad e^{i\pi(v+\tau)} = hz,$$

i.e. for example, the addition of ω to u is equivalent to the addition of $1/2$ to v or to the multiplication of z by i; the addition of ω' to u is equivalent to the addition of $\tau/2$ to v or to the multiplication of z by $h^{1/2}$. Notice in conclusion that we shall always denote the powers of h^ϱ and z^ϱ by $e^{i\pi\tau\varrho}$ and $e^{i\pi v\varrho}$,

175. The function $\vartheta_1(v)$. Using the new notation we can express the fundamental property of the function $\sigma(u)$ as follows:

$$\sigma(u+2\omega) = -e^{2\eta(u+\omega)}\sigma(u); \quad \sigma(u+2\omega') = -e^{2\eta'(u+\omega')}\sigma(u). \tag{84}$$

Add to the function $\sigma(u)$ the exponential factor

$$\varphi(u) = e^{au^2+bu}\sigma(u) \tag{85}$$

and select the numbers a and b so that the new function $\varphi(u)$ has a period 2ω. We have from (84)

$$\varphi(u+2\omega) = -\,e^{a(u+2\omega)^2+b(u+2\omega)+2\eta(u+\omega)}\,\sigma(u) =$$

$$= -\,e^{4a\omega u+4a\omega^2+2b\omega+2\eta(u+\omega)}e^{au^2+bu}\,\sigma(u),$$

or

$$\frac{\varphi(u+2\omega)}{\varphi(u)} = -\,e^{2(2a\omega+\eta)(u+\omega)+2b\omega} \tag{86}$$

and similarly

$$\frac{\varphi(u+2\omega')}{\varphi(u)} = -\,e^{2(2a\omega'+\eta')(u+\omega')+2b\omega'}\,. \tag{87}$$

In formula (86) the index on the right-hand side is a polynomial of the first degree in u. In order that the right-hand side should be unity for any value of u it is necessary to equate to zero the coefficient of u in the index while the constant term in the index should be equated to $k\pi i$, where k is an odd integer. In relation to this we assume that

$$a = -\frac{\eta}{2\omega}\,, \qquad b = \frac{\pi i}{2\omega}\,.$$

Substituting in the right-hand side of formula (87) we have from (83):

$$\frac{\varphi(u+2\omega')}{\varphi(u)} = -\,e^{-\frac{\pi i}{\omega}(u+\omega')+\pi i\frac{\omega'}{\omega}} = -\,e^{-\frac{\pi i u}{\omega}} = -\,z^{-2}\,.$$

We thus see that the function

$$\varphi(u) = e^{-\frac{\eta u^2}{2\omega}+\frac{i\pi u}{2\omega}}\,\sigma(u) = e^{-\frac{\eta u^2}{2\omega}}\,z\sigma(u) \tag{88}$$

satisfies the following equations

$$\varphi(u+2\omega) = \varphi(u), \quad \varphi(u+2\omega') = -\,z^{-2}\,\varphi(u). \tag{89}$$

Since $\varphi(u)$ is an integral function with a period 2ω, we can expand it in the form [173]

$$\varphi(u) = \sum_{n=-\infty}^{+\infty} a_n e^{\frac{2\pi i u}{2\omega}n} = \sum_{n=-\infty}^{+\infty} a_n z^{2n}.$$

Also the addition of $2\omega'$ to u is equivalent to the multiplication of z by h, i.e.

$$\varphi(u+2\omega') = \sum_{n=-\infty}^{+\infty} a_n h^{2n} z^{2n},$$

and the second of the formulae (89) gives

$$\sum_{n=-\infty}^{+\infty} a_n h^{2n} z^{2n} = -\sum_{n=-\infty}^{+\infty} a_n z^{2n-2},$$

or, replacing in the latter sum the variable of summation n by $n+1$,

$$\sum_{n=-\infty}^{+\infty} a_n h^{2n} z^{2n} = -\sum_{n=-\infty}^{+\infty} a_{n+1} z^{2n}.$$

Comparing the coefficients of like powers of z we obtain

$$a_{n+1} = -h^{2n} a_n = -h^{\left(n+\frac{1}{2}\right)^2 - \left(n-\frac{1}{2}\right)^2} a_n,$$

which can also be written as follows:

$$(-1)^{n+1} h^{-\left(n+\frac{1}{2}\right)^2} a_{n+1} = (-1)^n h^{-\left(n-\frac{1}{2}\right)^2} a_n.$$

We thus see that the expression

$$(-1)^n h^{-\left(n-\frac{1}{2}\right)^2} a_n$$

must have the same value for all integral values of n. Suppose that

$$(-1)^n h^{-\left(n-\frac{1}{2}\right)^2} a_n = Ci,$$

where C is a constant. This gives us the following expression for the coefficients in the expansion of the function $\varphi(u)$:

$$a_n = (-1)^n h^{\left(n-\frac{1}{2}\right)^2} Ci,$$

and therefore:

$$\varphi(u) = Ci \sum_{n=-\infty}^{+\infty} (-1)^n h^{\left(n-\frac{1}{2}\right)^2} z^{2n}. \tag{90}$$

Formula (88) gives, at the same time, the following expression for the Weierstrass function $\sigma(u)$:

$$\sigma(u) = e^{\frac{\eta u^2}{2\omega}} z^{-1} \varphi(u). \tag{91}$$

Here we must introduce a new function

$$\vartheta_1(v) = i \sum_{n=-\infty}^{+\infty} (-1)^n h^{\left(n-\frac{1}{2}\right)^2} z^{2n-1}, \tag{92}$$

which is connected with the function $\sigma(u)$ by the following relationship:

$$\sigma(u) = e^{\frac{\eta u^2}{2\omega}} C \vartheta_1(v). \tag{93}$$

Let us now determine the constant C. Bearing in mind that $u = 2\omega v$ which, from (93), gives $\vartheta_1(0) = 0$, and that the relationship $\vartheta_1(v)/v$, when $v = 0$, is equal to $\vartheta_1'(0)$, we obtain, on dividing both sides of formula (93) by u and then putting $u = 0$,

$$1 = \frac{1}{2\omega} C\vartheta_1'(0),$$

and finally

$$\sigma(u) = e^{\frac{\eta u^2}{2\omega}} \frac{2\omega}{\vartheta_1'(0)} \vartheta_1(v). \tag{94}$$

Let us now transform the power series (92) for the function $\vartheta_1(v)$ into a trigonometric series. To do so we must group together terms of the above expansion referring to powers of z which are equal in value but opposite in sign. Denoting by ν the odd positive number $\nu = 2n - 1$ $(n = 1, 2, 3, \ldots)$, whence $n = -\nu + 1/2$ we can write:

$$\vartheta_1(v) = i\left[\sum_{\nu}^{1,3,5,\ldots} (-1)^{\frac{\nu+1}{2}} h^{\frac{\nu^2}{4}} z^{\nu} + \sum_{\nu}^{1,3,5,\ldots} (-1)^{\frac{-\nu+1}{2}} h^{\frac{\nu^2}{4}} z^{-\nu}\right],$$

where each sum is over the odd positive numbers i.e. $\nu = 1, 3, 5, \ldots$ Bearing in mind that

$$(-1)^{\frac{\nu+1}{2}} = (-1)^{\nu}(-1)^{\frac{-\nu+1}{2}} = -(-1)^{\frac{-\nu+1}{2}} = -(-1)^{\frac{\nu-1}{2}}$$

and

$$z^{\nu} - z^{-\nu} = e^{i\nu\pi v} - e^{-i\nu\pi v} = 2i \sin \nu\pi v,$$

we can rewrite the above formula as follows:

$$\vartheta_1(v) = i \sum_{\nu}^{1,3,5,\ldots} (-1)^{\frac{\nu-1}{2}} h^{\frac{\nu^2}{4}} (z^{-\nu} - z^{\nu}),$$

or

$$\vartheta_1(v) = 2 \sum_{\nu}^{1,3,5,\ldots} (-1)^{\frac{\nu-1}{2}} h^{\frac{\nu^2}{4}} \sin \nu\pi v =$$

$$= 2\left[h^{\frac{1}{4}} \sin \pi v - h^{\frac{9}{4}} \sin 3\pi v + h^{\frac{25}{4}} \sin 5\pi v - \ldots\right]. \tag{95}$$

The function $\vartheta_1(v)$, which is usually known as *the first theta function is an odd integral function of* v. To construct this function we only use one complex number τ which, according to the given conditions, lies in the upper half-plane, i.e. its imaginary part must be positive, where $h = e^{i\pi\tau}$. Therefore the theta-function is sometimes denoted by $\vartheta_1(v; \tau)$,

176. The function $\vartheta_k(v)$. Above together with the function $\sigma(u)$, we introduced the three integral functions $\sigma_k(u)$. This, naturally, indicates that we should also introduce three theta-functions together with the function $\vartheta_1(v)$.

Using our new notation we have

$$\sigma_3(u) = e^{\eta' u} \frac{\sigma(\omega' - u)}{\sigma(\omega')}$$

or from (93)

$$\sigma_3(u) = \frac{C}{\sigma(\omega')} e^{\eta' u + \frac{\eta(\omega' - u)^2}{2\omega}} \vartheta_1\left(\frac{\omega' - u}{2\omega}\right).$$

Removing the brackets of the power index and replacing $u/2\omega$ by v and ω'/ω by τ we obtain:

$$\sigma_3(u) = C_3 e^{\frac{\eta u^2}{2\omega}} e^{(\eta'\omega - \eta\omega')\frac{u}{\omega}} \vartheta_1\left(\frac{\tau}{2} - v\right),$$

where C_3 is a new constant. Finally, using the relationship (83) we obtain an expression for the function $\sigma_3(u)$ in terms of the new theta-function:

$$\sigma_3(u) = C_3 e^{\frac{\eta u^2}{2\omega}} z^{-1} \vartheta_1\left(\frac{\tau}{2} - v\right). \tag{96}$$

Similarly we obtain for $\sigma_2(u)$:

$$\sigma_2(u) = e^{\bar{\eta} u} \frac{\sigma(\tilde{\omega} - u)}{\sigma(\tilde{\omega})},$$

where, for the sake of briefness, we have put

$$\bar{\eta} = \eta + \eta'; \quad \tilde{\omega} = \omega + \omega'.$$

Formula (93) then gives us

$$\sigma_2(u) = \frac{C}{\sigma(\tilde{\omega})} e^{\bar{\eta} u + \eta \frac{(\tilde{\omega} - u)^2}{2\omega}} \vartheta_1\left(\frac{\tilde{\omega} - u}{2\omega}\right),$$

and making calculations analogous with those above we finally obtain

$$\sigma_2(u) = C_2 e^{\frac{\eta u^2}{2\omega}} z^{-1} \vartheta_1\left(\frac{1}{2} + \frac{\tau}{2} - v\right). \tag{97}$$

We have similarly

$$\sigma_1(u) = C_1 e^{\frac{\eta u^2}{2\omega}} \vartheta_1\left(\frac{1}{2} - v\right). \tag{98}$$

Let us now deduce the expansion into a power series from the values of the theta-functions entering the expressions for the functions $\sigma_k(u)$. We have

$$\vartheta_1\left(\frac{1}{2}-v\right) = -\vartheta_1\left(v-\frac{1}{2}\right).$$

But, from (81), the subtraction of 1/2 from v is equivalent to the multiplication of z by $(-i)$ and, therefore, recalling (92)

$$\vartheta_1\left(\frac{1}{2}-v\right) = -i\sum_{n=-\infty}^{+\infty}(-1)^n h^{\left(n-\frac{1}{2}\right)^2}(-iz)^{2n-1} = \\ = \sum_{n=-\infty}^{+\infty} h^{\left(n-\frac{1}{2}\right)^2} z^{2n-1}. \tag{99}$$

Similarly

$$\vartheta_1\left(\frac{1}{2}+\frac{\tau}{2}-v\right) = -\vartheta_1\left(v-\frac{1}{2}-\frac{\tau}{2}\right),$$

and the subtraction of $(1/2+\tau/2)$ from v is equivalent to the multiplication of z by $-i\cdot h^{-1/2}$.

It follows that

$$\vartheta_1\left(\frac{1}{2}+\frac{\tau}{2}-v\right) = -i\sum_{n=-\infty}^{+\infty}(-1)^n h^{\left(n-\frac{1}{2}\right)^2}(-ih^{-\frac{1}{2}}z)^{2n-1} = \\ = h^{-\frac{1}{4}}z\sum_{n=-\infty}^{+\infty} h^{(n-1)^2}z^{2n-2}$$

or, replacing the variable of summation n by $n+1$,

$$\vartheta_1\left(\frac{1}{2}+\frac{\tau}{2}-v\right) = h^{-\frac{1}{4}}z\sum_{n=-\infty}^{+\infty} h^{n^2}z^{2n}, \tag{100}$$

and similarly

$$\vartheta_1\left(\frac{\tau}{2}-v\right) = h^{\frac{1}{4}}iz\sum_{n=-\infty}^{+\infty}(-1)^n h^{n^2}z^{2n}. \tag{101}$$

We introduce the three new theta-functions

$$\left.\begin{aligned}
\vartheta_2(v) &= \sum_{n=-\infty}^{+\infty} h^{\left(n-\frac{1}{2}\right)^2}z^{2n-1},\\
\vartheta_3(v) &= \sum_{n=-\infty}^{+\infty} h^{n^2}z^{2n},\\
\vartheta_4(v) &= \sum_{n=-\infty}^{+\infty}(-1)^n h^{n^2}z^{2n}.
\end{aligned}\right\} \tag{102}$$

In this case the above formulae for $\sigma_k(u)$ can be wrtten in the form

$$\sigma_1(u) = C_1 e^{\frac{\eta u^2}{2\omega}} \vartheta_2(v); \quad \sigma_2(u) = \widetilde{C}_2 e^{\frac{\eta u^2}{2\omega}} \vartheta_3(v); \quad \sigma_2(u) = \widetilde{C}_3 e^{\frac{\eta u^2}{2\omega}} \vartheta_4(v),$$

where $\widetilde{C}_2$ and $\widetilde{C}_3$ are new constants. To determine the constants we put $v = 0$. In this case $u = 0$ and $\sigma_k(0) = 1$; hence

$$C_1 = \frac{1}{\vartheta_2(0)}; \quad \widetilde{C}_2 = \frac{1}{\vartheta_3(0)}; \quad \widetilde{C}_3 = \frac{1}{\vartheta_4(0)},$$

and we have finally

$$\sigma_1(u) = e^{\frac{\eta u^2}{2\omega}} \frac{\vartheta_2(v)}{\vartheta_2(0)}; \quad \sigma_2(u) = e^{\frac{\eta u^2}{2\omega}} \frac{\vartheta_3(v)}{\vartheta_3(0)}; \quad \sigma_3(u) = e^{\frac{\eta u^2}{2\omega}} \frac{\vartheta_4(v)}{\vartheta_4(0)}. \tag{103}$$

Sometimes $\vartheta_0(v)$ is written instead of $\vartheta_4(v)$.

The power series (102) for theta-functions can easily be transformed into trigonometric series in the same way as those of the functions $\vartheta_1(v)$. We thus obtain:

$$\left.\begin{aligned} \vartheta_2(v) &= 2h^{\frac{1}{4}} \cos \pi v + 2h^{\frac{9}{4}} \cos 3\pi v + 2h^{\frac{25}{4}} \cos 5\pi v + \dots \\ \vartheta_3(v) &= 1 + 2h \cos 2\pi v + 2h^4 \cos 4\pi v + 2h^9 \cos 6\pi v + \dots \\ \vartheta_4(v) &= 1 - 2h \cos 2\pi v + 2h^4 \cos 4\pi v - 2h^9 \cos 6\pi v + \dots \end{aligned}\right\} \tag{104}$$

In future, to simplify notation, we shall not write the argument $v = 0$, i.e. instead of $\vartheta_1'(0)$ we shall simply write ϑ_1' and ϑ_k instead of $\vartheta_k(0)$. It follows from (95) and (104) that we can write the following expansions for these values

$$\left.\begin{aligned} \vartheta_1' &= 2\pi(h^{\frac{1}{4}} - 3h^{\frac{9}{4}} + 5h^{\frac{25}{4}} - 7h^{\frac{49}{4}} + \dots) \\ \vartheta_2 &= 2h^{\frac{1}{4}} + 2h^{\frac{9}{4}} + 2h^{\frac{25}{4}} + 2h^{\frac{49}{4}} + \dots \\ \vartheta_3 &= 1 + 2h + 2h^4 + 2h^9 + \dots \\ \vartheta_4 &= 1 - 2h + 2h^4 - 2h^9 + \dots \end{aligned}\right\} \tag{105}$$

These series converge very rapidly since it is given that $|h| < 1$ and the sum of these series are regular functions of τ which are determined in the upper half-plane.

It is not difficult to establish the connection between the Weierstrass function $\wp(u)$ and the theta-functions. We had earlier

$$\sqrt{\wp(u) - e_k} = \frac{\sigma_k(u)}{\sigma(u)},$$

and, bearing in mind the expression for the functions $\sigma(u)$ and $\sigma_k(u)$ in terms of theta-functions we obtain

$$\sqrt{\wp(u) - e_k} = \frac{1}{2\omega} \frac{\vartheta_1'}{\vartheta_{k+1}} \frac{\vartheta_{k+1}(v)}{\vartheta_1(v)}. \tag{106}$$

177. The properties of theta-functions. All theta-functions are integral functions of the argument v and the fundamental element used in their construction is the complex number τ in the upper half-plane. To emphasize this fact these functions are sometimes written as follows: $\vartheta_k(v; \tau)$. As we have said already the function $\vartheta_1(v)$ is odd and the remaining functions are even. We shall now investigate the behaviour of theta-functions when 1/2 is added to the argument v. Remembering the expansion of theta-functions into trigonometric series and using the formulae for the conversion of theta-functions, we obtain immediately

$$\vartheta_1\left(v + \frac{1}{2}\right) = \vartheta_2(v); \quad \vartheta_2\left(v + \frac{1}{2}\right) = -\vartheta_1(v);$$

$$\vartheta_3\left(v + \frac{1}{2}\right) = \vartheta_4(v); \quad \vartheta_4\left(v + \frac{1}{2}\right) = \vartheta_3(v).$$

Let us now investigate the behaviour of theta-functions when $\tau/2$ is added to the argument v. This, as we know, is equivalent to multiplying z by $h^{1/2}$. Using the power series for theta-functions we obtain, for example, from (92)

$$\vartheta_1\left(v + \frac{\tau}{2}\right) = i \sum_{n=-\infty}^{+\infty} (-1)^n h^{\left(n-\frac{1}{2}\right)^2} h^{\frac{2n-1}{2}} z^{2n-1} =$$

$$= ih^{-\frac{1}{4}} z^{-1} \sum_{n=-\infty}^{+\infty} (-1)^n h^{n^2} z^{2n},$$

or from (102)

$$\vartheta_1\left(v + \frac{\tau}{2}\right) = im\vartheta_4(v),$$

where

$$m = h^{-\frac{1}{4}} z^{-1} = h^{-\frac{1}{4}} e^{-i\pi v}, \tag{107}$$

and it can be shown similarly that

$$\vartheta_2\left(v + \frac{\tau}{2}\right) = m\vartheta_3(v); \quad \vartheta_3\left(v + \frac{\tau}{2}\right) = m\vartheta_2(v); \quad \vartheta_4\left(v + \frac{\tau}{2}\right) = im\vartheta_1(v).$$

From this more general transformation formulae can be obtained. Thus, for example:

$$\vartheta_1(v+\tau) = \vartheta_1\left(v+\frac{\tau}{2}+\frac{\tau}{2}\right) = ih^{-\frac{1}{4}}\,e^{-i\pi\left(v+\frac{\tau}{2}\right)}\vartheta_4\left(v+\frac{\tau}{2}\right) =$$

$$= ih^{-\frac{1}{4}}\,e^{-i\pi\left(v+\frac{\tau}{2}\right)}ih^{-\frac{1}{4}}\,e^{-i\pi v}\vartheta_1(v) = -l\vartheta_1(v),$$

where

$$l = h^{-1}z^{-2}. \tag{108}$$

The results obtained can be collected in a table as shown below:

	$v+\frac{1}{2}$	$v+\frac{\tau}{2}$	$v+\frac{1}{2}+\frac{\tau}{2}$	$v+1$	$v+\tau$	$v+1+\tau$
ϑ_1	ϑ_2	$im\vartheta_4$	$m\vartheta_3$	$-\vartheta_1$	$-l\vartheta_1$	$l\vartheta_1$
ϑ_2	$-\vartheta_1$	$m\vartheta_3$	$-im\vartheta_4$	$-\vartheta_2$	$l\vartheta_2$	$-l\vartheta_2$
ϑ_3	ϑ_4	$m\vartheta_2$	$im\vartheta_1$	ϑ_3	$l\vartheta_3$	$l\vartheta_3$
ϑ_4	ϑ_3	$im\vartheta_1$	$m\vartheta_2$	ϑ_4	$-l\vartheta_4$	$-l\vartheta_4$

(109)

If we want, for example, to express $\vartheta_3(v+1/2+\tau/2)$ in terms of theta functions of the fundamental argument v, then we must find ϑ_3 in the first column and take the expression under $v+1/2+\tau/2$, in the corresponding row, i.e.

$$\vartheta_3\left(v+\frac{1}{2}+\frac{\tau}{2}\right) = im\vartheta_1(v).$$

We shall also give a table of the zeros of theta-functions. The function $\vartheta_1(v)$ differs from the function $\sigma(u)$ by an exponential factor which cannot vanish and, consequently $\vartheta_1(v)$ vanishes when, and only when $\delta(u)$ vanishes; this latter fact takes place when

$$u = n2\omega + n'2\omega',$$

where n and n' are arbitrary integers. Dividing by 2ω we obtain the following expression for the zeros of the function $\vartheta_1(v)$:

$$v = n + n'\tau.$$

The zeros of the remaining theta-functions can be obtained by using the first row of the above table. Thus, for example, we have $\vartheta_3(v) = m^{-1}\vartheta_1(v+1/2+\tau/2)$, and therefore the zeros of $\vartheta_3(v)$ are determined by the condition

$$v+\frac{1}{2}+\frac{\tau}{2} = n + n'\tau,$$

since $m^{-1} = h^{\frac{1}{4}}e^{i\pi v}$ does not vanish, or

$$v = \left(n - \frac{1}{2}\right) + \left(n' - \frac{1}{2}\right)\tau,$$

where n and n' are arbitrary integers. We thus obtain the following table for the zeros of theta-functions:

	v
ϑ_1	$n + n'\tau$
ϑ_2	$n + n'\tau + \frac{1}{2}$
ϑ_3	$n + n'\tau + \frac{1}{2} + \frac{\tau}{2}$
ϑ_4	$n + n'\tau + \frac{\tau}{2}$

(110)

Notice also that it follows from the fifth column of the table (106) that the functions ϑ_3 and ϑ_4 have a unit period, and the functions ϑ_1 and ϑ_2 have a period of two. The above table shows that different theta-functions have different zeros.

Theta-functions can be regarded as functions of the two arguments v and τ. For any given τ in the upper half-plane they are integral functions of v, and for any given v they are regular functions of τ in the upper half-plane. This latter circumstance is directly due to the fact that the series (92) and (102) converge uniformly when $|h| < \varrho < 1$. We shall now show that all four theta-functions, by virtue of being functions of two arguments, satisfy one and the partial same differential equation of the second order:

$$\frac{\partial^2 \vartheta_k(v)}{\partial v^2} = 4\pi i \frac{\partial \vartheta_k(v)}{\partial \tau}. \tag{111}$$

This equation formally resembles the heat conductance equation with which we dealt earlier [II, 203]. We shall test the equation (111), for example, for the function $\vartheta_3(v)$. Differentiating the general term of the series (104), equal to $2h^{n^2}\cos 2n\pi v = 2e^{i\pi\tau n^2}\cos 2n\pi v$, twice with respect to v we obtain

$$-8n^2\pi^2 e^{i\pi\tau n^2}\cos 2n\pi v,$$

and the same result is obtained by differentiating once with respect to τ and multiplying the result by $4\pi i$

$$4\pi i\,(2i\pi n^2\,e^{i\pi\tau n^2}\cos 2n\pi v) = -8n^2\pi^2 e^{i\pi\tau n^2}\cos 2n\pi v.$$

This equation can be tested similarly for other theta-functions.

178. An expression for the numbers e_k in terms of ϑ_s. In the study of the Weierstrass function $\wp(u)$ we introduced the numbers e_k which, in our new notation, are defined as follows:

$$e_1 = \wp(\omega);\quad e_2 = \wp(\omega+\omega');\quad e_3 = \wp(\omega'), \tag{112}$$

and we obtained the following fundamental relationship for the function $\wp(u)$

$$\wp'^2(u) = 4(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3). \tag{113}$$

The numbers e_k, as we saw, satisfy the condition

$$e_1+e_2+e_3 = 0 \tag{114}$$

and are all different. These numbers are of fundamental importance in the theory of the function $\wp(u)$. They can be taken as the basis for the construction of the function $\wp(u)$ instead of 2ω or $2\omega'$. In this case the function $\wp(u)$ will be obtained as a result of the conversion of an elliptic integral of the first kind

$$u = \int_{\infty}^{x} \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}. \tag{115}$$

We shall now express the numbers e_k in terms of theta-functions, the argument of which is zero. Take formula (106)

$$\sqrt{\wp(u)-e_k} = \frac{1}{2\omega}\frac{\vartheta_1'}{\vartheta_{k+1}}\frac{\vartheta_{k+1}(v)}{\vartheta_1(v)} \qquad \left(v=\frac{u}{2\omega}\right)$$

and put $u=\omega$, i.e. $v=1/2$, and subsequently, put $u=\omega+\omega'$, i.e. $v=1/2+\tau/2$. We thus obtain from (112):

$$\sqrt{e_1-e_k} = \frac{1}{2\omega}\frac{\vartheta_1'}{\vartheta_{k+1}}\frac{\vartheta_{k+1}\left(\frac{1}{2}\right)}{\vartheta_1\left(\frac{1}{2}\right)};$$

$$\sqrt{e_2-e_k} = \frac{1}{2\omega}\frac{\vartheta_1}{\vartheta_{k+1}}\frac{\vartheta_{k+1}\left(\frac{1}{2}+\frac{\tau}{2}\right)}{\vartheta_1\left(\frac{1}{2}+\frac{\tau}{2}\right)}.$$

Using the table (109) which gives the conversion formulae for theta-functions we have

$$\sqrt{e_1 - e_2} = \frac{1}{2\omega}\frac{\vartheta_1'}{\vartheta_3}\frac{\vartheta_4}{\vartheta_2},$$

$$\sqrt{e_1 - e_3} = \frac{1}{2\omega}\frac{\vartheta_1}{\vartheta_4}\frac{\vartheta_3}{\vartheta_2},$$

$$\sqrt{e_2 - e_3} = \frac{1}{2\omega}\frac{\vartheta_1\vartheta_2}{\vartheta_4\vartheta_3}.$$

We shall prove below the following important identity

$$\vartheta_1' = \pi\vartheta_2\vartheta_3\vartheta_4, \tag{116}$$

the use of which makes it possible to rewrite the above formulae in a very simple form:

$$\sqrt{e_1 - e_2} = \frac{\pi}{2\omega}\vartheta_4^2; \quad \sqrt{e_1 - e_3} = \frac{\pi}{2\omega}\vartheta_3^2; \quad \sqrt{e_2 - e_3} = \frac{\pi}{2\omega}\vartheta_2^2. \tag{117}$$

Let us now prove the identity (116). We have from (106)

$$\sqrt{\wp(2\omega v) - e_k} = \frac{1}{2\omega}\frac{\vartheta_1'}{\vartheta_{k+1}}\frac{\vartheta_{k+1}(v)}{\vartheta_1(v)},$$

whence, expanding the functions $\vartheta_1(v)$ and $\vartheta_{k+1}(v)$ into McLaurin's series and bearing in mind that $\vartheta_1(v)$ is odd while the remaining functions are even we, obtain

$$\sqrt{\wp(2\omega v) - e_k} = \frac{1}{2\omega}\frac{1 + \frac{\vartheta_{k+1}''}{\vartheta_{k+1}}\frac{v^2}{2} + \dots}{v + \frac{\vartheta_1'''}{\vartheta_1'}\frac{v^3}{6} + \dots}$$

or, taking the factor v out of the denominator and dividing one series by the other series

$$\sqrt{\wp(2wv) - e_k} = \frac{1}{2\omega v}\left[1 + \left(\frac{\vartheta_{k+1}''}{\vartheta_{k+1}} + \frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'}\right)\frac{v^2}{2} + \dots\right],$$

or

$$\wp(2wv) - e_k = \frac{1}{4\omega^2 v^2}\left[1 + \left(\frac{\vartheta_{k+1}''}{\vartheta_{k+1}} - \frac{1}{3}\frac{\vartheta_k'''}{\vartheta_1'}\right)\frac{v^2}{2}\right) + \dots\Big]^2.$$

As we know, the expansion of $\wp(u)$ near $u = 0$ contains no constant term and, consequently, squaring the bracket on the right-hand side, and collecting together all the constant terms we should obtain $(-e_k)$, and so

$$e_k = \frac{1}{4\omega^2}\left(\frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'} - \frac{\vartheta_{k+1}''}{\vartheta_{k+1}}\right). \tag{118}$$

This and (114) gives us the following relationship

$$\frac{\vartheta_1'''}{\vartheta_1'} = \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_4''}{\vartheta_4} . \tag{119}$$

In all these formulae the dashes above ϑ indicate partial differentiation with respect to v so that, for example, ϑ_1''' is $\partial^3\vartheta_1(v)/\partial v^3$, when $v = 0$. The equation (111) when $v = 0$ gives

$$\vartheta_k'' = 4\pi i \frac{\partial \vartheta_k}{\partial \tau} \quad (k = 2, 3, 4)$$

and, similarly, assuming in the equation (111), that $k = 1$, differentiating with respect to v and subsequently putting $v = 0$ we obtain

$$\vartheta_1''' = 4\pi i \frac{\partial \vartheta_1'}{\partial \tau} .$$

By using the last two relationships we can rewrite the formula (119) as follows

$$\frac{1}{\vartheta_1'}\frac{\partial \vartheta_1'}{\partial \tau} = \frac{1}{\vartheta_2}\frac{\partial \vartheta_2}{\partial \tau} + \frac{1}{\vartheta_3}\frac{\partial \vartheta_3}{\partial \tau} + \frac{1}{\vartheta_4}\frac{\partial \vartheta_4}{\partial \tau} .$$

Integrating with respect to τ we have

$$\vartheta_1' = C\vartheta_2\vartheta_3\vartheta_4,$$

where C is a constant independent of τ, i.e. of h. To determine this constant we substitute on both sides of the above identity the expansion (105) writing out only its first terms

$$2\pi(h^{\frac{1}{4}} - \ldots) = C\,(2h^{\frac{1}{4}} + \ldots)\,(1 + \ldots)\,(1 - \ldots).$$

Comparing the coefficients of terms containing $h^{1/4}$, we obtain $C = \pi$ which gives the identity (116).

179. The elliptic Jacobian functions. Instead of the elliptic Weierstrass function $\wp(u)$ other elliptic functions are frequently used which have an earlier origin historically and date back to Jacobi. Assume, as always, that τ is an arbitrary number in the upper half-plane, while ω and ω' are two numbers, the ratio of which is $\omega'/\omega = \tau$. By using these elements we can construct theta-functions. Let us

define three new functions which are the ratios of two integral functions, i.e. which are fractional functions:

$$\left.\begin{aligned} \operatorname{sn}(u) &= \frac{\sigma(u)}{\sigma_3(u)} = 2\omega \frac{\vartheta_4}{\vartheta_1'} \frac{\vartheta_1(v)}{\vartheta_4(v)} \\ \operatorname{cn}(u) &= \frac{\sigma_1(u)}{\sigma_3(u)} = \frac{\vartheta_4}{\vartheta_2} \frac{\vartheta_2(v)}{\vartheta_4(v)} \qquad \left(v = \frac{u}{2\omega}\right) \\ \operatorname{dn}(u) &= \frac{\sigma_2(u)}{\sigma^3(u)} = \frac{\vartheta_4}{\vartheta_3} \frac{\vartheta_3(v)}{\vartheta_4(v)}. \end{aligned}\right\} \tag{120}$$

According to the known formulae

$$\sqrt{\wp(u) - e_k} = \frac{\sigma_k(u)}{\sigma(u)},$$

these new functions are connected with the Weierstrass function $\wp(u)$ by the following three relationships:

$$\sqrt{\wp(u) - e_3} = \frac{1}{\operatorname{sn}(u)}; \quad \sqrt{\wp(u) - e_1} = \frac{\operatorname{cn}(u)}{\operatorname{sn}(u)}; \quad \sqrt{\wp(u) - e_2} = \frac{\operatorname{dn}(u)}{\operatorname{sn}(u)}. \tag{121}$$

Eliminating the function $\wp(u)$ from these relationships we obtain two relationships for the new functions:

$$\operatorname{cn}^2(u) + (e_1 - e_3)\operatorname{sn}^2(u) = 1; \quad \operatorname{dn}^2(u) + (e_2 - e_3)\operatorname{sn}^2(u) = 1. \tag{122}$$

The formulae (117) from the previous paragraph give us

$$e_1 - e_2 = \left(\frac{\pi}{2\omega}\right)^2 \vartheta_4^4; \quad e_1 - e_3 = \left(\frac{\pi}{2\omega}\right)^2 \vartheta_3^4; \quad e_2 - e_3 = \left(\frac{\pi}{2\omega}\right)^2 \vartheta_2^4. \tag{123}$$

Until now the complex numbers ω and ω' have remained fully arbitrary, the only essential condition being that the relationship $\omega'/\omega = \tau$ should lie in the upper half-plane. In the theory of the Weierstrass function these numbers are not subjected to any other limitations. In the theory of Jacobian functions the number ω for a given τ is determined by the condition that the difference $e_1 - e_3$ is unity. The second of the relationships (123) then gives ω:

$$\omega = \frac{\pi}{3}\vartheta_3^2 = \frac{\pi}{2}(1 + 2h + 2h^4 + 2h^9 + \ldots)^2 \quad (h = e^{i\pi\tau}), \tag{124}$$

which is fully defined by this formula for a given τ; ω' is determined subsequently from the formula $\omega' = \omega\tau$. Substituting the expression (124) into the relationship (123) we obtain:

$$e_1 - e_2 = \frac{\vartheta_4^4}{\vartheta_3^4}; \quad e_1 - e_3 = 1; \quad e_2 - e_3 = \frac{\vartheta_2^4}{\vartheta_3^4}, \tag{125}$$

where the right-hand sides depend only on τ. The relationships (122) can then be rewritten as follows:

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1; \quad \mathrm{dn}^2(u) + k^2\,\mathrm{sn}^2(u) = 1, \tag{126}$$

where it is assumed, for the sake of briefness, that

$$k^2 = \frac{\vartheta_2^4}{\vartheta_3^4}. \tag{127}$$

The Jacobian functions are constructed by using τ alone and therefore the following notation is sometimes used

$$\mathrm{sn}(u;\ \tau); \quad \mathrm{cn}(u;\ \tau); \quad \mathrm{dn}(u;\ \tau).$$

The number k given by formula (127) is known as the *modulus of the Jacobian function*. We shall also introduce the so-called additional modulus which is defined by the formula

$$k'^2 = \frac{\vartheta_4^4}{\vartheta_3^4}. \tag{128}$$

Adding the first and third of the relationships (125) we obtain

$$k^2 + k'^2 = 1. \tag{129}$$

The formulae (127) and (128) determine k^2 and k'^2 as the squares of certain single-valued functions of τ; by taking definite values of the radicals we can therefore write:

$$k = \frac{\vartheta_2^2}{\vartheta_3^2}; \quad k' = \frac{\vartheta_4^2}{\vartheta_3^2}. \tag{130}$$

Let us now return to the formulae (120). Factors on the right which are independent of v can be expressed in terms of k and k'. In fact, we have from (130)

$$\sqrt{k} = \frac{\vartheta_2}{\vartheta_3}; \quad \sqrt{k'} = \frac{\vartheta_4}{\vartheta_3}; \quad \sqrt{\frac{k'}{k}} = \frac{\vartheta_4}{\vartheta_2},$$

which, together with (124) and (116), gives

$$2\omega\,\frac{\vartheta_4}{\vartheta_1'} = \pi\vartheta_3^2\,\frac{\vartheta_4}{\vartheta_1'} = \frac{\vartheta_3}{\vartheta_2} = \frac{1}{\sqrt{k}},$$

and therefore the formulae (120) can be rewritten as follows:

$$\mathrm{sn}(u) = \frac{1}{\sqrt{k}}\,\frac{\vartheta_1(v)}{\vartheta_4(v)}; \quad \mathrm{cn}(u) = \sqrt{\frac{k'}{k}}\,\frac{\vartheta_2(v)}{\vartheta_4(v)}; \quad \mathrm{dn}(u) = \sqrt{k'}\,\frac{\vartheta_3(v)}{\vartheta_4(v)} \tag{131}$$

$$\left(v = \frac{u}{2\omega}\right).$$

180. The fundamental properties of Jacobian functions. The formulae (131) represent the Jacobian functions as the quotients of two integral functions. Using the fact that $\vartheta_1(v)$ is odd and that the remaining functions $\vartheta_k(v)$ are even, we can conclude that $\mathrm{sn}(u)$ *is an odd function and* $\mathrm{cn}(u)$ *and* $\mathrm{dn}(u)$ *are even functions.*

Also $\vartheta_1(0) = 0$ and we have

$$\left.\frac{\vartheta_1(v)}{u}\right|_{v=0} = \left.\frac{\vartheta_1(v)}{2\omega v}\right|_{v=0} = \frac{1}{2\omega}\vartheta_1',$$

and the formulae (120) give

$$\left.\frac{\mathrm{sn}(u)}{u}\right|_{u=0} = 1; \quad \mathrm{cn}(0) = \mathrm{dn}(0) = 1. \tag{132}$$

Let us now return to the table (109) which gives the conversion formulae for theta-functions. Bearing in mind the fact that the addition of 1/2 or $\tau/2$ to v is equivalent to the addition of ω or ω' to u, and using the fundamental relationships (131) we obtain the following table of conversion formulae for the Jacobian functions:

	$u+\omega$	$u+\omega'$	$u+\omega+\omega'$	$u+2\omega$	$u+2\omega'$	$u+2\omega+2\omega'$
sn	$\frac{\mathrm{cn}(u)}{\mathrm{dn}(u)}$	$\frac{1}{k}\frac{1}{\mathrm{sn}(u)}$	$\frac{1}{k}\frac{\mathrm{dn}(u)}{\mathrm{cn}(u)}$	$-\mathrm{sn}(u)$	$\mathrm{sn}(u)$	$-\mathrm{sn}(u)$
cn	$-k'\frac{\mathrm{sn}(u)}{\mathrm{dn}(u)}$	$-\frac{i}{k}\frac{\mathrm{dn}(u)}{\mathrm{sn}(u)}$	$-i\frac{k'}{k}\frac{1}{\mathrm{cn}(u)}$	$-\mathrm{cn}(u)$	$-\mathrm{cn}(u)$	$\mathrm{cn}(u)$
dn	$k'\frac{1}{\mathrm{dn}(u)}$	$-i\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)}$	$ik'\frac{\mathrm{sn}(u)}{\mathrm{cn}(u)}$	$\mathrm{dn}(u)$	$-\mathrm{dn}(u)$	$-\mathrm{dn}(u)$

(133)

The last three columns of this table show that the function $\mathrm{sn}(u)$ has periods 4ω and $2\omega'$, the function $\mathrm{cn}(u)$ has periods 4ω and $2\omega + 2\omega'$, and finally, the function $\mathrm{dn}(u)$ has periods 2ω and $4\omega'$.

The table (110) which gives the zeros of theta-functions leads us directly to a table giving the zeros and poles of Jacobian functions. Adding to this the periods shown above we obtain the following table:

	Zeros	Poles	Periods
$\mathrm{sn}(u)$	$2n\omega + 2n'\omega'$	$2n\omega + (2n'+1)\omega'$	4ω and $2\omega'$
$\mathrm{cn}(u)$	$(2n+1)\omega + 2n'\omega'$	$2n\omega + (2n'+1)\omega'$	4ω and $2\omega + 2\omega'$
$\mathrm{dn}(u)$	$(2n+1)\omega + (2n'+1)\omega'$	$2n\omega + (2n'+1)\omega'$	2ω and $4\omega'$

(134)

In Fig. 84 below, the parallelograms of periods for Jacobian functions are given. The circles denote the zeros and the crosses the poles of the corresponding function. Owing to the fact that both the theta-functions and the $\sigma(u)$ functions have simple zeros we can say that the Jacobian functions have simple poles. Below there are two poles in each parallelogram, i.e. all Jacobian functions are elliptic functions of the second order with simple poles.

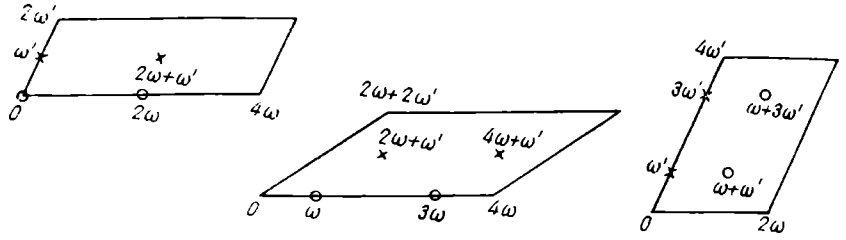

FIG. 84

This is directly due to the fact that all these functions can be obtained by converting certain elliptic integrals of the first kind to a polynomial of the fourth degree under the radical. We shall now try to explain this circumstance.

181. The differential equation for Jacobian functions. It follows directly from the formulae (113) and (121) that

$$\wp'(u) = \pm \frac{2\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{\mathrm{sn}^3(u)}.$$

To determine the sign on the right-hand side, we multiply both sides of the above equation by u^3 and then put $u = 0$. Bearing in mind the fact that the product $u^3\wp'(u)$ equals (-2) when $u = 0$, and using the formulae (132) we find that the minus sign must be taken on the right hand side of the above formula. This sign will remain unchanged during analytic continuation of the function, i.e. we have

$$\wp'(u) = -\frac{2\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{\mathrm{sn}^3(u)}.$$

On the other hand, differentiating the relationship

$$\wp(u) - e_3 = \frac{1}{\mathrm{sn}^2(u)},$$

we obtain

$$\wp'(u) = -\frac{2(\operatorname{sn}(u))'}{\operatorname{sn}^3(u)},$$

and comparing these two expressions for $\wp'(u)$ we obtain

$$[\operatorname{sn}(u)]' = \operatorname{cn}(u)\operatorname{dn}(u). \tag{135}$$

Differentiating the equations (126) and using (135) we obtain the derivatives of two other Jacobian functions:

$$(\operatorname{cn}(u))' = -\operatorname{sn}(u)\operatorname{dn}(u); \quad (\operatorname{dn}(u))' = -k^2\operatorname{sn}(u)\operatorname{cn}(u). \tag{136}$$

Squaring and using (126) we finally obtain the following differential equations for Jacobian functions:

$$\left.\begin{aligned}\left(\frac{\mathrm{d}\operatorname{sn}(u)}{\mathrm{d}u}\right)^2 &= (1-\operatorname{sn}^2(u))(1-k^2\operatorname{sn}^2(u)),\\ \left(\frac{\mathrm{d}\operatorname{cn}(u)}{\mathrm{d}u}\right)^2 &= (1-\operatorname{cn}^2(u))(k'^2+k^2\operatorname{cn}^2(u)),\\ \left(\frac{\mathrm{d}\operatorname{dn}(u)}{\mathrm{d}u}\right)^2 &= -(1-\operatorname{dn}^2(u))(k'^2-\operatorname{dn}^2(u)).\end{aligned}\right\} \tag{137}$$

Let us investigate the differential equation for the function $\operatorname{sn}(u)$ in greater detail. If we put $x = \operatorname{sn}(u)$ we can write

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \sqrt{(1-x^2)(1-k^2x^2)},$$

where it must be assumed that $x = 0$, when $u = 0$, and also that the radical on the right is equal to unity since $\operatorname{sn}'(0) = 1$ from (132). Separating the variables and integrating we obtain:

$$u = \int_0^x \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{138}$$

This shows that *the function* $\operatorname{sn}(u)$ *is obtained as a result of the conversion of an elliptic integral of the first kind in the Legendre form.* It can be shown, conversely, that by taking arbitrary complex values other than 0 and 1 for the number k^2, we obtain the Jacobian function $\operatorname{sn}(u)$ as a result of the conversion of the integral (138). Hence together with τ the number k can serve as an element in the construction of the Jacobian function. We investigated the integral (138) in detail from the point of view of conformal transformation in the particular case when k^2 is real and lies between zero and unity. We than had one real period, which in [167] we denoted by $4K$ and

another purely imaginary period $2iK'$. Comparing this with our new notation we obtain:

$$K = \omega = \frac{\pi}{2}\vartheta_3^2; \quad iK' = \omega' = \omega\tau = \frac{\pi}{2}\vartheta_3^2\tau.$$

182. Addition formulae. Let us consider three functions of the variable u: $\varphi_1(u) = \operatorname{sn}(u)\ \operatorname{sn}(u+v)$; $\varphi_2(u) = \operatorname{cn}(u)\ \operatorname{cn}(u+v)$; $\varphi_3(u) = \operatorname{dn}(u)\operatorname{dn}(u+v)$ where v is a fixed arbitrary number. By using the table (133) it can readily be seen that all these functions have periods 2ω and $2\omega'$. The function $\varphi_1(u)$ has simple poles at the points where $\operatorname{sn}(u)$ or $\operatorname{sn}(u+v)$ have poles. Using the table (134) we can see that these points differ from ω' or $-v+\omega'$ by a period, i.e. they differ by an expression of the form $n2\omega + n'2\omega'$, where n and n' are arbitrary integers. In the fundamental parallelogram of periods constructed on the vectors 2ω and $2\omega'$ there will, consequently, be only two such points. The same result will also be obtained for the remaining functions $\varphi_k(u)$, i.e. all these functions are elliptic functions of the second order with periods 2ω and $2\omega'$ and have two simple poles in the parallelogram of periods, one of which is equal to ω'. The constants A and B can be so chosen that the two functions

$$\varphi_2(u) + A\varphi_1(u) \quad \text{and} \quad \varphi_3(u) + B\varphi_1(u) \tag{139}$$

have no pole at $u = \omega'$. For this choice of constants the functions (139) will have only one pole of the first order in the parallelogram of periods, and since no elliptic functions of the first order exist [168] it follows that these functions are simply constants. We can therefore say that with a suitable choice of the constants A and B the following relationships apply:

$$\left.\begin{aligned} \operatorname{cn}(u)\operatorname{cn}(u+v) + A\operatorname{sn}(u)\operatorname{sn}(u+v) &= A_1 \\ \operatorname{dn}(u)\operatorname{dn}(u+v) + B\operatorname{sn}(u)\operatorname{sn}(u+v) &= B_1. \end{aligned}\right\} \tag{140}$$

The constants A, B, A_1 and B_1 are constants with respect to the argument u but their value depends on the choice of v. Let us determine these constants. On putting $u = 0$ in the formulae (140) we obtain:

$$A_1 = \operatorname{cn}(v); \quad B_1 = \operatorname{dn}(v).$$

Differentiating the relationships (140) and then putting $u = 0$ we obtain from (135), (136) and (132):

$$(\operatorname{cn}(v))' + A\operatorname{sn}(v) = 0; \quad (\operatorname{dn}(v))' + B\operatorname{sn}(v) = 0$$

and again, from (136)

$$A = \mathrm{dn}\,(v), \quad B = k^2\,\mathrm{cn}\,(v).$$

Substituting the constants in (140) we finally obtain the following two relationships:

$$\left.\begin{aligned} \mathrm{cn}\,(u)\,\mathrm{cn}\,(u+v) + \mathrm{dn}\,(v)\,\mathrm{sn}\,(u)\,\mathrm{sn}\,(u+v) &= \mathrm{cn}\,(v), \\ \mathrm{dn}\,(u)\,\mathrm{dn}\,(u+v) + k^2\,\mathrm{cn}\,(v)\,\mathrm{sn}\,(u)\,\mathrm{sn}\,(u+v) &= \mathrm{dn}\,(v), \end{aligned}\right\} \tag{141}$$

which can be regarded as identities with respect to u and v. Replacing u by $(-u)$ and v by $v+u$ we obtain

$$\begin{aligned} \mathrm{cn}\,(u)\,\mathrm{cn}\,(v) - \mathrm{dn}\,(u+v)\,\mathrm{sn}\,(u)\,\mathrm{sn}\,(v) &= \mathrm{cn}\,(u+v), \\ \mathrm{dn}\,(u)\,\mathrm{dn}\,(v) - k^2\,\mathrm{cn}\,(u+v)\,\mathrm{sn}\,(u)\,\mathrm{sn}\,(v) &= \mathrm{dn}\,(u+v). \end{aligned}$$

The last two formulae enable us to find $\mathrm{cn}(u+v)$ and $\mathrm{dn}(u+v)$ and on substitution in the first of the equations (141) we obtain $\mathrm{sn}(u+v)$. We thus arrive at the following *addition formulae* which express the Jacobian functions of the sum of two arguments in terms of Jacobian functions of the individual arguments

$$\left.\begin{aligned} \mathrm{sn}\,(u+v) &= \frac{\mathrm{sn}\,(u)\,\mathrm{cn}\,(v)\,\mathrm{dn}\,(v) + \mathrm{sn}\,(v)\,\mathrm{cn}\,(u)\,\mathrm{dn}\,(u)}{1 - k^2\,\mathrm{sn}^2\,(u)\,\mathrm{sn}^2\,(v)}, \\ \mathrm{cn}\,(u+v) &= \frac{\mathrm{cn}\,(u)\,\mathrm{cn}\,(v) - \mathrm{sn}\,(u)\,\mathrm{dn}\,(u)\,\mathrm{sn}\,(v)\,\mathrm{dn}\,(v)}{1 - k^2\,\mathrm{sn}^2\,(u)\,\mathrm{sn}^2\,(v)}, \\ \mathrm{dn}\,(u+v) &= \frac{\mathrm{dn}\,(u)\,\mathrm{dn}\,(v) - k^2\,\mathrm{sn}\,(u)\,\mathrm{cn}\,(u)\,\mathrm{sn}\,(v)\,\mathrm{cn}\,(v)}{1 - k^2\,\mathrm{sn}^2\,(u)\,\mathrm{sn}^2\,(v)}. \end{aligned}\right\} \tag{142}$$

The first two formulae resemble the addition formulae for ordinary trigonometric functions: the sines and the cosines. These latter functions do, in fact, appear to be special cases of Jacobian functions when $k = 0$. Thus if we put $k = 0$ in the integral (138) then its conversion gives $x = \sin u$; it follows from (126) and (132) that $\mathrm{cn}(u)$ becomes $\cos u$. Finally, the second of the formulae (126) shows that the function $\mathrm{dn}(u)$ simply becomes unity when $k = 0$ and therefore it has no analogous function among the trigonometric functions.

183. The connection between the functions $\wp(u)$ and $\mathrm{sn}(u)$. We shall now establish a direct connection between the functions $\wp(u)$ and $\mathrm{sn}(u)$. Let us investigate the function $\wp(u)$ with periods 2ω and $2\omega'$. Consider the number $\omega'/\omega = \tau$ in the upper half-plane and construct theta-functions and the function $\mathrm{sn}(u)$ by using this number and the first of the formulae (130) and (131). The numbers 2ω and $2\omega'$ which are

connected with the above Weierstrass function, will not, in general, satisfy the condition $e_1 - e_3 = 1$. According to the formulae (117) quoted above we have the following relationships:

$$\left.\begin{aligned} e_1 - e_2 = \left(\frac{\pi}{2\omega}\right)^2 \vartheta_4^4;\quad e_1 - e_3 = \left(\frac{\pi}{2\omega}\right)^2 \vartheta_3^4;\quad e_2 - e_3 = \left(\frac{\pi}{2\omega}\right)^2 \vartheta_2^4; \\ k^2 = \frac{\vartheta_2^4}{\vartheta_3^4} = \frac{e_2 - e_3}{e_1 - e_3}\,. \end{aligned}\right\} \tag{143}$$

We have new numbers $2\tilde{\omega}$ and $2\tilde{\omega}'$ for the function $\mathrm{sn}(u)$ instead of 2ω and $2\omega'$ which, as we know, are determined by the conditions:

$$2\tilde{\omega} = \pi\vartheta_3^2;\quad 2\tilde{\omega}' = 2\tilde{\omega}\tau. \tag{144}$$

Denoting by λ the relationship $\lambda = \tilde{\omega}/\omega = \tilde{\omega}'/\omega'$, we consider the function

$$f(u) = \frac{\lambda^2}{\mathrm{sn}^2(\lambda u)}\,.$$

We have $\lambda 2\omega = 2\tilde{\omega}$ and $\lambda 2\omega' = 2\tilde{\omega}'$ and, according to the table (133), the function $f(u)$ has periods 2ω and $2\omega'$. We see from the table (134) that the function $f(u)$ has poles at $n2\omega + n'2\omega'$, where n and n' are arbitrary integers.

Hence the function $f(u)$, like the function $\wp(u)$, has periods 2ω and $2\omega'$; it also has in the fundamental parallelogram of periods a single pole of the second order at $u = 0$. We shall show that the infinite part of the function $f(u)$ at this pole will be the same as of that the function $\wp(u)$ and will be equal to $1/u^2$. In fact, remembering that the function $\mathrm{sn}(u)$ is odd and also (132) we have the following expansion near the point $u = 0$:

$$\mathrm{sn}(u) = u + c_3u^3 + c_5u^5 + \ldots,$$

whence:

$$\frac{1}{\mathrm{sn}^2(u)} = \frac{1}{u^2}\cdot\frac{1}{(1 + c_3u^2 + c_5u^4 + \ldots)^2} = \frac{1}{u^2} + d_0 + d_2u^2 + \ldots,$$

or we have near $u = 0$:

$$f(u) = \frac{1}{\mathrm{sn}^2(\lambda u)} = \frac{1}{u^2} + \lambda^2 d_0 + \lambda^4 d_2\, u^2 + \ldots,$$

which we wanted to prove. Hence the functions $f(u)$ and $\wp(u)$ have in their common parallelogram of periods the same poles with equal infinite parts; it therefore follows that these functions differ only by their constant terms, i.e.

$$\wp(u) = \frac{\lambda^2}{\mathrm{sn}^2(\lambda u)} + C. \tag{145}$$

To determine the constant C we put $u = \omega$. We have $\wp(\omega) = e_1$ and from the table (133)

$$\operatorname{sn}(\lambda\omega) = \operatorname{sn}(\tilde{\omega}) = \frac{\operatorname{cn}(0)}{\operatorname{dn}(0)} = 1,$$

and formula (145) gives:

$$C = e_1 - \lambda^2. \tag{146}$$

According to the formulae (143) and (144) we can write

$$2\tilde{\omega} = 2\omega\sqrt{e_1 - e_3}; \quad 2\tilde{\omega}' = 2\omega'\sqrt{e_1 - e_3},$$

i.e.

$$\lambda = \frac{\tilde{\omega}}{\omega} = \frac{\tilde{\omega}'}{\omega'} = \sqrt{e_1 - e_3},$$

whence, by using (146) we have $C = e_3$.

Using the equations (143) and (114) we can write the constant C as follows:

$$C = -\frac{(1 + k^2)\lambda^2}{3}.$$

Hence we finally obtain the following connection between the functions $\wp(u)$ and $\operatorname{sn}(u)$:

$$\wp(u) = \frac{e_1 - e_3}{\operatorname{sn}^2(\sqrt{e_1 - e_3}u)} + e_3 \tag{147}$$

or

$$\wp(u) = \frac{\lambda^2}{\operatorname{sn}^2(\lambda u)} - \frac{(1 + k^2)\lambda^2}{3} \quad (\lambda = \sqrt{e_1 - e_3}). \tag{148}$$

184. Elliptic coordinates. Elliptic functions are frequently used, particularly in mechanics. Here we shall only deal with fundamental and very simple applications of these functions. One application is their use in finding elliptic coordinates in space. We have already met elliptic coordinates earlier [II, 137]. Here we shall repeat what we know already and give some additional properties of these functions. We shall somewhat modify our earlier notation, viz. we shall replace the numbers a^2, b^2 and c^2 by $-a^2$, $-b^2$ and $-c^2$. Let us write the equation

$$\frac{x^2}{\varrho - a^2} + \frac{y^2}{\varrho - b^2} + \frac{z^2}{\varrho - c^2} - 1 = 0. \tag{149}$$

This is an equation of the third degree with respect to ϱ. At any given point with cartesian coordin ates(x, y, z) the equation (149) has three real zeros λ, μ and ν, which satisfy the inequality

$$\lambda > a^2 > \mu > b^2 > \nu > c^2, \tag{150}$$

and these three numbers are known as the *elliptic coordinates of the given point*. So as not to affect the sign of the equality we assume that x, y and z are not

zero and that they are positive, say. If, in the equation (149) we put $\varrho = \lambda$, we obtain an ellipsoid which passes through the given point; when $\varrho = \mu$ this will be a one-sided hyperboloid and when $\delta = \nu$ a two sided hyperboloid. We have seen earlier that the coordinate surfaces $\lambda = \text{const}$ $\mu = \text{const}$ and $\nu = \text{const}$ are mutually orthogonal, i.e. elliptic coordinates are orthogonal coordinates. We shall introduce formulae which give the cartesian coordinates in terms of elliptic coordinates. Bringing the left-hand side of the equation (149) to a common denominator and remembering that the numerator is a polynomial of the third degree of ϱ with zeros λ, μ and ν, the first coefficient of which is (-1), we can write an identity for ϱ:

$$\frac{x^2}{\varrho - a^2} + \frac{y^2}{\varrho - b^2} + \frac{z^2}{\varrho - c^2} - 1 = \frac{-(\varrho - \lambda)(\varrho - \mu)(\varrho - \nu)}{(\varrho - a^2)(\varrho - b^2)(\varrho - c^2)}. \tag{151}$$

Multiplying by $(\varrho - a^2)$ and then putting $\varrho = a^2$ we obtain an expression for x^2 and analogous expressions for y^2 and z^2:

$$\left.\begin{aligned} x^2 &= \frac{(\lambda - a^2)(\mu - a^2)(\nu - a^2)}{(a^2 - b^2)(a^2 - c^2)}, \\ y^2 &= \frac{(\lambda - b^2)(\mu - b^2)(\nu - b^2)}{(b^2 - c^2)(b^2 - a^2)}, \\ z^2 &= \frac{(\lambda - c^2)(\mu - c^2)(\nu - c^2)}{(c^2 - a^2)(c^2 - b^2)}. \end{aligned}\right\} \tag{152}$$

We shall now deduce a formula for the square of an element of arc in elliptic coordinates. Taking logarithms and differentiating the formulae (152), we obtain:

$$\begin{aligned} 2\frac{dx}{x} &= \frac{d\lambda}{\lambda - a^2} + \frac{d\mu}{\mu - a^2} + \frac{d\nu}{\nu - a^2}, \\ 2\frac{dy}{y} &= \frac{d\lambda}{\lambda - b^2} + \frac{d\mu}{\mu - b^2} + \frac{d\nu}{\nu - b^2}, \\ 2\frac{dz}{z} &= \frac{d\lambda}{\lambda - c^2} + \frac{d\mu}{\mu - c^2} + \frac{d\nu}{\nu - c^2}. \end{aligned}$$

Multiplying by x, y, z, squaring each term separately and adding we have

$$ds^2 = L^2 d\lambda^2 + M^2 d\mu^2 + N^2 d\nu^2. \tag{153}$$

where, for example,

$$4L^2 = \frac{x^2}{(\lambda - a^2)^2} + \frac{y^2}{(\lambda - b^2)^2} + \frac{z^2}{(\lambda - c^2)^2}. \tag{154}$$

Notice that the right hand side of formula (153) does not contain products $d\lambda\, d\mu$ etc. since elliptic coordinates are orthogonal [II, 130]. We can obtain the right-hand side of the formula (154) by differentiating the left-hand side of the identity (151) with respect to ϱ, changing the sign and then putting $\varrho = \lambda$, i.e.

$$4L^2 = \frac{d}{d\varrho} \frac{(\varrho - \lambda)(\varrho - \mu)(\varrho - \nu)}{(\varrho - a^2)(\varrho - b^2)(\varrho - c^2)}\Bigg|_{\varrho = \lambda}.$$

We can therefore write the following formula for ds^2:

$$4ds^2 = \frac{(\lambda-\mu)(\lambda-\nu)}{(\lambda-a^2)(\lambda-b^2)(\lambda-c^2)}\,d\lambda^2 + \frac{(\mu-\lambda)(\mu-\nu)}{(\mu-a^2)(\mu-b^2)(\mu-c^2)}\,d\mu^2 + \\ + \frac{(\nu-\lambda)(\nu-\mu)}{(\nu-a^2)(\nu-b^2)(\nu-c^2)}\,d\nu^2. \qquad (155)$$

Knowing the expression for an element of length we can write the Laplace equation in elliptic coordinates [II, 119]. For the sake of simplicity we introduce the following notation:

$$f(\varrho) = (\varrho-a^2)(\varrho-b^2)(\varrho-c^2).$$

In the notation of [II, 119] we had:

$$2H_1 = \sqrt{\frac{(\lambda-\mu)(\lambda-\nu)}{f(\lambda)}};\quad 2H_2 = \sqrt{\frac{(\mu-\lambda)(\mu-\nu)}{f(\mu)}};\quad 2H_3 = \sqrt{\frac{(\nu-\lambda)(\nu-\mu)}{f(\nu)}},$$

where H_k must be positive; it must also be remembered that $f(\lambda)$ and $f(\nu)$ are positive and $f(\mu) < 0$. The Laplace equation in elliptic coordinates is as follows:

$$\frac{\nu-\mu}{\sqrt{f(\mu)f(\nu)}}\frac{\partial}{\partial\lambda}\left(\sqrt{f(\lambda)}\frac{\partial U}{\partial\lambda}\right) + \frac{\lambda-\nu}{\sqrt{f(\nu)f(\lambda)}}\frac{\partial}{\partial\mu}\left(\sqrt{f(\mu)}\frac{\partial U}{\partial\mu}\right) + \\ + \frac{\mu-\lambda}{\sqrt{f(\lambda)f(\mu)}}\frac{\partial}{\partial\nu}\left(\sqrt{f(\nu)}\frac{\partial U}{\partial\nu}\right) = 0, \qquad (156)$$

where the last two terms are obtained from the first as a result of the cyclic rearrangement of the letters λ, μ and ν.

185. The introduction of elliptic functions. Let us replace the variables λ, μ and ν by new variables α, β and γ, according to the formulae

$$\frac{d\lambda}{\sqrt{f(\lambda)}} = d\alpha; \qquad \frac{d\mu}{\sqrt{f(\mu)}} = d\beta; \qquad \frac{d\nu}{\sqrt{f(\nu)}} = d\gamma, \qquad (157)$$

i.e. α, β and γ are expressed by elliptic integrals of the first kind in terms of λ, μ and ν and, conversely, the latter are elliptic functions of the former. Thus we have, for example, from (157)

$$\sqrt{f(\lambda)}\frac{\partial}{\partial\lambda} = = \frac{\partial}{\partial\alpha},$$

and we can rewrite the equation (156) as follows:

$$(\nu-\mu)\frac{\partial^2 U}{\partial\alpha^2} + (\lambda-\nu)\frac{\partial^2 U}{\partial\beta^2} + (\mu-\lambda)\frac{\partial^2 U}{\partial\gamma^2} = 0. \qquad (158)$$

Let us now turn to the formulae (152) and show that x, y and z are single-valued functions of the new variables α, β and γ. In fact, consider the radical

$$\sqrt{f(\varrho)} = \sqrt{(\varrho-a^2)(\varrho-b^2)(\varrho-c^2)},$$

which enters the expressions (157). Replace ϱ by a new variable t according to the formula

$$\varrho = p + qt,$$

where p and q are constants. We obtain

$$(\varrho - a^2)(\varrho - b^2)(\varrho - c^2) = q^3 (t - e_1)(t - e_2)(t - e_3),$$

where e_k are the zeros of the polynomial with respect to t so that we have:

$$a^2 = p + qe_1; \quad b^2 = p + qe_2; \quad c^2 = p + qe_3.$$

Let us select the constant p so that the sum of the e_k should be zero, i.e.

$$e_1 + e_2 + e_3 = 0,$$

hence

$$p = \frac{a^2 + b^2 + c^2}{3}.$$

The preceeding formulae will determine the numbers e_k accurately except for q which we assume to be positive and denote by s^2. We therefore have:

$$\left.\begin{aligned} s^2 e_1 &= a^2 - \frac{a^2 + b^2 + c^2}{3}, \\ s^2 e_2 &= b^2 - \frac{a^2 + b^2 + c^2}{3}, \\ s^2 e_3 &= c^2 - \frac{a^2 + b^2 + c^2}{3}. \end{aligned}\right\} \tag{159}$$

It also follows that:

$$a^2 - b^2 = s^2 (e_1 - e_2); \quad a^2 - c^2 = s^2 (e_1 - e_3); \quad b^2 - c^2 = s^2 (e_2 - e_3). \tag{160}$$

Substituting $\varrho = p + qt$ we have:

$$\varrho - a^2 = s^2 (t - e_1); \quad \varrho - b^2 = s^2 (t - e_2); \quad \varrho - c^2 = s^2 (t - e_3).$$

The polynomial $f(\varrho)$ with the new variable can be written in the form

$$f(\varrho) = s^6 (t - e_1)(t - e_2)(t - e_3).$$

On putting

$$\lambda = \frac{a^2 + b^2 + c^2}{3} + s^2 t, \tag{161}$$

we can rewrite the first of these formulae in the form:

$$\frac{2}{s} \int_{\infty}^{t} \frac{\mathrm{d}t}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}} = a,$$

where the arbitrary constant on the right is omitted since it is of no significance.

Assuming, for the sake of simplicity, that $s = 2$ we obtain $t = \wp(a)$ as a result of the conversion of the integral since the polynomial under the radical has the form given in [178] owing to the fact that $e_1 + e_2 + e_3 = 0$.

Formula (161) gives

$$\lambda = \frac{a^2 + b^2 + c^2}{3} + 4\wp(a), \tag{162}$$

and we obtain similarly:

$$\mu = \frac{a^2+b^2+c^2}{3} + 4\wp(\beta); \quad \nu = \frac{a^2+b^2+c^2}{3} + 4\wp(\gamma). \tag{163}$$

Substituting these expressions in the formulae (152) and taking into account (159), (162) and (163) we have:

$$\left.\begin{aligned} x^2 &= 4\frac{(\wp(\alpha)-e_1)(\wp(\beta)-e_1)(\wp(\gamma)-e_1)}{(e_1-e_2)(e_1-e_3)}, \\ y^2 &= 4\frac{(\wp(\alpha)-e_2)(\wp(\beta)-e_2)(\wp(\gamma)-e_2)}{(e_2-e_3)(e_2-e_1)}, \\ z^2 &= 4\frac{(\wp(\alpha)-e_3)(\wp(\beta)-e_3)(\wp(\gamma)-e_3)}{(e_3-e_1)(e_3-e_2)}. \end{aligned}\right\} \tag{164}$$

As we know from [172] all the differences in the numerators are squares of single-valued functions of α, β, γ, so that, in fact, the above formulae give x, y and z as analytic functions of α, β and γ. According to (162) and (163) the Laplace equation (158) with the new variables will be

$$(\wp(\gamma)-\wp(\beta))\frac{\partial^2 U}{\partial\alpha^2} + (\wp(\alpha)-\wp(\gamma))\frac{\partial^2 U}{\partial\beta^2} + (\wp(\beta)-\wp(\alpha))\frac{\partial^2 U}{\partial\gamma^2} = 0. \tag{165}$$

186. The Lamé equation. Let us separate the variables in the Laplace equation and find its solution in the form of a product of three functions, one of which depends only on α, the second only on β and third only on γ:

$$U = A(\alpha)\,B(\beta)\,C(\gamma). \tag{166}$$

Substituting in the equation (165) and dividing by $A(\alpha)\,B(\beta)\,C(\gamma)$, we have

$$(\wp(\gamma)-\wp(\beta))\frac{A''(\alpha)}{A(\alpha)} + (\wp(\alpha)-\wp(\gamma))\frac{B''(\beta)}{B(\beta)} + (\wp(\beta)-\wp(\alpha))\frac{C''(\gamma)}{C(\gamma)} = 0.$$

This equation will be satisfied if it is assumed that the factors in the expression (166) are solutions of an equation of one and the same form viz.:

$$\frac{A''(\alpha)}{A(\alpha)} = -a\wp(\alpha) - b; \quad \frac{B''(\beta)}{B(\beta)} = -a\wp(\beta) - b; \quad \frac{C''(\gamma)}{C(\gamma)} = -a\wp(\gamma) - b,$$

where a and b are constants. We thus obtain an equation of the second order with a coefficient of dual periodicity

$$\frac{\mathrm{d}^2 R(u)}{\mathrm{d}u^2} + [a\wp(u) + b]\,R(u) = 0. \tag{167}$$

Let us determine, first of all, the constant a so that the general solution of the equation (167) is a single-valued function of u. The coefficient $a\wp(u)+b$ can be expanded near the point $u = 0$ as follows:

$$\frac{a}{u^2} + b + \dots,$$

and therefore the determining equation at this regular point will be

$$\varrho(\varrho-1) + a = 0. \tag{168}$$

If the integral is to be single-valued the zeros of this equation must be integers. The sum of the zeros is equal to $+1$ and therefore the equation (168) must have zeros at $-n$ and $n+1$, where n is a positive integer or zero. Hence the constant a can have the following possible values:

$$a_n = -n(n+1) \qquad (n = 0, 1, 2, \ldots). \tag{169}$$

Strictly speaking we have only shown above that the equation (169) gives the necessary condition for the general solution to be single-valued. We shall now show that this is also the sufficient condition. It follows from general theory that one of the solutions of the equation (167) when $a = -n(n+1)$ can be expanded as follows near the origin:

$$R(u) = u^{n+1}(c_0 + c_1 u + c_2 u^2 + \ldots) \qquad (c_0 \neq 0). \tag{170}$$

The equation (167) remains unaltered if u is replaced by $(-u)$; hence if we make the same substitution in formula (170) we should also obtain a solution which only differs by its constant term from the solution (170) since the second solution, linearly-independent with (170), has a completely different form near $u = 0$. It follows from these considerations that the power series in formula (170) contains only even powers of u, i.e.

$$R_1(u) = u^{n+1}(c_0 + c_2 u^2 + c_4 u^4 + \ldots) \qquad (c_0 \neq 0). \tag{171}$$

The second solution of the equation (167), as we know, can be obtained from the formula [II, 24]

$$R_2(u) = R_1(u) \int \frac{du}{R_1^2(u)},$$

or

$$R_2(u) = R_1(u) \int \frac{1}{u^{2n+2}} (c_0 + c_2 u^2 + c_4 u^4 + \ldots)^{-2} \, du.$$

The integrand can be expanded near the point $u = 0$ into a series containing only even powers of u and therefore the term in u^{-1} will be absent while the second solution of $R_2(u)$ will not contain $\log u$. We can thus see that both solutions will be single-valued near $u = 0$. The above arguments can be repeated word for word for every singularity of the equation (167). Its singularities lie at the points $u = m_1 \omega_1 + m_2 \omega_2$, where ω_1 and ω_2 are the periods of $\wp(u)$ and m_1 and m_2 are arbitrary integers. Hence any solution of the equation (167) can only have poles at the singularities and therefore it must, in fact, be a single-valued function of u.

Substituting the value of the constant (169) in the equation (167) we obtain the equation

$$\frac{d^2R(u)}{du^2} + [-n(n+1)\wp(u) + b] R(u) = 0, \tag{172}$$

which is generally known as the *Lamé equation*. The constant b is determined by the condition that the equation (172) should have a solution in the form of a polynomial in $\wp(u)$ or in the form of a product of this polynomial and a factor

of the following form

$$\sqrt{\wp(u) - e_1}; \quad \sqrt{\wp(u) - e_2}; \quad \sqrt{\wp(u) - e_3},$$

and there can be one, two or three additional factors. It is apparent that the constant b can have $(2n + 1)$ values which satisfy this condition. If $R_0(u)$ is a solution of the equation (172) of the form mentioned above then the product

$$R_0(\alpha)\, R_0(\beta)\, R_0(\gamma),$$

which is a solution of the Laplace equation, is a polynomial of the nth degree in the x, y, z coordinates. For a given n there will be $(2n + 1)$ such solutions, as explained above, and they are generally known as Lamé functions. These polynomials are obviously directly connected with spherical functions with which we dealt earlier.

187. The simple pendulum. We shall consider a simple pendulum as one of the simplest applications of Jacobian functions. Assume that a heavy material point of unit mass moves round a smooth circle. Let the coordinate axes

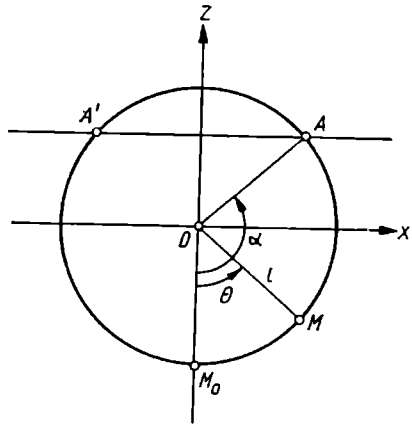

FIG. 85

X and Z be in the plane of this circle and let the Z axis be directed vertically upwards, let l be the radius of the circle. Suppose that when $t = 0$ our point is set free from the lowest point $M_0(z = -l)$ with an initial velocity v_0. The increment in kinetic energy is equal the work done by the force of gravity and we thus obtain the formula

$$\frac{1}{2} v^2 - \frac{1}{2} v_0^2 = -gz - gl,$$

or

$$v^2 = 2g(a - z) \qquad \left(a = -l + \frac{v_0^2}{2g}\right). \tag{173}$$

Suppose that the line $z = a$ intersects our circle at the points A and A', i.e. $a < l$ or $v_0 < 2\sqrt{lg}$. It follows from formula (173) that $z < a$, and therefore movement takes place along the arc AM_0A' (Fig. 85) of our circle. We have

$z = -l\cos\theta$; let us now introduce an angle α such that $a = -l\cos\alpha$ $(0 < \alpha < \pi)$. The velocity is then given by the formula

$$v = \frac{ds}{dt} = l\left|\frac{d\theta}{dt}\right|,$$

and therefore the equation (173) can be rewritten in the form

$$l^2\left(\frac{d\theta}{dt}\right)^2 = 2gl\ (\cos\theta - \cos\alpha)$$

or introducing half-angles

$$l\left(\frac{d\theta}{dt}\right)^2 = 4g\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right),$$

hence

$$2\sqrt{\frac{g}{l}}\,dt = \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}. \tag{174}$$

We are supposing that θ increases with t. Replace θ by the new variable τ according to the formula

$$\sin\frac{\theta}{2} = \tau\sin\frac{\alpha}{2}.$$

Differentiating this relationship we readily obtain

$$d\theta = \frac{2\sin\frac{\alpha}{2}\,d\tau}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\alpha}{2}\,d\tau}{\sqrt{1 - \sin^2\frac{\theta}{2}}}, \quad \text{i.e, } d\theta = \frac{2\sin\frac{\alpha}{2}\,d\tau}{\sqrt{1 - \sin^2\frac{\alpha}{2}\tau^2}},$$

and therefore, substituting in (174) and bearing in mind that, when $t = 0$, we have $\theta = \tau = 0$:

$$\sqrt{\frac{g}{l}}\,t = \int_0^\tau \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} \quad \left(k^2 = \sin^2\frac{\alpha}{2}\right),$$

hence

$$\tau = \mathrm{sn}\left(\sqrt{\frac{g}{l}}\,t\right), \tag{175}$$

and using the known property of Jacobian functions we obtain:

$$\left.\begin{aligned}\sin\frac{\theta}{2} &= \sin\frac{\alpha}{2}\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}\,t\right) = k\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}\,t\right),\\ \cos\frac{\theta}{2} &= \sqrt{1 - k^2\,\mathrm{sn}^2\left(\sqrt{\frac{g}{l}}\,t\right)} = \mathrm{dn}\left(\sqrt{\frac{g}{l}}\,t\right),\end{aligned}\right\} \tag{176}$$

and to extract the zero we take into account that when $t = 0$, $\theta = 0$. The last formulae make it possible to express the coordinates x and z in terms of single-valued functions of t.

Let us consider now the case when the constant a in formula (173) is greater than l. We can rewrite this formula as follows:

$$l^2\left(\frac{d\theta}{dt}\right)^2 = 2g(a + l\cos\theta) = 2g\left(a + l - 2l\sin^2\frac{\theta}{2}\right),$$

or

$$l^2\left(\frac{d\theta}{dt}\right)^2 = 2g(a+l)\left(1 - k^2\sin^2\frac{\theta}{2}\right), \tag{177}$$

where

$$k^2 = \frac{2l}{a+l}, \tag{178}$$

and, obviously, $k^2 < 1$. Integrating the relationship (177) we obtain:

$$\lambda t = \int_0^\theta \frac{d\theta}{\sqrt{1 - k^2\sin^2\frac{\theta}{2}}}, \quad \text{where } \lambda = \frac{\sqrt{2g(a+l)}}{l}.$$

Replacing θ by the new variable $\tau = \sin\theta/2$ we have:

$$\lambda t = \int_0^\tau \frac{2d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}}, \qquad \text{whence} \qquad \tau = \sin\frac{\theta}{2} = \operatorname{sn}\left(\frac{1}{2}\lambda t\right),$$

and similarly

$$\cos\frac{\theta}{2} = \sqrt{1 - \operatorname{sn}^2\left(\frac{1}{2}\lambda t\right)} = \operatorname{cn}\left(\frac{1}{2}\lambda t\right).$$

These formulae make it again possible to express the coordinates as single-valued functions of time.

188. An example of conformal transformation. As we saw above, when $0 < k < 1$ the function

$$u = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \tag{179}$$

transforms the upper half-plane z into a rectangle in the u-plane and, consequently, the reciprocal function $z = \operatorname{sn}(u; k)$ transforms a rectangle into a plane. The lengths of the sides of the rectangle are determined by the integrals [167]:

$$2\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}},$$

where $k^2 + k'^2 = 1$. In this way a rectangle with sides of any length can be obtained. By adding the constant $1/\lambda$ to the right-hand side of formula (179), a rectangle with arbitrary sides can be obtained, and such a rectangle will be transformed into a half-plane by the function $z = \operatorname{sn}(\lambda u; k)$. We shall now show that the function which transforms the rectangle into the circle can be simply expressed in terms of the Weierstrass function $\sigma(u)$. Take a rectangle K_1 in the plane, the apexes of which have the following coordinates: (0, 0), (0, a),

(a, b) and $(0, b)$. Let $z = f(u)$ be a function which transforms K_1 into a unit circle; a point (ξ, η) in K_1 becomes the centre of the circle. If we analytically continue $f(u)$ across the side which connects the apexes $(0, 0)$ and $(0, a)$ then as a result of the principle of symmetry, $f(u)$ transforms the rectangle K_2, symmetrical with K_1 with respect to the above side, into the outside of the unit circle, i.e. into the domain $|z| > 1$, while the point $(\xi, -\eta)$, symmetrical with the point (ξ, η), becomes the point at infinity. Bearing in mind the fact that the reflection is in one sheet we can say that $f(u)$ has a simple zero at the point $\xi + i\eta$ and a simple pole at the point $\xi - i\eta$. If we construct two more rectangles K_3 and K_4, symmetrical with K_1 and K_2 with respect to the imaginary axis, then one of these will be transformed into the domain $|z| < 1$ and the other into the domain $|z| > 1$; the function $f(u)$ will have a simple zero at the point $z = -\xi i\eta$ and a simple pole at the point $z = -\xi + i\eta$.

It can be shown in the same way as in [167] that $f(u)$ is an elliptic function with periods $2a$ and $2bi$. The fundamental parallelogram (rectangle) of periods consists of the above four rectangles and in this parallelogram of periods has the same zeros and poles as those mentioned above.

If we suppose that $\omega_1 = 2a$ and $\omega_2 = 2bi$ we can construct the Weierstrass function $\sigma(u)$ and also the new function:

$$\varphi(u) = \frac{\sigma(u - \xi - i\eta)\,\sigma(u + \xi + i\eta)}{\sigma(u - \xi + i\eta)\,\sigma(u + \xi - i\eta)}. \tag{180}$$

This function has the same simple zeros and poles in the above parallelogram of periods as our function $f(u)$. We shall show that the function (180) has periods ω_1 and ω_2; if this is so then $f(u)$ and $\varphi(u)$ only differ by a constant term. Using the property of the function $\sigma(u)$ as given by the equation (49) we can write:

$$\varphi(u + \omega_k) = \frac{\Theta^{\eta_k\left(u-\xi-i\eta+\frac{\omega_k}{2}\right)+\eta_k\left(u+\xi+i\eta+\frac{\omega_k}{2}\right)}}{\Theta^{\eta_k\left(u-\xi+i\eta+\frac{\omega_k}{2}\right)+\eta_k\left(u+\xi-i\eta+\frac{\omega_k}{2}\right)}} \frac{\sigma(u-\xi-i\eta)\,\sigma(u+\xi+i\eta)}{\sigma(u-\xi+i\eta)\,\sigma(u+\xi-i\eta)} =$$

$$= \frac{\sigma(u - \xi - i\eta)\,\sigma(u + \xi + i\eta)}{\sigma(u - \xi + i\eta)\,\sigma(u + \xi - i\eta)} = \varphi(u) \qquad (k = 1, 2),$$

which we had to prove. Hence

$$f(u) = C\,\frac{\sigma(u - \xi - i\eta)\,\sigma(u + \xi + i\eta)}{\sigma(u - \xi + i\eta)\,\sigma(u + \xi - i\eta)}.$$

To determine the constant C we put $u = 0$ and rewrite the above formula as follows

$$f(0) = C\,\frac{\sigma(-\xi - i\eta)}{\sigma(-\xi + i\eta)} \cdot \frac{\sigma(\xi + i\eta)}{\varrho(\xi - i\eta)}. \tag{181}$$

The definition of the function $\sigma(u)$ gives

$$\sigma(u) = u \prod'_{m_1, m_2} \left(1 - \frac{u}{w}\right) \Theta^{\frac{u}{w} + \frac{1}{2}\left(\frac{u}{w}\right)^2},$$

where $w = m_1\, 2a + m_2\, 2bi$. We assume that u is real. Owing to the fact that the product includes all integral values of m_1 and m_2, except $m_1 = m_2 = 0$, the factors are conjugate in pairs, i.e. those factors are conjugate which have the same m_1 but where m_2 has different signs. If $m_2 = 0$ then the corresponding factors are real.

Hence in the case under consideration where ω_1 is real and ω_2 is purely imaginary, the function $\sigma(u)$ is real when u is real. As a result of the principle of symmetry it will have conjugate values for conjugate values of u. It therefore follows that the numerator and the denominator of both fractions on the right-hand side of formula (181) are conjugate so that the modulus of each fraction is unity. Consider the left-hand side. The point $u = 0$ lies on the contour of the fundamental rectangle K_1 (at its apex) and therefore $f(0)$ lies on the unit circle, i.e. $|f(0)| = 1$. Hence formula (181) shows that $|C| = 1$, i.e. $C = e^{i\theta}$, where θ is real. We finally obtain the following formula for the function which transforms the rectangle K_1 into a unit circle:

$$f(u) = e^{i\vartheta}\, \frac{\sigma(u - \xi - i\eta)\, \sigma(u + \xi + i\eta)}{\sigma(u - \xi + i\eta)\, \sigma(u + \xi - i\eta)}\,. \tag{182}$$

The choice of θ is of no importance. When the values of θ change the unit circle rotates about its centre.

SUPPLEMENT

THE CONVERSION OF MATRICES INTO THE CANONICAL FORM

189. Auxiliary hypothesis. It is the aim of this supplement to prove the hypothesis which we stated without proof in [III$_1$, 27]: if A is a matrix then a non-singular matrix V can always be found such that the matrix VAV^{-1}, similar to the matrix A, will have a quasidiagonal (or diagonal) form

$$VAV^{-1} = [I_{\varrho_1}(\lambda_1),\ I_{\varrho_2}(\lambda_2),\ \ldots,\ I_{\varrho_p}(\lambda_p)], \tag{1}$$

where the matrices $I_\varrho(\lambda)$ are of the form

$$I_\varrho(\lambda) = \begin{Vmatrix} \lambda, & 0, & 0, & \ldots, & 0, & 0 \\ 1, & \lambda, & 0, & \ldots, & 0, & 0 \\ 0, & 1, & \lambda, & \ldots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & \lambda, & 0 \\ 0, & 0, & 0, & \ldots, & 1, & \lambda \end{Vmatrix}. \tag{2}$$

The letter ϱ indicates the rank of the matrix and the argument λ gives the value of each element on the main diagonal. When $\varrho = 1$ then the matrix $I_1(\lambda)$ becomes the number λ. In the proof of this hypothesis we shall enlarge upon some essential points.

Let us recall, first of all, the geometric meaning of the transformation to a similar matrix. The matrix A of order n is an operator in an n-dimensional space in the sense that it effects a given linear transformation of that space. As we know from [III$_1$, 21] the form of the matrix A depends on the choice of coordinates, i.e. on the choice of the main axes. If the matrix A gives a linear transformation for a definite choice of axes and if we transform the coordinates in the course of which the new components of every vector are given in terms of former components by means of the transformation V, then in the new system of coordinates our linear transformation will be given by the matrix VAV^{-1}. Hence our problem essentially involves

the choice of the axes; these are most important for the linear transformation which in the former system of coordinates was effected by the matrix A, viz. it involves a choice of axes for which our linear transformation is expressed by a matrix of the form shown on the right-hand side of equation (1).

Before solving this problem we shall explain some additional hypotheses which we shall use later. The majority of these assumptions were explained in earlier paragraphs but to obtain a complete picture we shall collect them together here.

First of all we shall explain the concept of the subspace which we have already met. If $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are k linearly independent vectors in space, where $k \leqslant n$, then the set of vectors given by the formula

$$c_1 \mathbf{x}_1 + \ldots + c_k \mathbf{x}_k, \tag{3}$$

where c_s are arbitrary numbers can be called the *subspace* of k dimensions formed by the above vectors. When $k = n$ the subspace coincides with the space. Another definition equivalent with the above definition can be used for defining the subspace viz. the subspace consists of a set of vectors which have the following two properties. If a vector $\mathbf{x}$ belongs to this set then the vector $c\mathbf{x}$, where c is arbitrary, will also belong to this set, and if two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to the set, then their sum $\mathbf{x}_1 + \mathbf{x}_2$ will also belong to this set. In other words by multiplying or adding vectors of the set we do not depart from the set.

In future we shall use two methods for determining a subspace which we shall now mention. Let P be a matrix of order n and $\mathbf{x}$ be an arbitrary vector in an n-dimensional space. The set of vectors defined by the formula

$$\xi = P\mathbf{x}, \tag{4}$$

is evidently a subspace which may coincide with the space. In fact if a vector $\xi_1 = P\mathbf{x}_1$ belongs to the set then the vector $c_1 \xi_1 = P(c_1 \mathbf{x}_1)$ also belongs to the set, and if two vectors $\xi_1 = P\mathbf{x}_1$ and $\xi_2 = P\mathbf{x}_2$ belong to our set then the vector $\xi_1 + \xi_2 = P(\mathbf{x}_1 + \mathbf{x}_2)$ evidently, also belongs to that set; therefore formula (4) for any arbitrary variable vector $\mathbf{x}$ does, in fact, define a subspace. As we said in [III_1, 15] the number of dimensions of this subspace is equal to the rank of the matrix P.

We shall now give the second method for defining a subspace. Let Q be a matrix of order n and consider a set of vectors which satisfy the equation

$$Q\mathbf{x} = 0. \tag{5}$$

We can show in the same way as we did above that this set of vectors forms a subspace. As we saw in [III$_1$, 14] this subspace will have k dimensions, where $(n - k)$ is the rank of the matrix.

When we talk about a subspace we are, of course, always assuming that it is not empty, i.e. that it does, in fact, contain vectors other than zero vectors. Let us consider the case when formula (4) gives an empty subspace, i.e. the case when our space formula (4) gives a zero vector for every $\mathbf{x}$. Bearing in mind the form of the linear transformation we can see that this will be so if, and only if, the matrix P is zero, i.e. when all its elements are zeros.

Let $E_1, \ldots, E_m$ be certain subspaces. We say that they form the complete system of the subspace if every vector $\mathbf{x}$ of the space can be represented uniquely as the sum of the vectors

$$\mathbf{x} = \xi_1 + \ldots + \xi_m, \tag{6}$$

of the above subspaces. Let us explain the condition for this unique representation. It follows directly from the condition that a zero vector cannot be represented as the sum (6) which contains terms other than zero and this, in its turn, is equivalent to the fact that there can be no linear dependence among vectors in the above subspaces. As an example consider the usual real three-dimensional space formed by vectors originating at a point O. We can define the complete system of subspaces by means of a plane L which passes through O and a line l through O, not lying in L. Let the first subspace, which is two-dimensional, be defined by two vectors in L which do not lie on the same straight line and the second subspace, which is one-dimensional, by a vector along l, then any vector in our three dimensional space can be represented in a unique way as the sum of the vectors in the L-plane and the vector along l.

Let A be a matrix which defines a linear transformation of the space. Suppose that we succeeded in finding a complete system of subspaces $E_1, \ldots, E_m$ of dimensions $\varrho_1, \ldots, \varrho_m$, so that each of these subspaces should be invariant with respect to the linear transformation defined by the matrix A, in other words, any vector of the subspace E_s $(s = 1, \ldots, m)$ will, as a result of the linear transformation defined by the matrix A, be transformed into a vector of that same subspace. In this case we have the following natural choice of axes for which the matrix A assumes the quasidiagonal structural form, viz. $\{\varrho_1, \ldots, \varrho_m\}$. We take for the first ϱ_1 axes any ϱ_1 linearly-independent vectors which form the subspace E_1; for the next ϱ_2 axes we

choose any ϱ_2 linearly-independent vectors which form the subspace E_2, etc. Since E_s forms the complete system of subspaces we have, evidently, $\varrho_1 + \ldots + \varrho_m = n$. It can readily be seen that for this choice of axes our matrix A will, in fact, be of the quasidiagonal form. We shall investigate this in greater detail but, to simplify our notation, we shall only deal with the case when $m = 2$. Let $(x_1, \ldots, x_n)$ be a certain vector and $(x_1', \ldots, x_n')$ another vector obtained from the first as a result of a linear transformation. Since the subspace E_1 is invariant and for the given choice of axes, when $x_{\varrho_1+1} = x_{\varrho_1+2} = x_n = 0$, we must have $x'_{\varrho_1+1} = x'_{\varrho_1+2} = \ldots = x_n' = 0$. Similarly, as a result of the invariance of the subspace E_2, when $x_1 = \ldots = x_{\varrho_1} = 0$ we must have $x_1' = \ldots = x_{\varrho_1}' = 0$. It follows that for our choice of axes a linear transformation is effected by a quasidiagonal matrix of the form

$$\left\| \begin{array}{cccccccc} a_{11}, & a_{12}, & \ldots, & a_{1\varrho_1}, & 0, & 0, & \ldots, & 0 \\ a_{21}, & a_{22}, & \ldots, & a_{2\varrho_1}, & 0, & 0, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{\varrho_1 1}, & a_{\varrho_1 2}, & \ldots, & a_{\varrho_1\varrho_1}, & 0, & 0, & \ldots, & 0 \\ 0, & 0, & \ldots, & 0, & b_{11}, & b_{12}, & \ldots, & b_{1\varrho_2} \\ 0, & 0, & \ldots, & 0, & b_{21}, & b_{22}, & \ldots, & b_{2\varrho_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & 0, & b_{\varrho_2 1}, & b_{\varrho_2 2}, & \ldots, & b_{\varrho_2\varrho_2} \end{array} \right\| = [A', B']. \qquad (7)$$

Notice that the choice of the fundamental axes in each subspace remains fully arbitrary and in future we shall use this freedom to convert each individual matrix A' and B', which is part of the quasidiagonal matrix (7), into the simplest form in a certain sense.

We shall now make the following assumptions which we shall use later.

Let $f(z)$ be the polynomial

$$f(z) = a_0 z^p + a_1 z^{p-1} + a_{p-1} z + a_p.$$

Replacing z by a matrix A we obtain a matrix polynomial

$$f(A) = a_0 A^p + a_1 A^{p-1} + \ldots + a_{p-1} A + a_p. \qquad (8)$$

By performing the operations shown on the right-hand side we obtain another matrix, i.e. any matrix polynomial $f(A)$ is also a matrix. Notice that the coefficient a_s of the polynomial is numerical. Owing to the fact that positive integral powers of one and the same matrix A_k commute with each other and with all constants, we can say that both the addition and the multiplication of polynomials of the same

matrix A are done in accordance with the usual laws of algebra in the same way as with polynomials in numerical arguments. Hence if an identity connects several polynomials in numerical arguments which have to be added or multiplied then this identity is still valid if the argument z is replaced by a matrix A.

The following characteristic equation is of fundamental importance in the conversion of a matrix into the canonical form:

$$\varphi(\lambda) = \begin{vmatrix} a_{11} - \lambda, & a_{12} & \dots, & a_{1n} \\ a_{21}, & a_{22} - \lambda & \dots, & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots, & a_{nn} - \lambda \end{vmatrix} = 0, \tag{9}$$

where a_{lk} are the elements of the matrix A. This equation can be written in the form

$$D(A - \lambda) = 0, \tag{10}$$

where the symbol $D(U)$ denotes the determinant of the matrix U. As we have shown before [90] the following Cayley identity applies:

$$\varphi(A) = 0, \tag{11}$$

i.e. if in the characteristic polynomial $\varphi(\lambda)$ of the matrix A the argument λ is replaced by the matrix A then a zero matrix results.

We shall now state two more simple hypotheses. As we know the zeros of the equation (9) are known as the *characteristic zeros of the matrix A*. We shall now prove the following theorem: *if* $\lambda_1, \dots \lambda_n$ *are the characteristic zeros of the matrix A then the matrix* A^s, *where s is a positive integer, will have the following characteristic zeros:* $\lambda_1^s, \dots \lambda_n^s$.

Remembering that the term of highest degree in the polynomial $\varphi(\lambda)$ is equal to $(-\lambda)^n$ we can write the following identity for λ:

$$D(A - \lambda) = \prod_{k=1}^{n} (\lambda_k - \lambda). \tag{12}$$

Let $\varepsilon = e^{2\pi i/s}$ be the sth zero of unity. We have the obvious identity [I, 175]:

$$(z - \lambda)(z - \varepsilon\lambda) \dots (z - \varepsilon^{s-1}\lambda) = z^s - \lambda^s. \tag{13}$$

Bearing in mind the fact that the determinant of a product of matrices is equal to the product of the determinants and also the identities (12) and (13) we can write

$$D(A^s - \lambda^s) = \prod_{k=1}^{n} (\lambda_k - \lambda) \prod_{k=1}^{n} (\lambda_k - \varepsilon\lambda) \dots \prod_{k=1}^{n} (\lambda_k - \varepsilon^{s-1}\lambda)$$

or

$$D(A^s - \lambda^s) = \prod_{k=1}^{n} \left[(\lambda_k - \lambda)(\lambda_k - \varepsilon\lambda) \ldots (\lambda_k - \varepsilon^{s-1}\lambda)\right].$$

As a result of the identity (13), this gives:

$$D(A^s - \lambda^s) = \prod_{k=1}^{n} (\lambda_k^s - \lambda^s),$$

i.e.

$$D(A^s - \mu) = \prod_{k=1}^{n} (\lambda_k^s - \mu),$$

which proves the above theorem

In future we shall have to evaluate a matrix in the quasidiagonal form

$$A = [A_1, A_2, \ldots, A_k].$$

It can easily be seen that it is equal to the product of the determinants of the matrices A_k, i.e.

$$D(A) = D(A_1) D(A_2) \ldots D(A_k). \tag{14}$$

To simplify the notation we shall only deal with the case when $k = 2$. The multiplication law gives:

$$[A_1 A_2] = [A, I][I, A_2],$$

hence

$$D(A) = D([A_1, I]) D([I, A_2]).$$

Using the expansion of a determinant in terms of the elements of a certain row or column we obtain, for example,

$$D([A_1, I]) = D(A_1),$$

from which formula (14) follows directly.

To conclude this section we recall that similar matrices have similar characteristic zeros.

190. The case of simple zeros. Above we have investigated in full the conversion of a matrix into its canonical form when its characteristic zeros were distinct. Let us now formulate this in a somewhat different way so that we can subsequently make analogous arguments in the general case when the characteristic zeros coincide.

Let $\lambda_1, \ldots, \lambda_n$ be the characteristic zeros of the matrix A which are all distinct. As we know from above there are, in this case, n linearly-independent vectors $\mathbf{v}_k$ which satisfy the equations

$$A\mathbf{v}_k = \lambda_k \mathbf{v}_k \qquad (k = 1, 2, \ldots, n)$$

or

$$(A - \lambda_k)\mathbf{v}_k = 0\,. \tag{15}$$

Each one of the vectors $\mathbf{v}_k$ gives a one-dimensional subspace E_k and these subspaces E_k together generate the whole space. Each of the vectors $c_k \mathbf{v}_k$, where c_k is a constant, must satisfy the equation $Ac_k \mathbf{v}_k = \lambda_k c_k \mathbf{v}_k$, i.e. as a result of the transformation A, each vector is multiplied by λ_k. In other words, each of the subspaces E_k is invariant with respect to the linear transformation effected by the matrix A. By taking the vectors $\mathbf{v}_k$ for the axes we convert the matrix A not into the quasidiagonal form but simply into the diagonal form since each of the subspaces E_k is one-dimensional.

We now consider an equation of the form

$$(A - \lambda_k)\mathbf{x} = 0 \qquad (k = 1, 2, \ldots, n)\,. \tag{16}$$

This equation is satisfied by vectors of the subspace E_k. It can readily be seen that it has no other solutions, i.e. that the equation (16) defines a one-dimensional subspace. In fact, if this equation determined a subspace of higher dimension e.g. a two-dimensional subspace, then as we have shown in [III$_1$, 27], each vector of this subspace would be linearly-independent of the vectors in the remaining subspaces E_k and we would then obtain $(n + 1)$ linearly-independent vectors in an n-dimensional space which is impossible. Hence in the case under consideration the equation (16) defines a one-dimensional subspace E_k.

These subspaces can be defined in a different way. To do so take the expansion into partial fractions

$$\frac{1}{\varphi(z)} = \sum_{k=1}^{n} \frac{a_k}{z - \lambda_k} \quad \text{or} \quad \sum_{k=1}^{n} a_k \frac{\varphi(z)}{z - \lambda_k} = 1\,,$$

where a_k are constants other than zero. Replacing z by the matrix A we obtain:

$$\sum_{k=1}^{n} a_k \frac{\varphi(A)}{A - \lambda_k} = 1\,. \tag{17}$$

Consider now the subspaces E'_k given by the formulae

$$\xi = a_k \frac{\varphi(A)}{A - \lambda_k}\mathbf{x} \qquad (k = 1, 2, \ldots, n)\,, \tag{18}$$

where $\mathbf{x}$ is an arbitrary vector in space. The constant factor a_k in formula (18) is, obviously, of no significance. Formula (17) gives the expansion for any vector $\mathbf{x}$:

$$\mathbf{x} = \sum_{k=1}^{n} a_k \frac{\varphi(A)}{A - \lambda_k} \mathbf{x}, \tag{19}$$

where the terms on the right-hand side belong to the subspaces E_k'. We will show that these subspaces E_k', given by formula (18), are the same as E_k, which is given by the equation (16). In fact, if ξ is a vector in E_k and is given by the formula (18) then, as a result of Cayley's formula, we have:

$$(A - \lambda_k)\,\xi = a_k \varphi(A)\, x = 0,$$

i.e. any vector E_k' belongs to E_k. It remains for us to show that, conversely, any vector η_k in E_k can be obtained from formula (18) provided $\mathbf{x}$ is suitably chosen. To do so we write η_k in place of $\mathbf{x}$ in (19). Owing to the fact that each polynomial $\varphi(A)/(A - \lambda_s)$, when $s \neq k$ contains the factor $(A - \lambda_k)$ we have, from (16) which determines E_k, the following:

$$\frac{\varphi(A)}{A - \lambda_s} \eta_k = 0 \quad \text{when} \quad s \neq k,$$

and therefore by replacing $\mathbf{x}$ by η_k we obtain from (19)

$$\eta_k = a_k \frac{\varphi(A)}{A - \lambda_k} \eta_k,$$

i.e. the vector η_k can, in fact, be obtained from formula (18) if we take the vector η_k instead of $\mathbf{x}$.

We will now use exactly the same arguments as when above for the case the zeros of the characteristic equation are repeated. This will make it possible to divide the space into a complete system of subspaces, invariant in relation to the transformation effected by the matrix A. For each of these subspaces all zeros of the characteristic equation are the same; the second step in our transformation will be to choose axes in the space which will bring us to the fundamental formula (1).

191. The first stage of the transformation in the case of repeated zeros. Let us suppose that the characteristic equation (9) has a zero a_1 which occurs r_1 times, a zero a_2 which occurs r_2 times etc. and finally, a

zero a_s which occurs r_s times. Expanding into partial fractions we obtain the following formula

$$\frac{1}{\varphi(z)} = \sum_{k=1}^{s} \frac{g_k(z)}{(z-a_k)^{r_k}},$$

where $g_k(z)$ is a polynomial in z of degree not higher than $(r_k - 1)$, and $g_k(a_k) \neq 0$. Consider the polynomials

$$f_k(z) = g_k(z) \frac{\varphi(z)}{(z-a_k)^{r_k}}. \tag{20}$$

We obviously have the identity

$$1 = \sum_{k=1}^{s} f_k(z),$$

or, replacing the argument z by the matrix A

$$1 = \sum_{k=1}^{s} f_k(A).$$

Thus any vector $\mathbf{x}$ can be represented as the sum of s vectors

$$\mathbf{x} = \sum_{k=1}^{s} f_k(A)\,\mathbf{x}. \tag{21}$$

Let us define certain subspaces $E_1, \ldots, E_s$, viz. assume that E_k is a subspace which is given by the formula

$$\xi = f_k(A)\,\mathbf{x} \qquad (k = 1, 2, \ldots, s). \tag{22}$$

We shall see later that none of these subspaces E_k is empty. Denote by $\mathbf{x}_k$ any vector in the subspace E_k. We shall prove, first of all, the following two formulae:

$$f_p(A)\,\mathbf{x}_q = 0 \text{ when } p \neq q \text{ and } f_p(A)\,\mathbf{x}_p = \mathbf{x}_p. \tag{23}$$

In fact, we have by definition

$$\mathbf{x}_q = f_q(A)\,\mathbf{x},$$

where $\mathbf{x}$ is any vector in the whole space. We thus obtain from (20):

$$f_p(A)\,\mathbf{x}_q = g_p(A)\,g_q(A)\,\frac{[\varphi(A)]^2}{(A-a_p)^{r_p}(A-r_q)^{r_q}}\,\mathbf{x}.$$

If p and q are different then the fraction on the right-hand side of the equation is a polynomial containing $\varphi(A)$ and therefore, as a result of Cayley's identity, this polynomial will be a zero matrix; this proves the first of the formulae (23). To prove the second

formula it is sufficient to assume in formula (21) that $\mathbf{x} = \mathbf{x}_p$ and make use of the first of the formulae (23). We then obtain directly the second of these formulae. We will now show that these subspaces form a complete system of subspaces. Formula (21) shows that any vector can be represented as the sum of vectors from the subspace E_k. Hence we have only to show that there can be no linear dependence among vectors in these subspaces. Assume that this linear dependence does exist

$$C_1 \mathbf{x}_1 + C_2 \mathbf{x}_2 + \ldots + C_s \mathbf{x}_s = 0\,, \tag{24}$$

where the vector $\mathbf{x}_k$ belongs to the subspace E_k. We must show that if $\boldsymbol{x}_k$ is not zero then the coefficient C_k must be zero. Applying the linear transformation $f_k(A)$ to both sides of the equation (24) we obtain from (23)

$$C_k \mathbf{x}_k = 0\,,$$

which proves our hypothesis.

Hence the constructed subspaces E_k do, in fact, form a complete system of subspaces and the sum of their dimensions must be n, i.e. the same as the dimension of the complete space.

Each of the subspaces E_k can be defined in a different way from the above definitions, viz. it can be shown that the subspace E_k is defined by an equation of the form

$$(A - a_k)^{r_k} \mathbf{x} = 0\,, \tag{25}$$

i.e. it represents a set of vectors which satisfy this equation. In fact, suppose that we have a vector ξ given by formula (22) and let us show that it will satisfy the equation (25). In fact, substituting the expression

$$\xi = f_k(A)\,\mathbf{x}$$

instead of $\mathbf{x}$ in the equation (25) we obtain on the left-hand side of this equation an expression

$$(A - a_k)^{r_k} f_k(A)\,\mathbf{x} = g_k(A)\,\varphi(A)\,\mathbf{x}\,,$$

and as a result of Cayley's identity $[\varphi(A) = 0]$ the result will, in fact, be zero. We now have to prove the converse, viz. that any solution η of the equation (25) can be obtained from formula (22) for a given choice of $\mathbf{x}$. Furthermore, we will show that from the equation

$$(A - a_k)^{r_k} \eta = 0 \tag{26}$$

it follows that

$$\eta = f_k(A)\,\eta\,. \tag{27}$$

In fact, we have from (21)

$$\eta = \sum_{p=1}^{s} f_p(A)\,\eta\,.$$

But each of the polynomials $f_p(A)$, when $p \neq k$, contains the factor $(A - a_k)^{r_k}$ and therefore, from (26), we have $f_p(A)\,\eta = 0$ when $p \neq k$ from which formula (27) follows directly.

Let us now return to [III$_1$, 27]. If $\lambda = a_k$ is a zero of the characteristic equation then substituting this in place of λ in the coefficients of the system (105) we obtain a homogeneous system with a zero determinant and we can therefore construct for it a solution which is not zero. This solution $\mathbf{v}_k$ will satisfy the equation

$$(A - a_k)\,\mathbf{v}_k = 0\,,$$

and hence the equation (25) will also be satisfied by it, i.e. it forms part of the subspace E_k which therefore cannot be empty.

It follows from the form of the equation (25) that each of the subspaces E_k will be invariant in relation to the transformation by the matrix A. In fact, if a vector $\mathbf{x}$ satisfies the equation (25) then it is clear that the vector $A\boldsymbol{x}$ will also satisfy this equation, since

$$(A - a_k)^{r_k} A\mathbf{x} = A(A - a_k)^{r_k}\,\mathbf{x}\,.$$

Let $q_1, \ldots, q_s$ be the dimensions of the subspaces $E_1, \ldots, E_s$. Selecting the fundamental axes in these subspaces in the way described in the previous section we obtain instead of the matrix A a similar matrix in the quasidiagonal form

$$S_1 A S_1^{-1} = [A_1, A_2, \ldots, A_s]\,, \tag{28}$$

and the component matrices will be of rank q_k. We will now show that the numbers q_k are the same as the order r_k of the zeros of the characteristic equation and that every matrix A_k has a single characteristic zero a_k which occurs r_k times.

To prove this take any vector ξ in the subspace E_k. It must satisfy the equation (25). With the new choice of axes this equation can be rewritten in the form

$$S_1(A - a_k)^{r_k} S_1^{-1}\,\xi = 0\,.$$

But we have, for example

$$S_1(A - a_k)^2 S_1^{-1} = S_1(A - a_k)\,S_1^{-1}\,S_1(A - a_k)\,S_1^{-1} = (S_1 A S_1^{-1} - a_k)^2\,,$$

so that the above equation can be rewritten as follows:

$$(S_1 A S_1^{-1} - a_k)^{r_k} \xi = 0 ,$$

or

$$[A_1 - a_k,\ A_2 - a_k,\ \ldots,\ A_s - a_k]^{r_k} \xi = 0 . \tag{29}$$

Consider, for example, the case when $k = 1$.

In this case all components of the vector ξ, except the first q_1, will be equal to zero and instead of the equation (29) we can write:

$$(A_1 - a_1)^{r_1} \xi' = 0 , \tag{30}$$

where by ξ' we denote an arbitrary vector in a q_1-dimensional space and the matrix $(A_1 - a_1)^{r_1}$ is of order q_1. Owing to the fact that the equation (30) holds for any vector ξ' we have

$$(A_1 - a_1)^{r_1} = 0 .$$

It follows that all the characteristic zeros of the matrix $(A_1 - a_1)^{r_1}$ must be equal to zero. But they are obtained from the characteristic zeros of the matrix $A_1 - a_1$ by raising them to the power of r_1; consequently all characteristic zeros of the matrix $A_1 - a_1$ are equal to zero and all characteristic zeros of the matrix A are equal to a_1. It can be shown similarly that in the general case all the characteristic zeros of the matrix A_k of rank q_k are equal to a_k. But the matrix (28), which is similar to the matrix A, must have the same characteristic zeros as the matrix A. Its characteristic equation has the form:

$$D([A_1 - \lambda,\ A_2 - \lambda,\ \ldots,\ A_s - \lambda]) = 0$$

or [189]:

$$D(A_1 - \lambda)\, D(A_2 - \lambda) \ldots D(A_s - \lambda) = 0 .$$

It follows that q_k must coincide with r_k and that the matrix A_k has a single characteristic zero a_k which occurs r_k times.

192. Conversion into the canonical form. We saw that each of the matrices A_k has a single characteristic zero a_k which occurs r_k times. To convert this matrix to the canonical form mentioned at the beginning of this chapter it is sufficient to select in a definite manner the axes in the subspace E_k. We thus have to investigate a particular case, viz. a matrix with a single characteristic zero. Suppose that a matrix D of order r has a single characteristic zero a which occurs r times.

The matrix $B = D - a$ will have a single characteristic zero equal to zero of order r and it is this matrix which we shall now consider.

As a result of Cayley's identity we have $B^r = 0$, since the left-hand side of the characteristic equation must be the matrix B of equal to $(-1)^r \lambda^r$. It may happen that $B^l = 0$, where l is a positive integer smaller than r. Take the smallest positive integer l to which following formula applies

$$B^l = 0 . \tag{31}$$

If, for example, the matrix B itself is equal to zero then $l = 1$. For the matrix

$$B = \begin{Vmatrix} 0, & 0, & 0, & 0 \\ 0, & 0 & 0, & 0, \\ 0, & 0, & 0, & 0 \\ 1, & 0, & 0, & 0 \end{Vmatrix}$$

it can easily be shown that $B^2 = 0$.

If the matrix B is equal to zero then $D = B + a$ is a diagonal matrix

$$D = [a, \ a, \ \ldots, \ a] ,$$

and therefore we have the canonical form already. Hence it is only important to consider the case when $l > 1$.

As a result of the condition (31) the equation

$$B^l \mathbf{x} = 0$$

describes the whole space of r dimensions. We shall in future denote this by ω. Let us now construct the equation

$$B^{l-1} \mathbf{x} = 0 .$$

Since the matrix B^{l-1} is not equal to zero this equation gives a subspace the dimensions of which are smaller than r. We shall generally definite the sequence of subspaces by the following equations:

$$B^l \mathbf{x} = 0; \quad B^{l-1} \mathbf{x} = 0; \ \ldots; \quad B\mathbf{x} = 0 , \tag{32}$$

and denote by F_m a subspace given by the equation $B^m \mathbf{x} = 0$. Let τ_m give the dimensions of this subspace. As we have already said above, F_l coincides with the whole space ω, and $\tau_l = r$, where $\tau_{l-1} < \tau_l$. If a vector ξ belongs to the subspace F_m, i.e. satisfies the equation $B^m \xi = 0$, then the vector $B\xi$ satisfies the equation $B^{m-1}(B\xi) = 0$, i.e. it forms part of the subspace F_{m-1}. It is also obvious that any vector

of the subspace F_m also belongs to the subspace F_{m+1}, i.e. the subspace F_m forms part of the subspace F_{m+1}. We shall see later that the subspace F_m always has smaller dimensions than the subspace F_{m+1}, i.e. the subspace F_m forms a proper part of the subspace F_{m+1} but does not coincide with it. We have, for the moment, the following inequalities:

$$\tau_l > \tau_{l-1} \geqslant \tau_{l-2} \geqslant \ldots \geqslant \tau_1, \tag{33}$$

and we will show that in each of these equations the strict inequality holds.

Let us put $\tau_l - \tau_{l-1} = r_l$, where r_l is a positive integer. In the subspace F_l (in other words, in the whole space ω) we can construct r_l linearly independent vectors $\xi_1, \ldots, \xi_{rl}$, so that none of their linear combinations belongs to F_{l-1}. In this case any vector of F_l can be represented as a linear combination of the vectors $\xi_1, \ldots, \xi_{rl}$ and a vector of F_{l-1}. To construct these vectors $\xi_1, \ldots, \xi_{r_1}$ we select τ_{l-1} linearly independent vectors in any way we choose, for example, in the subspace F_{l-1}. In this case the vectors $\xi_1, \ldots, \xi_{rl}$ will complement these latter vectors and form a complete system of linearly-independent vectors in the space ω. Denote similarly $\tau_{l-1} - \tau_{l-2} = r_{l-1}$, where r_{l-1} is a positive integer, and in the subspace F_{l-1} construct r_{l-1} linearly independent vectors so that none of their linear combinations belongs to the subspace F_{l-2}. Denote these vectors and any of their linear combinations by η. Consider now the vectors

$$B\xi_1, \ldots, B\xi_{r_l}. \tag{34}$$

They all belong to the subspace F_{l-1}. Let us show that none of their linear combinations can belong to the subspace F_{l-2}. For otherwise we should have

$$B^{l-2}(c_1 B\xi_1 + \ldots + c_{r_l} B\xi_{r_l}) = 0$$

or

$$B^{l-1}(c_1 \xi_1 + \ldots + c_{r_l} \xi_{r_l}) = 0,$$

i.e. it appears that the linear combination of the vectors $\xi_1, \ldots, \xi_{rl}$ belongs to the subspace B^{l-1} which contradicts the definition of these vectors. We thus see that the vectors (34) are linearly-independent vectors belonging to the subspace F_{l-1} and that they are the vectors η of this subspace, i.e. none of their linear combinations belongs to the subspace F_{l-2}. It follows directly that $r_{l-1} \geqslant r_l$. Similarly, denoting

$\tau_{l-2} - \tau_{l-3}$ by r_{l-2} we obtain $r_{l-2} \geqslant r_{l-1}$ and, in general putting $\tau_m - \tau_{m-1} = r_m$, we have:

$$0 < r_l \leqslant r_{l-1} \leqslant r_{l-2} \leqslant \ldots \leqslant r_1 \qquad (r_1 = \tau_1). \tag{35}$$

It also follows directly that in formula (33) we have everywhere the sign $>$, i.e.

$$\tau_l > \tau_{l-1} > \ldots > \tau_1. \tag{36}$$

The number r_l can be called the number of dimensions of the subspace F_l in relation to the subspace F_{l-1} which is contained in it. More strictly r_l is the number of linearly-independent vectors of F_l which are such that none of their linear combinations belongs to F_{l-1}. These vectors form a subspace G_l which is part of F_l. Similarly, r_{l-1} gives the dimensions of F_{l-1} in relation to F_{l-2} and we obtain, in the same way as above, a subspace G_{l-1} which is contained in F_{l-1}. In general, r_m gives the dimensions of F_m in relation to F_{m-1} and there are r_m linearly-independent vectors of F_m having the characteristic property that none of their linear combinations belongs to F_{m-1}; they form a subspace G_m which belongs to F_m. The subspace G_1 coincides with F_1. If ξ is a vector of G_m, and therefore also of F_m, then $B\xi$ belongs to F_{m-1}. It can, however, no longer belong to F_{m-2} since we would otherwise have $B^{m-2}(B\xi) = 0$ and, consequently, the vector ξ would belong not only to F_m but also to F_{m-1}, which contradicts the definition of the subspace G_m. Hence, by applying the linear transformation B to the subspace G_m, we obtain part of the subspace G_{m-1} (or all the subspace G_{m-1}) and the linearly-independent vectors in G_m are transformed into other linearly-independent vectors in G_{m-1}. From the formulae

$$r_m = \tau_m - \tau_{m-1} \qquad (r_1 = \tau_1)$$

and since $\tau_l = r$, it follows directly that:

$$r_l + r_{l-1} + \ldots + r_1 = r,$$

and the subspaces $G_l, \ldots, G_1$ evidently form the complete system of subspaces.

We finally come to the last stage of the construction, viz. to the construction of the final subspaces which are invariant in relation to the linear transformation by the matrix B. Take a vector ξ_1 in G_1 for the first axis and construct $(l-1)$ more axes according to the following equations:

$$\xi_2 = B\xi_1; \quad \xi_3 = B\xi_2; \quad \ldots; \quad \xi_l = B\xi_{l-1} \qquad (B\xi_l = B^l \xi_1 = 0).$$

It follows from the above arguments that these axes are linearly-independent and belong successively to the subspaces G_l, G_{l-1}, ..., G_1. It can readily be seen that they form a subspace, invariant in relation to the linear transformation B. In fact, it follows from (37) that for any choice of the constants c_k:

$$B(c_1\xi_1 + c_2\xi_2 + \ldots + c_l\xi_l) = c_1\xi_2 + c_2\xi_3 + \ldots + c_{l-1}\xi_l .$$

It follows directly that the matrix of the linear transformation of the invariant subspace so formed, when the ξ_k are taken for the axes, will be a matrix of order l in canonical form:

$$I_l(0) = \begin{Vmatrix} 0, & 0, & 0, \ldots, & 0, & 0 \\ 1, & 0, & 0, \ldots, & 0, & 0 \\ 0, & 1, & 0, \ldots, & 0, & 0 \\ \cdot & \cdot & \cdot \cdot \cdot & \cdot & \cdot \\ 0, & 0, & 0, \ldots, & 1, & 0 \end{Vmatrix}$$

Using in this way one of the vectors of G_l we can take any other vector η_1 of G_l, which is linearly independent of ξ_1, and add to it $(l-1)$ more vectors, according to the formulae:

$$\eta_2 = B\eta_1;\ \eta_3 = B\eta_2;\ \ldots;\ \eta_l = B\eta_{l-1} .$$

The l vectors so constructed will be linearly-independent not only of each other but also of the vectors ξ_k. This is directly due to the fact that linearly-independent vectors of G_m are transformed into linearly-independent vectors of G_{m-1}, by the transformation B. Taking the vectors η_k for the axes we obtain an invariant subspace corresponding to the linear transformation effected by the matrix $I_l(0)$. Using all r_l vectors of the subspace G_l we thus construct r_l invariant subspaces of l dimensions for each of which the linear transformation is effected by a matrix of the form $I_l(0)$.

We now pass on to the next subspace G_{l-1} of r_{l-1} dimensions, where $r_{l-1} \geqslant r_l$. We have already used r_l vectors of this subspace for the construction of the axes. Using the remaining $(r_{l-1} - r_1)$ vectors in the same way as we did above we construct $(r_{l-1} - r_l)$ invariant subspaces, in each of which our linear transformation for the given choice of axes, will be effected by a canonical matrix $I_{l-1}(0)$ of order $(l-1)$.

In general, when we reach the subspace G_m it will contain $(r_m - r_{m+1})$ unused linearly-independent vectors. Selecting these vectors in any way we please and applying to each in succession the transformation B we obtain from each vector $(m-1)$ additional vectors and, taking

these vectors as the axes, we thus obtain $(r_m - r_{m+1})$ sets of axes, where each set contains m axes and defines an invariant subspace of m dimensions corresponding to the transformation due to a canonical matrix $I_m(0)$ of order m.

Finally, when we reach the last subspace G_1, it contains $(r_1 - r_2)$ linearly-independent vectors satisfying the formula $B\xi = 0$. Taking these vectors as axes we obtain $(r_1 - r_2)$ invariant one-dimensional subspaces, corresponding to the linear transformation effected by a zero matrix of the first order. As a consequence of the new choice of axes we have a linear transformation σ of the component vector and the linear transformation caused by the matrix B, will now be due to a similar matrix in the quasidiagonal form

$$\sigma B \sigma^{-1} = [I_{\beta_1}(0),\ I_{\beta_2}(0),\ \ldots,\ I_{\beta_r}(0)]\,, \tag{37}$$

where among the lower suffixes in the square brackets there are r_l equal to l, $(r_{l-1} - r_1)$ equal to $(l-1)$ etc. and, finally, $(r_1 - r_2)$ equal to 1. We obviously have for the matrix $D = B + a$:

$$\sigma D \sigma^{-1} = \sigma B \sigma^{-1} + \sigma a \sigma^{-1} = \sigma B \sigma^{-1} + a\,,$$

i.e. a is added to the diagonal elements and we thus obtain

$$\sigma D \sigma^{-1} = [I_{\beta_1}(a), I_{\beta_2}(a), \ldots, I_{\beta_r}(a)]\,. \tag{38}$$

Let us return, finally, to our initial matrix A. In the previous section, according to formula (28), we represented it as a quasi-diagonal matrix in which every component matrix A_k had a single characteristic zero a_k which occurred r_k times. According to the above section each matrix A_k can be converted to the canonical form (38) with the aid of a matrix σ_k of order r_k. If we consider the matrix

$$S_2 = [\sigma_1, \sigma_2, \ldots, \sigma_s]\,,$$

then we have

$$S_2[A_1, A_2, \ldots, A_s]\, S_2^{-1} = [\sigma_1 A_1 \sigma_1^{-1}, \sigma_2 A_2 \sigma_2^{-1}, \ldots, \sigma_s A_s \sigma_s^{-1}]\,,$$

and our matrix A is finally obtained in the canonical form

$$(S_2 S_1)\, A (S_2 S_1)^{-1} = [I_{\varrho_1}(\lambda_1), I_{\varrho_2}(\lambda_2), \ldots, I_{\varrho_p}(\lambda_p)]\,. \tag{39}$$

The numbers λ_j will be equal to a_k and the sum of the lower suffixes of the component matrices $I_{\varrho_j}(\lambda_j)$, where $\lambda_j = a_k$ must be r_k.

Formula (39) completes the conversion of a given matrix to a canonical form. The question of the uniqueness of such a representation arises, i.e. it has to be proved that for any method of conversion to the

canonical form there will be inside the square bracket on the right-hand side of formula (39) a definite number of matrices $I\varrho_j(\lambda_j)$ for the given ϱ_j and λ_j. For example, suppose that the matrix A is converted to the canonical form in an arbitrary way

$$VAV^{-1} = [I_{\varrho_1}(\lambda_1), I_{\varrho_2}(\lambda_2), \ldots, I_{\varrho p}(\lambda_p)] .$$

Bearing in mind the fact that similar matrices have the same characteristic equation we can write the characteristic equation of the matrix A as follows:

$$D([I_{\varrho_1}(\lambda_1), I_{\varrho_2}(\lambda_2), \ldots, I_{\varrho p}(\lambda_p)] - \lambda) = 0$$

or

$$D([I_{\varrho_1}(\lambda_1 - \lambda),\ I_{\varrho_2}(\lambda_2 - \lambda), \ldots,\ I_{\varrho p}(\lambda_p - \lambda)]) = 0 ,$$

which is equivalent to the following [189]:

$$D(I_{\varrho_1}(\lambda_1 - \lambda))\, D(I_{\varrho_2}(\lambda_2 - \lambda)) \ldots D(I_{\varrho p}(\lambda_p - \lambda)) = 0 .$$

but because of the form of the matrix $I\varrho(a)$ it follows that

$$D[I_\varrho(a)] = (a - \lambda)^\varrho . \tag{40}$$

Hence the numbers λ_j must be equal to the characteristic zeros a_k of the matrix A and the sum of the symbols ϱ_j, for which $\lambda_j = a_k$, must equal the order r_k of the characteristic zero a_k. It remains to show that all the numbers ϱ_j must have a definite value. This can be proved by using the same geometric concepts as in the proof of the conversion of a matrix to the canonical form. In doing this the consideration of invariant subspaces will be of great importance We shall not give this proof here but in the following section we shall indicate an algebraic criterion which determines the values of all the symbols ϱ_j for the given matrix A. This criterion, based on considering the highest common factor for determinants of a given order of the matrix $(A - \lambda)$, is given without proof in the first part of this volume [III_1, 27]. It will also establish the uniqueness of the representation of a given matrix in canonical form.

193. The determination of the structure of the canonical form. As a preliminary let us prove two auxiliary lemmas.

LEMMA 1. *If A and B are two square matrices of order n and $C = AB$ is their product, then any determinant of the matrix C of order t, where $t \leqslant n$, can be represented as the sum of products of certain determinants of order t of the matrix A and determinants of order t of the matrix B.*

This lemma follows directly from the theorem proved in [III$_1$, 6].

COROLLARY. Suppose that the elements of the matrix $A(\lambda)$ are polynomials of λ and that the elements of the matrix B do not contain λ, where the determinant of the matrix B is not zero. Denote by $d_t(\lambda)$ the highest common factor of all determinants of order t which belong to the matrix $A(\lambda)$ and by $d_t'(\lambda)$ the highest common factor of the matrix $A(\lambda)B$. It follows directly from the theorem that $d_t(\lambda)$ must be a factor of $d_t'(\lambda)$. But we can write

$$A(\lambda) = [A(\lambda)\, B]\, B^{-1},$$

and the above lemma also gives us directly that $d_t'(\lambda)$ must be a factor of $d_t(\lambda)$, i.e. $d_t(\lambda)$ and $d_t'(\lambda)$ are equal. We would obtain the same result if instead of the matrix $A(\lambda)B$ we had constructed the matrix $BA(\lambda)$.

It also follows that *the highest common factors of the matrix $A(\lambda)$ and of the similar matrix $BA(\lambda)B^{-1}$ will be equal.*

Let us explain one more property of the highest common factor $d_t(\lambda)$. To do so we shall require a new definition.

DEFINITION. *By an elementary transformation of the matrix $A(\lambda)$, the elements of which are polynomials of λ, we understand a transformation of this matrix by means of a finite number of the following three operations:*

(1) the transposition of two rows (or columns);

(2) the multiplication of all the elements of a row (or column) by a certain constant, other than zero;

(3) the addition to the elements of a certain row (or column) the corresponding elements of another row (or column) which are al multiplied by a certain constant or bz a certain polynomial of λ.

If the matrix $A_1(\lambda)$ is obtained from $A(\lambda)$ by means of an elementary transformation then, evidently, the reverse will also be true and $A(\lambda)$ can be obtained from $A_1(\lambda)$ by means of an elementary transformation. If two matrices can be transformed into one another by means of an elementary transformation they are said to be *equivalent.*

LEMMA 2. *Equivalent matrices have the same highest common factors* $d_t(\lambda)$ $(t = 1, 2, \ldots, n)$.

It is sufficient to show that when all the determinants of order t of the matrix $A(\lambda)$ contain a common factor which is a polynomial of $\varphi(\lambda)$ then all the determinants of order t of the equivalent matrix $A_1(\lambda)$ will contain the same factor. The first and second of the above three transformations add a numerical factor other than zero to determinants of

order t and for these two transformations the lemma is evident. It remains to show that the common factor $\varphi(\lambda)$ will also remain in the third transformation. Suppose, for example, that this transformation involves the addition to elements of the p row of the corresponding elements of the qth row, $q \neq p$ which were previously multiplied by the polynomial $\psi(\lambda)$. All determinants of order t which do not contain a pth row or which do contain the pth and qth rows will not alter during this transformation because of the property VI of a determinant [III$_1$, 3]. Determinants of order t which contain a pth row but which do not contain a qth row will have the following form after the transformation: $A'(\lambda) \pm \psi(\lambda)A''(\lambda)$, where $A'(\lambda)$ and $A''(\lambda)$ are determinants of the matrix $A(\lambda)$ of order t. It follows from what was said above that the factor $\varphi(\lambda)$ of the determinant $A(\lambda)$ of order t will, in fact, be a factor of all determinants of order t of the matrix $A_1(\lambda)$.

LEMMA 3. *Any matrix of the form*

$$I_\varrho(a-\lambda) = \begin{Vmatrix} a-\lambda, & 0, & 0, & \ldots, & 0, & 0 \\ 1, & a-\lambda, & 0, & \ldots, & 0, & 0 \\ 0, & 1, & a-\lambda, & \ldots, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & \ldots, & 1, & a-\lambda \end{Vmatrix} \tag{41}$$

of order ϱ can be obtained in the form of a diagonal matrix $[1,1, \ldots, 1, (a-\lambda)^\varrho]$ *by means of an elementary transformation.*

When $\varrho = 1$ this lemma is trivial. Consider the case when $\varrho = 2$. Interchanging the rows and multiplying subsequently the elements of the first column by $-(a-\lambda)$, adding the products so obtained to elements of the second column and doing likewise with rows we obtain the required result after dividing the last column by (-1)

$$\begin{Vmatrix} a-\lambda, & 0 \\ 1, & a-\lambda, \end{Vmatrix} \to \begin{Vmatrix} 1, & a-\lambda \\ a-\lambda, & 0 \end{Vmatrix} \to \begin{Vmatrix} 1, & 0 \\ a-\lambda, & -(a-\lambda)^2 \end{Vmatrix} \to$$

$$\to \begin{Vmatrix} 1, & 0 \\ 0, & -(a-\lambda)^2 \end{Vmatrix} \to [1, (a-\lambda)^2].$$

For a matrix of the third order we obtain the following result after performing the above transformations:

$$\begin{Vmatrix} a-\lambda, & 0, & 0 \\ 1, & a-\lambda, & 0 \\ 0, & 1, & a-\lambda \end{Vmatrix} \to \begin{Vmatrix} 1, & 0 & 0 \\ 0, & (a-\lambda)^2, & 0 \\ 0, & -1, & a-\lambda \end{Vmatrix}.$$

Interchanging the second and third rows and performing further elementary transformations we obtain:

$$\left\|\begin{matrix}1, & 0, & 0\\ 0, & (a-\lambda)^2, & 0\\ 0, & -1, & a-\lambda\end{matrix}\right\| \to \left\|\begin{matrix}1, & 0, & 0\\ 0, & -1, & (a-\lambda)\\ 0, & (a-\lambda)^2, & 0\end{matrix}\right\| \to$$

$$\to \left\|\begin{matrix}1, & 0, & 0\\ 0, & -1, & 0\\ 0, & (a-\lambda)^2, & (a-\lambda)^3\end{matrix}\right\| \to \left\|\begin{matrix}1, & 0, & 0\\ 0, & -1, & 0\\ 0, & 0, & (a-\lambda)^3\end{matrix}\right\|,$$

and after dividing the second column by (-1) we obtain $[1, 1, (a-\lambda)^3]$. In this way we can gradually prove our lemma for a matrix of any order.

We shall now prove the algebraic criterion of structure for the canonical form of a matrix A which was given in [III$_1$, 27]. The corollary from Lemma 1 enables us to find the highest common factors $d_t(\lambda)$ not for the matrix $A-\lambda$ but for a similar matrix

$$V(A-\lambda)V^{-1} = VAV^{-1} - \lambda = [I_{\varrho_1}(\lambda_1-\lambda),\ I_{\varrho_2}(\lambda_2-\lambda), \ldots, I_{\varrho_p}(\lambda_p-\lambda)]. \tag{42}$$

Applying Lemma 3 to each matrix in the quasidiagonal form we can, by choosing the highest common factors $d_t(\lambda)$, replace the matrix (42) by a purely diagonal matrix which has along its main diagonal $(\varrho_1 - 1)$ units, $(\lambda_1-\lambda)^{\varrho_1}$ units, $(\varrho_2 - 1)$ units, $(\lambda_2-\lambda)^{\varrho_2}$ units etc. in succession. Notice also that if, during the construction of a determinant of order t belonging to this matrix we rule out a set of rows, then in the determinant so obtained at least one row and one column will consist of zeros and, this determinant will be equal to zero. Hence when constructing determinants belonging to a diagonal matrix we must always rule out the same rows and columns which simply involves the ruling out of diagonal elements, the product of which gives the value of the determinant.

Consider one zero $\lambda = a$ of the characteristic equation which occurs k times. The determinant of order n must contain the factor $(\lambda-a)^k$. Suppose that the highest common factor of the determinants of order $(n-1)$ contains only the factor $(\lambda-a)^{k_1}$. This means that the highest power of $(\lambda-a)$ which belongs to the constructed diagonal matrix is equal to $(k-k_1)$, i.e. the canonical representation of our matrix includes a matrix $I_{k-k_1}(a)$, and so on. If the highest common factor of determinants of the order $(n-2)$ is equal to $(\lambda-a)^{k_2}$ it means that after $(\lambda-a)^{k-k_1}$ the highest power of $(\lambda-a)$ belonging

to the constructed diagonal matrix is equal to $(\lambda - a)^{k_1 - k_2}$, i.e. the canonical form includes both matrices $I_{k-k_1}(a)$ and $I_{k_1-k_2}(a)$. When we finally arrive at determinants of a certain order of which at least one does not contain the factor $(\lambda - a)$ then we absorb all the component parts of the canonical form A for which $\lambda = a$. Hence the algebraic criterion for the canonical structure of a matrix mentioned in [III$_1$, 27] is proved. Notice that it not only follows from the above arguments that

$$k > k_1 > k_2 > \ldots > k_m, \quad \text{but also that} \quad l_1 \geqslant l_2 \geqslant \ldots \geqslant l_m \geqslant l_{m+1},$$

where

$$l_1 = k - k_1;\ l_2 = k_1 - k_2;\ \ldots;\ l_m = k_{m-1} - k_m;\ l_{m+1} = k_m.$$

194. Examples. We can solve the characteristic equation of a given matrix A and therefore the canonical form can be directly determined, for example, by means of the algebraic criterion mentioned in the previous paragraph. The problem remains of how to construct a matrix V with a determinant other than zero which would convert the given matrix A to the canonical form. In deducing the transformation to the canonical form we successively choose new axes. The choice of these axes finally converted the given matrix to the canonical form. But we know from [III$_1$, 21] how, for a given transformation of axes, we can construct a transformation U to convert the matrix A, as an operator, to the new form. If T is the linear transformation of axes then the matrix A will be converted to a new form UAU^{-1}, where $U = T^{*-1}$ i.e. to obtain U from T we must interchange the rows and columns in T and take the inverse matrix.

Let us now systematize the solution of the problem. First we select the new axes so as to obtain our matrix in the quasi-diagonal form, taking into account its various characteristic zeros as described in [191]. In this case the choice of the new axes involves the solution of an equation of the first degree in the following form: $(A - a_k)^{r_k}\mathbf{x} = 0$. Then we have to convert the matrix B with a single characteristic zero equal to zero to the canonical form. Here we have, at first, to determine the smallest number l, such that $B^l = 0$, and then we can construct a system of equations of the first degree of the form

$$B^{l-1}\mathbf{x} = 0.$$

To determine the order of this system of equations we take those vectors which do not satisfy it and, on subjecting them to the trans-

formation B, we construct a new series of axes, etc. We thus obtain a second transformation of axes and, at the same time, a second transformation of a matrix similar to the fundamental matrix A; this finally gives us the matrix in the canonical form. We shall explain these general ideas by a numerical example.

Consider a matrix of order 5:

$$A = \begin{Vmatrix} -2, & -1, & -1, & 3, & 2 \\ -4, & 1, & -1, & 3, & 2 \\ 1, & 1, & 0, & -3, & -2 \\ -4, & -2, & -1, & 5, & 1 \\ 4, & 1, & 1, & -3, & 0 \end{Vmatrix}.$$

Constructing its characteristic equation in accordance with the usual rules we obtain it in the following form:

$$(\lambda - 2)^3 (\lambda + 1)^2 = 0,$$

i.e. this equation has a zero $\lambda = 2$ of order three and a zero $\lambda = -1$ of order. Let us now construct the matrices $(A - 2)^3$ and $(A + 1)^2$. The equation $(A-2)^3 \mathbf{x} = 0$ must give a three-dimensional subspace, i.e. the matrix $(A - 2)^3$ is of the second rank.

Similarly the matrix $(A + 1)^2$ is of the third rank. By an elementary calculation we find:

$$(A-2)^3 = \begin{Vmatrix} -54, & 0, & -27, & 27, & 27 \\ -54, & 0, & -27, & 27, & 27 \\ 27, & 0, & 0, & -27, & -27 \\ -54, & 0, & -27, & 27, & 27 \\ 54, & 0, & 27, & -27, & -27 \end{Vmatrix}$$

and the system $(A - 2)^3 \mathbf{x} = 0$ can be written as the two equations:

$$\begin{aligned} -54x_1 - 27x_3 + 27x_4 + 27x_5 &= 0 \\ 27x_1 \qquad\quad - 27x_4 - 27x_5 &= 0, \end{aligned}$$

where $(x_1, x_2, x_3, x_4, x_5)$ are the components of the vector $\mathbf{x}$.

We thus have:

$$x_1 = x_4 + x_5, \quad x_3 = -x_4 - x_5,$$

where x_2, x_4 and x_3 remain arbitrary. On supposing that one is equal to unity and the others to zero we obtain three new axes which have the following components in terms of the former system:

$$(0, 1, 0, 0, 0); \quad (1, 0, -1, 1, 0); \quad (1, 0, -1, 0, 1). \tag{43}$$

Similarly, as a result of elementary transformations we obtain:

$$(A+1)^2 = \begin{Vmatrix} 0, & -6, & 0, & 9, & 3 \\ -9, & 3, & 0, & 9, & 3 \\ 0, & 6, & 0, & -9, & -3 \\ -9, & -12, & 0, & 18, & -3 \\ 9, & 6, & 0, & -9, & 6 \end{Vmatrix}$$

and equation $(A+1)^2 \mathbf{x} = 0$ can be written as a system of three equations:

$$\begin{aligned} -2x_2 + 3x_4 + x_5 &= 0 \\ -3x_1 + x_2 + 3x_4 + x_5 &= 0 \\ 3x_1 + 2x_2 - 3x_4 + 2x_5 &= 0, \end{aligned}$$

or

$$x_2 = x_1; \quad x_4 = x_1; \quad x_5 = -x_1,$$

where x_1 and x_3 remain arbitrary. This gives us two new axes:

$$(1, 1, 0, 1, -1) \text{ and } (0, 0, 1, 0, 0). \tag{44}$$

The new axes (43) and (44) will be expressed in terms of the former axes by the formulae:

$$\begin{aligned} e_1' &= e_2 \\ e_2' &= e_1 - e_3 + e_4 \\ e_3' &= e_1 - e_3 + e_5 \\ e_4' &= e_1 + e_2 + e_4 - e_5 \\ e_5' &= e_3 \end{aligned}$$

The matrix of this linear transformation has the form

$$T = \begin{Vmatrix} 0, & 1, & 0, & 0, & 0 \\ 1, & 0, & -1, & 1, & 0 \\ 1, & 0, & -1, & 0, & 1 \\ 1, & 1, & 0, & 1, & -1 \\ 0, & 0, & 1, & 0, & 0 \end{Vmatrix},$$

and by interchanging the rows and columns and taking the inverse matrix we obtain:

$$S_1^{-1} = T^{(*)} = \begin{Vmatrix} 0, & 1, & 1, & 1, & 0 \\ 1, & 0, & 0, & 1, & 0 \\ 0, & -1, & -1, & 0, & 1 \\ 0, & 1, & 0, & 1, & 0 \\ 0, & 0, & 1, & -1, & 0 \end{Vmatrix}, \quad S_1 = T^{(*)-1} = \begin{Vmatrix} -1, & 1, & 0, & 1, & 1 \\ -1, & 0, & 0, & 2, & 1 \\ 1, & 0, & 0, & -1, & 0 \\ 1, & 0, & 0, & -1, & -1 \\ 0, & 0, & 1, & 1, & 1 \end{Vmatrix}.$$

Multiplying the matrices in accordance in the usual way we convert the matrix A to the quasidiagonal form which consists of matrices of the third and second ranks:

$$S_1 A S_1^{-1} = \begin{Vmatrix} 1, & 0, & -1, & 0, & 0 \\ -2, & 2, & -2, & 0, & 0 \\ 1, & 0, & 3, & 0, & 0 \\ 0, & 0, & 0, & -2, & -1 \\ 0, & 0, & 0, & 1, & 0 \end{Vmatrix}. \tag{45}$$

The matrix of the third rank

$$D_1 = \begin{Vmatrix} 1, & 0, & -1 \\ -2, & 2, & -2 \\ 1, & 0, & 3 \end{Vmatrix}$$

has a characteristic zero $\lambda = 2$ occurring three times. Let us construct the new matrix:

$$B_1 = D_1 - 2 = \begin{Vmatrix} -1, & 0, & -1 \\ -2, & 0, & -2 \\ 1, & 0, & 1 \end{Vmatrix}.$$

with the characteristic zero $\lambda = 0$ occurring three times. By squaring it we obtain $B_1^2 = 0$. Hence we have, in this case, $l = 2$ and the system $B_1 \mathbf{x} = 0$ can be written as the single equation

$$x_1 + x_3 = 0\,.$$

The subspace F_2 in our former notation coincides with a complete three-dimensional space and the subspace F_1 can be formed by the vectors $(1, 0, -1)$ and $(0, 1, 0)$. We take the vector $(1, 0, 0)$ which does not form part of F_1. It forms the subspace G_2. Subjecting this vector to the operation B_1 we obtain

$$B_1(1, 0, 0) = (-1, -2, 1)\,.$$

The vectors $(1, 0, 0)$ and $(-1, -2, 1)$ form the first pair of new axes to which the following canonical matrix corresponds:

$$\begin{Vmatrix} 0, & 0 \\ 1, & 0 \end{Vmatrix}.$$

For the third axis we can take any vector F_1 which is linearly-independent of the vector $(-1, -2, 1)$. Let us take the vector $(0, 1, 0)$. The zero canonical matrix of the first order corresponds to this vector.

The new axes will be expressed in terms of the former axes by the formulae:

$$\begin{aligned} e_1'' &= e_1' \\ e_2'' &= -e_1' - 2e_2' + e_3' \\ e_3'' &= e_2'. \end{aligned}$$

We now take a matrix of the second rank which belongs to the quasidiagonal matrix (45):

$$D_2 = \left\| \begin{matrix} -2, & -1 \\ 1, & 0 \end{matrix} \right\|.$$

This matrix has a characteristic zero $\lambda = -1$ occurring twice. We construct the matrix

$$B_2 = D_2 + 1 = \left\| \begin{matrix} -1, & -1 \\ 1, & 1 \end{matrix} \right\|$$

with a characteristic zero $\lambda = 0$ occurring twice. Obviously $B_2^2 = 0$, as it should be in accordance with Cayley's formula. The equation $B_2 \mathbf{x} = 0$ is equivalent to $x_4 + x_5 = 0$, where x_4 and x_5 are the components of $\mathbf{x}$ in the two-dimensional space under consideration. We take for the first axis the vector (1, 0), i.e. $x_4 = 1$ and operating by B_2 on $x_5 = 0$, which does not satisfy the equation $B_2 \mathbf{x} = 0$, we obtain

$$B_2(1, 0) = (-1, 1).$$

Hence the two new axes will be (1, 0) and (−1, 1) so that we obtain the following expressions which give the new axes in terms of the former axes:

$$e_4'' = e_4'; \quad e_5'' = -e_4' + e_5'$$

or, taking into account earlier formulae:

$$\begin{aligned} e_1'' &= e_1' \\ e_2'' &= -e_1' - 2e_2' + e_3' \\ e_3'' &= e_2' \\ e_4'' &= e_4' \\ e_5'' &= -e_4' + e_5'. \end{aligned} \tag{46}$$

The cannoical matrix $I_2(0)$ will correspond to the last two axes.

We must add two to the first two matrices along the main diagonal and (-1) to the last canonical matrix. The final canonical form for the matrix A is:

$$\begin{Vmatrix} 2, & 0, & 0, & 0, & 0 \\ 1, & 2, & 0, & 0, & 0 \\ 0, & 0, & 2, & 0, & 0 \\ 0, & 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & 1, & -1 \end{Vmatrix} = [I_2(2), I_1(2), I_2(-1)]. \tag{47}$$

We finally construct the matrix V which transforms A to the form (47). As we know from above it is a product of $S_2 S_1$, where S_1 has been obtained above and S_2 can be determined from the formula

$$S_2 = T_1^{(*)-1},$$

where T_1 is the matrix of the linear transformation (46). We have:

$$T_1^{(*)} = \begin{Vmatrix} 1, & -1, & 0, & 0, & 0 \\ 0, & -2, & 1, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & -1 \\ 0, & 0, & 0, & 0, & 1 \end{Vmatrix} \text{ and } S_2 = T_1^{(*)-1} = \begin{Vmatrix} 1, & 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ 0, & 1, & 2, & 0, & 0 \\ 0, & 0, & 0, & 1, & 1 \\ 0, & 0, & 0, & 0, & 1 \end{Vmatrix}.$$

Multiplying these two matrices we obtain:

$$V = S_2 S_1 = \begin{Vmatrix} 0, & 1, & 0, & 0, & 1 \\ 1, & 0, & 0, & -1, & 0 \\ 1, & 0, & 0, & 0, & 1 \\ 1, & 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 1, & 1 \end{Vmatrix}.$$

Finally

$$VAV^{-1} = [I_2(2),\ I_1(2),\ I_2(-1)].$$

INDEX

Abel's theorem
  first 45, 317
  second 101
Absolutely convergent series 40
Algebra fundamental theorem of 89
Analytic
  continuation 63
    along curve 67
    chain of regions 66
    of function of two variables 323
    of matrix 342
    theorem of uniqueness 69
    uniqueness 66
  function 67
Asymptotic expansion 407

Bernoulli numbers 289, 291
Bessel
  equation 379, 424
  functions 427, 537
    addition theorem of 551
    asymptotic representation of 434
    integral representation of 550
    orthogonality of 545
    of purely imaginary argument 560
    relationships of 539
Biharmonic
  expression (Hurse's formula) 201
  functions 200
Bilinear transformation 125
  of upper half-plane into itself 131
Branch-point 73
  order of 81
  type of 81

Cauchy
  formula 27
    for infinite region 29
    for two variables 315
  –Riemann equation 11
  test, existence of limit 4
  theorem 21, 22
    on number of roots 88
Cayley's identity 339
Characteristic
  equation 674
  roots of matrix 674
Christoffel's formula 146
  outside
    of polygon 157
    of square 159
  rectangle 154
  regular n-sided polygon 157
  triangle 154
Circle of convergence 46
Complex number, interpretation of 3
Conformal transformation 16, 120
  conjugate trigonometric series 164
  Gauss's equation 398
  of second kind 123
Conjugate trigonometric series 103
Continuity of functions 5
Contour
  of Cauchy's type 32
    limiting values of 113
  integral 17
    assessments of modulus 18
    conditions for independence 19
Convergent series of matrices 325
Conversion function
  Bessel functions 549
  Hermitian 589
  Laguerre polynomials 595
  Legendre 506
Cotangent, expansion into simple fractions 251
Curve in plane 16

Derivative 7
  geometric meaning
    of argument 15
    modulus 14
  of matrix 350
Determining equation 370
Differential equation of second order 357
  analytic continuation of solutions 636
  determining equation of 370
  existence and uniqueness of 361
  irregular singularity of 368, 403
  periodic coefficients 445
  regular singularity of 367, 369, 375
  solutions, singularities 366–367
Diffraction of two-dimensional wave 212
Dirichlet's problem 518

Elliptic
  coordinates 661
  function 618
    of order m 622
  integrals 604
    complete 611
    first, second
      of Legendre's form 611
      and third kind 609
  Jacobian functions 650
    addition formulae of 656
    differential equations of 654
    modulus of 652
Equations of Fuch's class 376, 377
Equivalent matrices 689
Essential singularity 38, 61, 62
Euler
  constant 276
  integral
    of first kind 272
    of second kind 267
      representation by contour integral 279
  summation formula 290
Exponential function 2, 12, 54
  of matrix 328

Flow of liquid, two-dimensional 172
Fourier–Bessel formula 552
Fractional function (meromorphic) 248
  construction of 254
  expansion into simple fractions 248, 255
Fresnel's integral 223
Full flow round contour 178
Function of complex variable 6

Gauss's equation 381

Hankel functions 428, 553
  asymptotic formulae 432
Harnak's theorem 34
Hermitian polynomials 586
  asymptotic expression of 597
    orthogonality of 587
Hilbert
  core 193
  formula 193
  problem 196
Hurse's formula 202
Hypergeometric series 381

Infinite product 258
  for bilinear transformations 133
  convergent 258
  for $[\Gamma(z)]^{-1}$ 276
Integral(s)
  calculus, fundamental formula 24
  depending on parameters 264
  function 256
    construction from roots 261
    periodic, expansion of 636
  indefinite 266
Integration
  of rational fraction 227
  of trigonometric expressions 225, 228
Interior point 5
Interpolation polynomials 337
Invariant functions $\wp(u)$ 631
Irregular singularity, differential equation 368, 403
Isolated singularity 36, 80
Isothermic nets 122

Jacobian
  formula 454
  polynomials 394
Jordan's lemma 232

Laguerre
  function 594
    orthogonality of 594
  polynomials 439
    asymptotic expressions of 604
Lamé equation 663
Laplace transformation 410, 435
Laurent's series 50
Legendre
  function 530
    of second kind 534
      expressed as integral 535
  polynomials 391, 504
    addition theorem 513
    asymptotic expression of 600
    orthogonality of 392
  relationship 628
Limit
  of complex variable 4
  sequences of matrices 324
Linear transformation 124
Liouville's theorem 35
Lipschitz condition of order 104
Logarithm 2, 55, 78
  matrix 331, 345

Magnetic field 188
  electrostatic 181
Meromorphic *see* Fractional Function
Modulus, principle of 35

Neighbourhood of matrix 325
Neumann
  function 553
    expression in terms of Bessel functions 555
  problem
    exterior 519
    interior 519
Newton's binomial 56

Picard's theorem 39
Point at infinity 61
Pole 38
  of order m 60
  point at infinity 61
Potential
  of complex flow 172
  of spherical shell 522
  of voluminous masses 520
Power
  function of matrix 332
  series 45, 296
    conversion of 92
    differentiation and integration of 47
    of matrix 325
      conversion of 329
      multiplication 330
      theorem of uniqueness 328
    of two variables 316
Principal value of integral 103
  contour, of Cauchy's type 107
  in infinite interval 231
Principle of permanency 69
Pseudoelliptic integral 605

Radius of convergence 45
Reflection
  in circle 134
  of elastic waves 216
  in straight line 134
Region 5
  bounded 6
  closed 5
  connected 20
  positive direction of describing 21
Regular
  function (holomorphic) 8
    in closed region 20
    of matrix 328
    at point at infinity 61
    of two variables 313
    uniqueness 63
  singularity of differential equation 367, 369, 375
Residue
  calculation of 87

of function
at essential singularity 62
at infinity 62
at pole 60
sum of rational fraction 245
theorem 85
Riemann
surface 74
symbol 380
theorem 144
Roots of regular function 64
Rouche's theorem 89

Schrödinger's equation 584
Schwarz's formula 189
Sigma function 632
Silvester's formula 339
Similarity transformation 125
Singularity 59
Sokhotski, Iu. V., theorem 38
Spherical function
of order n (surface) 496
definite expression of 499
integral of square of 501
orthogonality of 500
of volume 496
Steepest descent method 293
Subspace 671
Symmetrical points in relation to
circle 98
Symmetry, principle of 95
general 99, 133
System of linear differential equations
453
canonical solution 469
complete solution 455
constant coefficients 242, 348
existence and uniqueness of
solution 453
expansion, irregular singularity 477
integral matrix 456
regular 459
singularity 456

Taylor
coefficients, assessment of 49
series 49
Theta-functions 644
first 642
Trigonometric functions 2
Tschebyshev's polynomial 398

Uniform convergence of series 41

Wallis's formula 287, 288
Wave equation 203
in cylindrical coordinates 577
fundamental theorem for solutions
of 205
homogeneous solution of 206
in spherical coordinates 580
Weierstrass
function 624
infinite product 263
theorem 42
Whittaker
equation 443
function 444

Zhukovskii's, N. E., formula 179